Wave Functions of Nonlocal Potentials: The Perey Effect*

N. AUSTERN
University of Pittsburgh, Pittsburgh, Pennsylvania
(Received 28 September 1964)

An eigenfunction of an attractive nonlocal single-particle potential is always smaller inside the region of the potential than outside; the converse occurs for repulsive potentials. This is the Perey effect. In the present article explicit formulas for the effect are derived for the case of motion in one dimension, and interpretive discussions of the effect are given. The derivation does not employ series expansions. It is argued that the effect can be understood in terms of the fundamental many-body theory from which the single-particle potential has (in principle) been derived. Some of the wave function lies in the channels which have been eliminated in the course of that derivation.

1. INTRODUCTION

THE single-particle wave functions which are used in the analysis of many-body systems are eigenfunctions computed for the problem of a particle moving in a potential well. In most practical calculations this potential is assumed to be "local," i.e., to be diagonal in configuration space. This assumption is in part motivated by mathematical convenience, in part by theoretical demonstrations of its reasonableness in a number of circumstances, and in part by analogy with the classical potentials of electrostatics and gravitation.

Nevertheless the present widespread use of local single-particle potentials should probably be regarded as an interim procedure. The single-particle potentials which actually arise as parts of more complete many-body theories are chosen to meet conditions of self-consistency, as in the Hartree theory, and generally are found to be to some extent nonlocal. This nonlocality may be caused by antisymmetrization, as in the Hartree-Fock theory, or it may be caused by virtual excitations of the medium in which the particle propagates. Thus nonlocality of the self-consistent potentials is expected, and it is interesting to explore what its physical consequences might be.

Phenomenologically chosen single-particle potentials, such as the nuclear optical potential, or the shell model potential, probably should resemble the self-consistent potentials of many-body theory, and should be nonlocal.

Perey and Buck[1] recently investigated by numerical means the effects caused by introducing into the nuclear-optical potential a particular sort of nonlocality which is physically not unreasonable; it resembles the nonlocalities of the self-consistent potentials. The interaction which they considered has the separable form

$$V(\mathbf{r},\mathbf{r}')=U(\tfrac{1}{2}|\mathbf{r}+\mathbf{r}'|)H(|\mathbf{r}-\mathbf{r}'|). \quad (1)$$

The Schrödinger equation for single-particle motion then becomes

$$-(\hbar^2/2M)\nabla^2\Psi(\mathbf{r})+\int V(\mathbf{r},\mathbf{r}')\Psi(\mathbf{r}')d^3r'=E\Psi(\mathbf{r}). \quad (2)$$

It was found that in many respects the eigenfunctions of Eq. (2) could be duplicated with the eigenfunctions of a suitably chosen "equivalent local potential." However, Perey later pointed out[2] that it is an interesting special feature of the eigenfunctions of the nonlocal potential that they are systematically smaller in the nuclear interior than are the eigenfunctions of the equivalent local potential.

The Perey effect is found to be remarkably systematic. In all cases studied, the wave function in the nuclear interior, for nucleons, is reduced by about 20% from the amplitude which is computed using the equivalent local potential. This reduction is found both for continuum wave functions and for bound state wave functions. It is an important effect. Matrix elements of many-body systems typically involve products of three or four single-particle functions, and a reduction of 20% in each of four factors implies a reduction of 60% in their product.

The present article presents an interpretative discussion of the Perey effect. Explicit mathematical expressions for the effect are derived in Sec. 2. The derivation is designed so as to emphasize at every step the physical motivations for the procedures which are followed. Questions of flux conservation are discussed in Sec. 3. In Sec. 4 there is raised the question of why the results of Secs. 2 and 3 are reasonable, on the basis of a qualitative comparison with effects known from many-body theory. A brief Appendix pursues further a "lattice model" which is introduced in Sec. 3.

2. DEMONSTRATION OF THE PEREY EFFECT

What is mainly considered in this section is the meaning of introducing a local potential which is "equivalent" to a given nonlocal potential.

