High-Energy γ-Ray Source from Electron-Positron Pair Annihilation*

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Problems associated with the use of the process $e^{+}+e^{-}\rightarrow 2\gamma$ as a source for high-energy γ rays are investigated. The polarization of the γ's is shown to be negligible at high energies. The spread of γ spectra at a fixed angle due to radiative corrections is investigated. Terms like $(\alpha/\pi)\ln^2(4E^2/m^2)$ are associated with infrared divergence and disappear with a reasonable energy resolution. The result is very similar to Schwinger's corrections without vacuum polarization. Our result was found to be also usable in the colliding-beam experiment without any modification, under certain conditions.

1. INTRODUCTION

IN the usual experiments involving photons as incident particles, the photon source is from the ordinary bremsstrahlungs which are not only nonmonochromatic but also strongly dominated by low-energy γ's due to the $1/k$ dependence in the bremsstrahlung cross sections. This fact not only imposes some complications on the kinematical analysis of the subsequent photoproduction processes but also sometimes renders the experiments impossible because of the enormous background produced by the low-energy γ's; for example in the bubble-chamber experiments there will be too many Compton electrons and electron pairs produced. Ballam and Guiragossián[1] made a proposal to use the fact that when positrons are injected into the hydrogen target one obtains monochromatic γ's from the $e^{+}+e^{-}\rightarrow 2\gamma$ process in addition to the ordinary bremsstrahlung from $e^{+}+e^{-}\rightarrow e^{+}+e^{-}+\gamma$ and $e^{+}+p\rightarrow e^{+}+p+\gamma$. One hopes that the γ's thus produced will have components of sufficiently high energy to overcome part of the difficulties mentioned above. In order to obtain resultant gamma spectra from e^{+} hydrogen atom collisions one has to calculate the following four processes in detail:

(i) $e^{+}+e^{-}\rightarrow 2\gamma$ and its radiative corrections,
(ii) $e^{+}+e^{-}\rightarrow 3\gamma$,
(iii) $e^{+}+e^{-}\rightarrow e^{+}+e^{-}+\gamma$,
(iv) $e^{+}+p\rightarrow e^{+}+p+\gamma$.

In this paper we treat only the first process and part of the second process where the third photon emitted is limited to a low energy. The exact treatments of processes (ii), (iii), and (iv) can be done by an electronic computer and will be published separately.[2] Figure 1 shows the anticipated γ spectrum at a fixed angle. The spike in the figure is due to processes (i) and (ii). The processes (ii), (iii), and (iv) contribute to the smooth curve in the background. The troublesome low-energy end of the spectrum can usually be greatly reduced by passing the γ rays through some material using the fact that the absorption coefficient for the low-energy γ rays is larger than that for the high-energy γ rays.

In Sec. 2, we treat the lowest order process, define the notation, give many useful kinematical relations and formulas, and show that the polarization of the γ rays produced in the process $e^{+}+e^{-}\rightarrow 2\gamma$ is negligibly small at high energies. In Sec. 3 the spreading of the γ spectra due to radiative corrections is calculated. It is found that terms like $(\alpha/\pi)\ln^2(4E^2/m^2)$ do not occur in our final expression for the radiative corrections δ. It is concluded that terms like $(\alpha/\pi)\ln^2(4E^2/m^2)$ are closely related to the infrared phenomena and disappear when the phase space for the third photon k_3 is taken to be such that $\omega_3^{\max}\gg m$ in the center-of-mass system. (See Sec. 5.) In Sec. 4 some numerical examples are given. In Sec. 5, the physical significance of the existence or absence of terms like $(\alpha/\pi)\ln^2(4E^2/m^2)$ is discussed. The resemblance between our results and Schwinger's formula is pointed out, and finally the adaptation of our result to the colliding-beam experiment is considered. In the Appendix we discuss some precautions needed in the calculation of cross sections involving many identical particles in the final states.

2. $e^{+}+e^{-}\rightarrow 2\gamma$

In this section we treat the lowest order cross section. Since we are interested only in the very high energy e^{+} beam we may treat the electron in the hydrogen atom

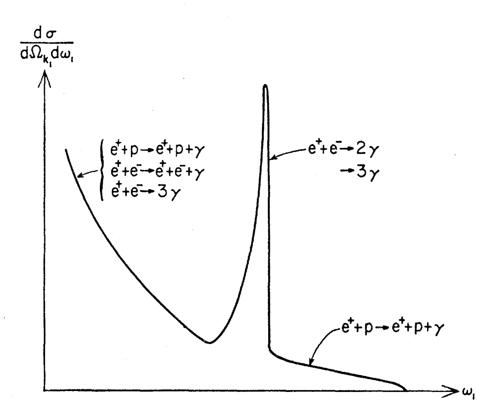

FIG. 1. A typical gamma spectrum of positron hydrogen atom collision at a fixed angle.