Qualitative discussions are most easily constructed for the case of one-dimensional motion. This is both interesting in itself, and it provides understanding of the behavior of individual partial waves of a three-dimensional problem. We therefore consider the Schrödinger

* Supported in part by the National Science Foundation.
[1] F. G. Perey and B. Buck, Nucl. Phys. 32, 353 (1962).

[2] F. G. Perey, in *Direct Interactions and Nuclear Reaction Mechanisms*, edited by E. Clementel and C. Villi (Gordon and Breach, Science Publishers, Inc., New York, 1963), p. 125.

equation

$$-\psi''(x)+I(x)=k^2\psi(x), \tag{3}$$

$$I(x)=\int_{-\infty}^{\infty} U(\tfrac{1}{2}(x+x'))H(x-x')\psi(x')dx'. \tag{4}$$

Here we assume $H(x-x')=H(x'-x)$, and also that

$$\int_{-\infty}^{\infty} H(x-x')dx'=1. \tag{5}$$

The potential function U has the dimensions (length)$^{-2}$. In order to avoid considerations of boundary conditions for a half space, we choose Eq. (3) to apply over the full one-dimensional space, $-\infty<x<\infty$.

Now, one kind of local potential which can be associated with the above $\psi(x)$ is the one which Perey[2] calls the "trivially-equivalent (TE) local potential." It is generated merely by demanding that $\psi(x)$ satisfy a Schrödinger equation in which there is a local potential, and then solving for that potential. Thus,

$$-\psi''(x)+U_{\mathrm{TE}}(x)\psi(x)=k^2\psi(x), \tag{6}$$

$$U_{\mathrm{TE}}(x)=I(x)/\psi(x). \tag{7}$$

This $U_{\mathrm{TE}}(x)$ is the only local potential of which $\psi(x)$ is an exact eigenfunction. Clearly each different state $\psi(x)$ leads to a different $U_{\mathrm{TE}}(x)$ function. This is not objectionable. However, it is very objectionable that $U_{\mathrm{TE}}(x)$ has a pole wherever $\psi(x)$ has a zero.[2] The function $U_{\mathrm{TE}}(x)$ is not a satisfactory equivalent local potential.

On the other hand, it is possible to seek an equivalent local potential $U_L(x)$ which reproduces only certain selected properties of $\psi(x)$, and which can therefore be chosen to be of a congenial mathematical form. It is this point of view that is followed whenever local optical potentials of usual types (Saxon shape, etc.) are fitted by numerical means to match the scattering predicted by a nonlocal potential.[1,2] The potential $U_L(x)$ which is found in those calculations predicts an eigenfunction $\psi_L(x)$ which exactly matches $\psi(x)$ everywhere outside the nucleus, but which differs from $\psi(x)$ in the nuclear interior.

Let us therefore introduce a function $F(x)$ in terms of which to formalize the study of the modifications of $\psi(x)$ which are considered to be permissible. We define

$$\psi(x)\equiv F(x)\psi_L(x). \tag{8}$$

In order that $\psi(x)$ and $\psi_L(x)$ describe the same scattering it is necessary that

$$F(x)\to 1 \quad \text{as} \quad |x|\to\infty. \tag{9}$$

This is a boundary condition for $F(x)$, and will be used in conjunction with a differential equation that will be derived below.

The function $\psi_L(x)$ is a Schrödinger eigenfunction, governed by the potential $U_L(x)$. Therefore

$$-\psi_L''(x)+U_L(x)\psi_L(x)=k^2\psi_L(x), \tag{10}$$

and

$$U_L(x)=\{I(x)-\psi_L(x)F''(x)-2\psi_L'(x)F'(x)\}/\psi(x). \tag{11}$$

Comparison of Eq. (11) with Eq. (7) shows that introduction of the function $F(x)$, which modifies the wave function, has added two additional terms in the numerator. Clearly we must choose these terms so that the zeros of the numerator lie over the zeros of the denominator, so that $U_L(x)$ will have no poles. Perhaps the easiest mathematical procedure for ensuring that the numerator and denominator of Eq. (11) oscillate in phase, so that there are no poles, is found by considering a "local exponential approximation" for $\psi_L(x)$, namely

$$\psi_L(x)\approx Ae^{ixk(x)}+Be^{-ixk(x)}. \tag{12}$$

Here the momentum at point x is $k(x)$, and is assumed to be constant over a sufficient interval in x so that the integral $I(x)$ in Eq. (11) can be computed easily. Upon substitution of Eq. (12) into Eq. (11) we find a result of the form

$$U_L(x)=\{Ce^{ixk(x)}+De^{-ixk(x)}\}/\{Ae^{ixk(x)}+Be^{-ixk(x)}\}, \tag{13}$$

where the coefficients C and D are computed from Eq. (11). It is in Eq. (13) that we introduce the requirement that $U_L(x)$ be of simple mathematical form. In order that $U_L(x)$ have no poles, or any other mathematical awkwardnesses which are caused by the oscillations of the wave function, it is clearly necessary that

$$(C/A)=(D/B). \tag{14}$$

We therefore find

$$U_L(x)F(x)\approx\int_{-\infty}^{\infty} U(x+\tfrac{1}{2}s)H(s)F(x-\tfrac{1}{2}s)e^{\pm isk(x)}ds - F''(x)\mp 2ik(x)F'(x), \tag{15}$$

where $s=x'-x$. The two signs in Eq. (15) correspond to the two directions of the momentum that was introduced in Eq. (12). Equation (15) must hold for both signs. Equation (15) is simplified if it is considered that $F(x)$ should vary slowly over the interval of nonlocality, i.e., if $F(x)$ should vary much more slowly than $H(s)$. In this case $F(x)$ may be factored out from the integral, and $F''(x)$ may be dropped, giving

$$U_L(x)F(x)\approx F(x)\int_{-\infty}^{\infty} U(x+\tfrac{1}{2}s)H(s)e^{\pm isk(x)}ds \mp 2ik(x)F'(x). \tag{15'}$$

Of course, because Eq. (15′) must hold for both signs, it really is two equations. In their most convenient form

these two equations are

$$[F'(x)/F(x)]=[2k(x)]^{-1} \times\int_{-\infty}^{\infty} U(x+\tfrac{1}{2}s)H(s)\sin[sk(x)]ds, \quad (16)$$

$$U_L(x)=\int_{-\infty}^{\infty} U(x+\tfrac{1}{2}s)H(s)\cos[sk(x)]ds. \quad (17)$$

The equivalent local potential is easily generated by using Eq. (17). It is seen to be somewhat weaker than the original $U(x)$, as expected.

Equation (16) is the equation of the double-signed imaginary terms of Eq. (15). This equation is actually an immediate restatement of Eq. (14). Integration of this equation gives

$$\ln F(x)=\int_{-\infty}^{x}[2k(x')]^{-1} \times\int_{-\infty}^{\infty} U(x'+\tfrac{1}{2}s)H(s)\sin[sk(x')]dsdx'. \quad (18)$$

Equation (9) was used to determine the integration constant in Eq. (18). Equation (18) is closely related to an equation derived by Perey and Saxon.[3] We see that nonvanishing contributions appear in Eq. (18) only if $U(x'+\frac{1}{2}s)$ should be rapidly varying, therefore only in the region of the nuclear surface. If $U(x)$ should be an attractive potential, then for x' near the left-hand edge of the nucleus we have that $dU(x'+\frac{1}{2}s)/ds$ is negative. Then $\ln F(x)$ picks up a negative increment as we enter the nucleus at the left-hand side. It picks up an equal and opposite positive increment as we depart at the right-hand side. Because $F(x)=1$ if x is a point outside the nucleus, we see that $F(x)<1$ everywhere inside the nucleus. This result is the Perey effect.