* Work supported by U. S. Atomic Energy Commission.
[1] J. Ballam and Z. G. T. Guiragossian, in Proceedings of the International Conference on High-Energy Physics, Dubna, 1964 (unpublished).
[2] S. Swanson, C. K. Iddings, and Y. S. Tsai (to be published).

as free. P_1 and P_2 represent the four-momenta of the target electron and the incident positron, respectively. k_1 and e_1 represent the four-momenta and polarization vector of the detected photon and k_2 and e_2 represent those of the undetected photon. Quantities with a bar on top represent the center-of-mass quantities. Our metric is defined such that if $k_1=(\omega_1,\mathbf{k}_1)$ and $P_1=(E_1,\mathbf{P}_1)$, then $k_1\cdot P_1=\omega_1E_1-\mathbf{k}_1\cdot\mathbf{P}_1$. The units used are $e^2/4\pi=\alpha$, and $\hbar=c=1$. We choose a gauge such that in the laboratory system $[P_1=(m,0)]$, $e_1\cdot P_1=e_2\cdot P_1=0$ and $e_1\cdot k_1=e_2\cdot k_2=0$. By calculating the lowest order Feynman diagrams, one obtains the differential cross section

$$\frac{d\sigma_0}{d\Omega_k}(e_1,e_2)=\frac{r_0^2}{8}\frac{m^2(m+E_2)}{P_2(m+E_2-P_2\cos\theta)^2}\times\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+2-4(e_1\cdot e_2)^2\right],\quad(2.1)$$

where $r_0^2=7.95\times10^{-26}$ cm^2 and θ is the angle between $\mathbf{P}_2$ and $\mathbf{k}_1$.

For convenience in discussing the polarization, let us choose a coordinate in which the direction of $\mathbf{k}_1$ is the z axis and that of $\mathbf{k}_1\times\mathbf{k}_2$ is the y axis. Then the summation over e_2 can be carried out with respect to two transverse directions

$$e_{2_{\parallel}}\equiv(0,\,+\cos\theta_{12},\,0,\,-\sin\theta_{12})$$
$$e_{2_{\perp}}\equiv(0,0,1,0)$$

and the result is

$$\frac{d\sigma_0}{d\Omega_{k_1}}(e_1)=\frac{A}{2}\left\{\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+2-2(e_{1x}^2\cos^2\theta_{12}+e_{1y}^2)\right\},\quad(2.2)$$

where

$$A\equiv\frac{r_0^2}{2}\frac{m^2(m+E_2)}{P_2(m+E_2-P_2\cos\theta)^2}.$$

From Eq. (2.2) we can construct the density matrix for the photon beam k_1,

$$X_{ij}=\frac{A}{2}\left\{\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+2\right)\delta_{ij}-2(\delta_{ix}\delta_{jx}\cos^2\theta_{12}+\delta_{iy}\delta_{jy})\right\},\quad(2.3)$$

where $i=x$, y and $j=x$, y.

The density matrix X completely specifies the quantum-mechanical description of a monochromatic photon beam k_1 because any subsequent interaction of k_1 can be written as $\mathrm{Tr}(MXM^\dagger)$, where M is the matrix element of the subsequent interaction (provided one uses the same gauge as used here). However, it is more convenient to give an equivalent description in terms of intensity of the beam and the three Stokes parameters S_x, S_y, and S_z.

The trace of X gives the differential cross section summed over the polarization e_1 and is given by

$$\frac{d\sigma_0}{d\Omega_k}=\mathrm{Tr}X=A\left\{\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+\sin^2\theta_{12}\right\}.\quad(2.4)$$

The definitions of the Stokes parameters, their functional form in our particular problem, and their physical meanings can be given by the following equations[3]:

$$S_x=\mathrm{Tr}\sigma_xX/\mathrm{Tr}X=(X_{yx}+X_{xy})/\mathrm{Tr}X=0$$
$$=\frac{d\sigma(\hat{e}_1=(\hat{e}_x+\hat{e}_y)/\sqrt{2})-d\sigma(\hat{e}_1=(\hat{e}_x-\hat{e}_y)/\sqrt{2})}{d\sigma(\hat{e}_1=(\hat{e}_x+\hat{e}_y)/\sqrt{2})+d\sigma(\hat{e}_1=(\hat{e}_x-\hat{e}_y)/\sqrt{2})},\quad(2.5)$$