If $U(x)$ should be a repulsive potential it is clear that the Perey effect would be reversed; we would then have $F(x)>1$ in the nuclear interior.

No numerical evaluations of Eqs. (17) and (18) will be attempted in the present article. Sufficiently good approximate evaluations of Eq. (18) are available from the work of Perey and Saxon,[3] and we may refer to their article for details. However, it is interesting to note that if the range of nonlocality of $H(s)$ is sufficiently small, then the sine function in Eq. (18) may be replaced with the linear approximation, so that

$$\ln F(x)\approx\frac{1}{2}\int_{-\infty}^{x}\int_{-\infty}^{\infty} U(x'+\tfrac{1}{2}s)H(s)sdsdx'. \quad (18')$$

Equation (18′) does not depend on the local momentum, so that in this approximation the Perey effect in the wave function is independent of the quantum mechanical state that is being considered.

It may be remarked that the work thus far, up to the results of Eqs. (17) and (18) does not make use of the separable property of the nonlocal potential of Eq. (1), or of Eq. (4).

It may also be remarked that a detailed correspondence with the work of Perey and Saxon[3] is obtained by expanding U of Eq. (18′) in a power series in the variable s, in the form

$$U(x'+\tfrac{1}{2}s)=U(x')+(\tfrac{1}{2}s)U'(x')+\tfrac{1}{2}(\tfrac{1}{2}s)^2U''(x')+\cdots.$$

Because $H(s)$ in Eq. (18′) is even in s, only the odd powers of the series contribute in the integral. The result of Perey and Saxon is obtained by carrying only the first nonvanishing term, giving

$$\ln F(x)\approx\tfrac{1}{4}U(x)\int_{-\infty}^{\infty} s^2H(s)ds. \quad (18'')$$

This formula is very convenient, and may well provide sufficient accuracy for most applications. The derivation of this very simple formula now does make use of the separable property of the nonlocal potential.

3. CONSERVATION OF FLUX

Further insight can be gained by examining the equation of conservation of quantum-mechanical flux.[4] This equation is obtained from Eq. (3) by multiplying each term of that equation by $\psi^*(x)$, and then extracting the equation of the imaginary parts. The rate of change of flux is found to be

$$\psi^*(x)\psi''(x)-\psi(x)\psi''^*(x)\equiv J'(x), \quad (19)$$

$$J'(x)=\int_{-\infty}^{\infty} H(x-x')\{\psi^*(x)\psi(x')U(\tfrac{1}{2}(x+x')) -\psi(x)\psi^*(x')U^*(\tfrac{1}{2}(x+x'))\}dx'. \quad (20)$$

It is seen in Eq. (20) that $J'(x)$ can be nonvanishing for two reasons, either because of the nonlocality, or because U is complex. If U should be real, then the rate of change of flux caused by nonlocality alone is

$$J'(x)=\int_{-\infty}^{\infty} H(x-x')U(\tfrac{1}{2}(x+x')) \times\{\psi^*(x)\psi(x')-\psi(x)\psi^*(x')\}dx'. \quad (21)$$

It is clear that integration of this $J'(x)$ over the entire space, $-\infty<x<\infty$ yields a zero result, so that in a problem governed by a real nonlocal potential there is over-all conservation of flux.

Flux also is conserved locally, *on the average*. This is seen by averaging $J'(x)$ about the point $\bar{x}$, using a

[3] F. G. Perey and D. S. Saxon, Phys. Letters **10**, 107 (1964), and to be published.

[4] Professor D. S. Saxon suggested to the author that it would be useful to study this topic.

symmetrical weight function $W(x-\bar{x})=W(\bar{x}-x)$. Then

$$\int_{-\infty}^{\infty} J'(x)W(x-\bar{x})dx$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'H(x-x')U(\tfrac{1}{2}(x+x'))$$
$$\times[\psi^*(x)\psi(x')-\psi(x)\psi^*(x')]$$
$$\times[W(x-\bar{x})-W(x'-\bar{x})]. \quad (22)$$

If the rate of change of W is much slower than that of H, then the integrand of the right-hand side of Eq. (22) is seen to be vanishingly small.

It is interesting to introduce a square well "lattice model" for the one-dimensional nonlocal potential

$$U(x)=-\kappa^2, \quad |x|<x_0,$$
$$=0, \quad |x|>x_0, \quad (23)$$

$$H(x-x')=\tfrac{1}{2}\{\delta(x'-x-\xi)+\delta(x'-x+\xi)\}. \quad (24)$$

The Schrödinger equation of Eq. (3) takes on different simple forms in each of the five regions which are shown in Fig. 1, when Eqs. (23), (24) are inserted into Eq. (3). These forms are

$$\psi''(x)+k^2\psi(x)=0, \quad (25a)$$
$$\psi''(x)+k^2\psi(x)+\tfrac{1}{2}\kappa^2\psi(x+\xi)=0, \quad (25b)$$
$$\psi''(x)+k^2\psi(x)+\tfrac{1}{2}\kappa^2[\psi(x+\xi)+\psi(x-\xi)]=0, \quad (25c)$$
$$\psi''(x)+k^2\psi(x)+\tfrac{1}{2}\kappa^2\psi(x-\xi)=0, \quad (25d)$$
$$\psi''(x)+k^2\psi(x)=0. \quad (25e)$$

Evidently the differential equation in region (b) depends on the solution function in region (c), and conversely; likewise the differential equations in regions (c) and (d) depend on each other. There is no coupling to regions (a) and (e).

Provided x_0/ξ is an integer we may divide region (c) into subintervals of the same length as that of regions (b) and (d). Each subinterval is coupled only to the two neighboring subintervals. An orderly set of equations to replace (25b)–(25d) is obtained by labeling ψ with the label of the subinterval in which it is being considered, so that $\psi(x)\equiv\psi_l(x)$ in subinterval l. Then region (b) is labeled *1*, region (d) is labeled n, and

$$0=\psi_1''(x)+k^2\psi_1(x)+\tfrac{1}{2}\kappa^2\psi_2(x+\xi), \quad (26.1)$$
$$0=\psi_2''(x)+k^2\psi_2(x)+\tfrac{1}{2}\kappa^2[\psi_1(x-\xi)+\psi_3(x+\xi)], \quad (26.2)$$
$$\vdots$$
$$0=\psi_n''(x)+k^2\psi_n(x)+\tfrac{1}{2}\kappa^2\psi_{n-1}(x-\xi). \quad (26.n)$$

No general solution of Eqs. (25) has been found. Some further analysis which yields partial solutions of these equations is given in the Appendix.

Flux conservation in the lattice model is best studied by integrating $J'(x)$ from the point $(-x_0-\frac{1}{2}\xi)$, where it first becomes nonzero, to any other interior point x. It is found that

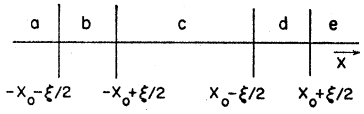

FIG. 1. Regions of x space in which the Schrödinger equation for the lattice model takes on simple forms.

$$J(x)-J(-x_0-\tfrac{1}{2}\xi)$$
$$=-\tfrac{1}{2}\kappa^2\int_{-x_0-1/2\xi}^{-x_0+1/2\xi}[\psi_1^*(x)\psi_2(x+\xi)-\psi_1(x)\psi_2^*(x+\xi)]dx$$
$$-\tfrac{1}{2}\kappa^2\int_{-x_0+1/2\xi}^{-x_0+3/2\xi}[\psi_2^*(x)\psi_1(x-\xi)+\psi_2^*(x)\psi_3(x+\xi)$$
$$-\psi_2(x)\psi_1^*(x-\xi)-\psi_2(x)\psi_3^*(x+\xi)]dx$$
$$-\cdots-\tfrac{1}{2}\kappa^2\int^{x}[\cdots]dx. \quad (27)$$

In Eq. (27) the integration has been broken up into a sum of integrals over the subintervals which lie between $-x_0-\frac{1}{2}\xi$ and x. It is clear that the two terms of the first integral are cancelled by two terms of the second integral, that the other two terms of the second integral cancel two terms of the third integral, and so forth. Thus as a wave penetrates into the nucleus, nonlocal effects remove flux from the first subinterval and replace it in the second; they remove flux from the second subinterval and replace it in the first and third, and so forth. We see why there is conservation on the average. Presumably in each subinterval except the first and the last there is also a fairly good balance between the flux gained and the flux lost.

4. CONCLUSIONS

The derivation given in Sec. 2 yields in Eq. (17) a local potential which is equivalent to a given nonlocal potential, and yields in Eq. (18) the reduction of the amplitude in the nuclear interior which occurs for the wave function of the nonlocal problem. Of course the fact that $F(x)<1$ inside the nucleus really only shows that $|\psi(x)|<|\psi_L(x)|$ inside the nucleus. However, except at very low energies, functions of the sort of $\psi_L(x)$ have about the same amplitude inside the nucleus as they have outside it. Therefore the Perey effect is an actual reduction of the amplitude of the nonlocal eigenfunction in the nuclear interior. To good approximation [see Eq. (18′)] this reduction is the same for all energy states of the nonlocal problem, and therefore affects both the wave functions used in reaction studies, and also the wave functions used in nuclear structure studies.

It should be emphasized again that some elements of

arbitrary taste go into the calculation of the local potential which is equivalent to the original nonlocal potential. The only physical condition we imposed on $U_L(x)$ was that $\psi_L(x)$ be identical with $\psi(x)$ outside the nucleus. However, this condition is already fulfilled if $\psi(x)$ and $\psi_L(x)$ have the same scattering phase shift, and there is an infinite variety of different potentials $U_L(x)$ which match a given phase shift at a single energy. To make the calculation definite we need to select a class of potentials among which to search for the "equivalent" potential. In the analysis of Sec. 2 this selection was introduced by imposing two mathematical conditions. The first condition was that $F(x)$ be smoothly varying, so that $\psi_L(x)$ and $\psi(x)$ should be as alike as possible. The second condition was that $U_L(x)$ must vary smoothly at the zeros of $\psi(x)$, *wherever those zeros might be.* This condition then took the form that the relative phase of the numerator and denominator of Eq. (11) not depend on the properties of $\psi(x)$. In fact this phase condition introduces very strong limitations upon the selection of $F(x)$, and makes our answer unique. This condition has a further useful consequence in the case that the one-dimensional Schrödinger equation we are analyzing happens to be a radial equation for a single partial wave of a three-dimensional problem. The condition that $U_L(x)$ not depend on the positions of the zeros of a given partial wave then signifies that the same $U_L(x)$ will be found for all the partial waves. It is then clear that the $U_L(x)$ which is found under our equivalence conditions is the same unique $U_L(x)$ which is found by numerical fits of the scattering cross section predicted by the nonlocal potential.[1,2]

Even if Eq. (11) is studied under much looser mathematical conditions, we see the qualitative fact that any $F(x)$ which displaces the zeros of the numerator to lie over those of the denominator must have the property $F(x)<1$ in the nuclear interior (if the potential is attractive). Thus the basic Perey effect is quite a stable one.

Now why does the Perey effect occur? To answer this question it is helpful to ask the further question, why should the potential which describes the motion of one particle in a many-particle system be nonlocal? We may assume that the basic two-particle interaction, the nucleon-nucleon interaction, is local. Then the nonlocality of the optical potential, say, is found when we project out from the over-all wave function of the system that part which describes the motion in the elastic channel. Nonlocality appears because the particle in the elastic channel moves under the influence of coupling to the other channels. Of course these other channels are not carried as such in an optical potential calculation. We may call them the "invisible channels." Elimination of the invisible channels is the basic step of the derivation of the optical potential. Nevertheless part of the physical wave function does lie in the invisible channels, and part of the flux which moves through the nucleus moves in the invisible channels. It is these aspects of the physical situation which are accounted for by the nonlocality of the single-particle potential, and by the Perey effect.

The role of the invisible channels is seen very clearly in Eq. (27), the flux equation of the lattice model. There flux seems to disappear mysteriously in interval *1*, and to reappear in interval *2*; it disappears in interval *2* and reappears in intervals *1* and *3*, etc. Consideration of the wave function of the full many-body problem shows that what in fact must be happening is that in interval *1* flux leaves the elastic channel and enters the invisible channels; it propagates for a while in the invisible channels and then reenters the elastic channel in the next interval, and so forth. Then the reason why flux in the elastic channel is conserved on the average is that we have implicitly assumed the invisible channels to be closed channels; therefore as much flux departs for the invisible channels as returns from them. They influence the motion in the elastic channel, but do not permanently remove any flux from it.

The reason for the Perey effect, that $\psi(x)$ is smaller inside the nucleus than it is outside the nucleus, then is that outside the nucleus $\psi(x)$ is the full physical wave function of the many-body system, whereas inside the nucleus part of the physical wave function lies in the invisible channels. It may be that all we want of the physical wave function is the piece $\psi(x)$ which lies in the elastic channel. Then the Perey effect is seen to cause an important reduction of the matrix elements which are computed with that piece of the wave function.

The physical interpretation of the Perey effect is more difficult in the case that the nonlocal potential is repulsive, inasmuch as in this case $\psi(x)$ is *larger* inside the nucleus than it is outside the nucleus. Of course a repulsive nonlocal potential is unusual; however, it is not necessarily unphysical. We therefore note that in the case of the attractive potential the virtual excitation of the invisible channels establishes a cloud of particles from the medium, which move with the incident particle and share the flux. Thereby the part carried in the elastic channel itself is reduced. In the repulsive case, the particles of the medium are driven away from the incident particle; nevertheless flux must be conserved, and therefore the incident particle carries more flux, in order to compensate for the part of the medium which has been suppressed. This effect resembles an effect which is known from the random phase approximation for bound states. In that theory the treatment of ground-state excitations forces certain amplitudes to be normalized to values greater than unity.

ACKNOWLEDGMENTS

I am grateful to F. G. Perey both for sending me copies of his work in advance of publication, and also for persuading me that the wave function reduction is a real effect. I am grateful to D. S. Saxon for suggesting

that flux conservation questions should be considered. I am grateful to R. M. Drisko for numerous discussions which helped in the preparation of the article.

APPENDIX

Equations (26) for the square well lattice model are a set of differential-difference equations. Despite the complications of these equations they possess solutions in normal modes. For normal mode number m the wave function in region l is

$$\psi_{l,m}(x)=B_{l,m}\exp[i\lambda_m(x-l\xi)],$$

where a convenient phase factor has been inserted into the definition. The secular equations are found to be

$$0=(k^2-\lambda_m^2)B_{1,m}+\tfrac{1}{2}\kappa^2B_{2,m}, \quad \text{for } l=1,$$
$$0=(k^2-\lambda_m^2)B_{n,m}+\tfrac{1}{2}\kappa^2B_{n-1,m}, \quad \text{for } l=n,$$
$$0=(k^2-\lambda_m^2)B_{l,m}+\tfrac{1}{2}\kappa^2(B_{l-1,m}+B_{l+1,m}), \quad \text{for } l\neq 1,n.$$

Although these equations are solved very easily in any explicit case, no solution for general n has yet been found.

The actual solution function $\psi(x)$ for the lattice model is a linear combination of the normal modes. The combination coefficients must be chosen so that $\psi(x)$ and $\psi'(x)$ are continuous across the boundaries between subintervals.

PHYSICAL REVIEW VOLUME 137, NUMBER 3B 8 FEBRUARY 1965

Possible Source Mechanism for Low-Energy Galactic Electrons

KARL A. BRUNSTEIN*
Goddard Space Flight Center, Greenbelt, Maryland
(Received 10 September 1964)

A calculation is made of the expected secondary electron flux resulting from the knock-on collisions of the primary nuclear beam with the interstellar gas. The model includes ionization losses and a statistical Fermi-mechanism energy gain. Comparison is made with recent satellite experimental data.

INTRODUCTION

RECENT interest in cosmic-ray electrons has been confined largely to higher energies. Specifically, experimental results[1–3] in the energy region of the order of 100 MeV to several BeV have been of interest because of their bearing on the problem of galactic radio emission. The study of lower energy electrons, although probably not of direct importance to the radio emission question, is of importance because of its relationship to the higher energy electron spectrum, and because of its bearing upon the questions of solar modulation and energetic electron production.

Several workers in the field have arrived at the conclusion that the primary cosmic-ray beam must traverse several g/cm^2 of interstellar material prior to being sampled at or near the earth.[4–6] This necessarily implies a flux of low-energy electrons in equilibrium with the primary beam due to the knock-on process in the interstellar gas. This problem has been extensively studied for knock-on electrons due to muons in various substances.[7–9] The equilibrium problem in the interstellar gas is somewhat different from the laboratory experiments described in Refs. 7 and 8 because of the absence of the cascading process in the interstellar gas and the enhanced ionization loss rate in the partially ionized hydrogen.[10] In addition, there is the possibility of further acceleration of the secondary electrons in the interstellar material.[11]

It is not clear that these galactic electrons of low rigidity could penetrate into the solar cavity; however, recent work by Palmeira and Balasubrahmanyan[12] suggests that, at least during solar minimum, they can. This question is not considered here. The question of solar modulation is a separate one. By considering the knock-on flux as expected in the absence of solar influence and comparing with experimental data obtained outside the magnetosphere, new information concerning solar influence may be inferred.

* National Aeronautics and Space Agency–National Academy of Sciences–National Research Council Regular Post-Doctoral Resident Research Associate.

[1] J. A. Earl, Phys. Rev. Letters **6**, 125 (1961).
[2] P. Meyer and R. Vogt, Phys. Rev. Letters **6**, 193 (1961).
[3] J. A. DeShong, R. H. Hildebrand, and P. Meyer, Phys. Rev. Letters **12**, 3 (1964).
[4] F. W. O'Dell, M. M. Shapiro, and B. Stiller, International Conference on Cosmic Rays and the Earth Storm, Kyoto, 1961 (unpublished).
[5] S. Hayakawa, K. Ito, and Y. Terashima, Progr. Theoret. Phys. (Kyoto) Suppl. **6**, (1958).
[6] H. Aizu, Y. Fujimoto, S. Hasegawa, M. Koshiba, I. Ito, J. Nishimura, and K. Yokoi, Progr. Theoret. Phys. Suppl. (Kyoto) **16**, 54 (1960).
[7] W. W. Brown, A. S. McKay, and E. D. Palmatier, Phys. Rev. **76**, 506 (1949).
[8] W. E. Hazen, Phys. Rev. **64**, 7 (1943).
[9] H. J. Bhabha, Proc. Roy. Soc. (London) **A164**, 257 (1938).
[10] S. Hayakawa and K. Kitao, Progr. Theoret. Phys. (Kyoto) **16**, 139 (1956).
[11] E. Fermi, Phys. Rev. **75**, 1169 (1949).
[12] R. Palmeira and V. K. Balasubrahmanyan, J. Geophys. Res. (to be published).