$$S_y=\mathrm{Tr}\sigma_yX/\mathrm{Tr}X=i(X_{xy}-X_{yx})/\mathrm{Tr}X=0$$
$$=\frac{d\sigma(\hat{e}_1=(\hat{e}_x-i\hat{e}_y)/\sqrt{2})-d\sigma(\hat{e}_1=(\hat{e}_x+i\hat{e}_y)/\sqrt{2})}{d\sigma(\hat{e}_1=(\hat{e}_x-i\hat{e}_y)/\sqrt{2})+d\sigma(\hat{e}_1=(\hat{e}_x+i\hat{e}_y)/\sqrt{2})},\quad(2.6)$$

$$S_z=\frac{\mathrm{Tr}\sigma_zX}{\mathrm{Tr}X}=\frac{X_{xx}-X_{yy}}{\mathrm{Tr}X}=\sin^2\theta_{12}\Big/\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+\sin^2\theta_{12}\right)$$
$$=\frac{d\sigma(\hat{e}_1=\hat{e}_x)-d\sigma(\hat{e}_1=\hat{e}_y)}{d\sigma(\hat{e}_1=\hat{e}_x)+d\sigma(\hat{e}_1=\hat{e}_y)}.\quad(2.7)$$

It is seen that the only nonvanishing Stokes parameter is S_z, and Eq. (2.7) shows that there are more photons plane-polarized in the production plane (x-z plane) than there are perpendicular to the production plane.[3a] The magnitude of S_z is roughly proportional to θ_{12}^2, the square of the angle between two photons produced. The differential cross section and the only nonvanishing Stokes parameter S_z of Eqs. (2.4) and (2.7) can be written in terms of θ and E_2 alone by the following kinematic relations:

$$\omega_1=\frac{(E_2+m)m}{m+E_2-P_2\ \mathrm{os}\theta}\approx\frac{E_2}{1+\frac{1}{2}\gamma\theta^2},\quad(2.8)$$

$$\omega_2=\frac{(E_2+m)(E_2-P_2\cos\theta)}{m+E_2-P_2\cos\theta}\approx\frac{\frac{1}{2}\gamma\theta^2E_2}{1+\frac{1}{2}\gamma\theta^2},\quad(2.9)$$

$$\sin^2\theta_{12}=\frac{2(E_2-m)\sin^2\theta}{E_2-P_2\cos\theta}-\frac{(E_2-m)^2\sin^4\theta}{(E_2-P_2\cos\theta)^2}$$
$$\approx\frac{4}{\gamma^2\theta^2}(1+\tfrac{1}{2}\gamma\theta^2)^2\approx0.\quad(2.10)$$

[3] σ_x, σ_y, and σ_z are usual 2×2 Pauli matrices.

[3a] *Note added in proof.* This is just opposite to the case of Compton scatterings where the photon is partially plane-polarized perpendicular to the (k_1k_2) plane. It should be noted that if the incident positrons are longitudinally polarized (such as obtained from β decays) the resultant photons will be almost completely circularly polarized, i.e., $|S_y|\sim1$. See L. A. Page, Phys. Rev. **106**, 394 (1957).

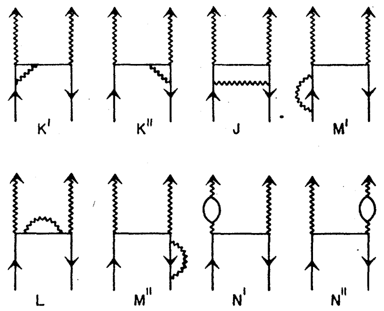

FIG. 2. Feynman diagrams for virtual radiative corrections.

The approximate relations $\approx$ hold only when

$$1\gg\theta\gg m/E_2\equiv 1/\gamma\,. \tag{2.11}$$

Hereafter we use the symbol $\approx$ to represent approximate relations true only if inequalities Eq. (2.11) are satisfied.

When $1\gg\theta$ and $\gamma\gg 1$ we may write[4]

$$\frac{d\sigma_0}{d\Omega_k}=\frac{r_0^2}{2}\frac{1}{[1+\frac{1}{2}\gamma\theta^2]^2}\left[\frac{2\gamma}{1+\gamma^2\theta^2}+\frac{1+\gamma^2\theta^2}{2\gamma}+\frac{4\gamma^2\theta^2}{1+\gamma^2\theta^2}-\frac{4\gamma^4\theta^4}{(1-\gamma^2\theta^2)^2}\right] \tag{2.12}$$

$$\approx\frac{r_0^2}{2}\frac{1}{[1+\frac{1}{2}\gamma\theta^2]^2}\left(\frac{2}{\gamma\theta^2}+\frac{1}{2}\gamma\theta^2\right). \tag{2.13}$$

The second relation holds only when inequalities Eq. (2.11) are satisfied. Similarly for S_z we have

$$S_z=8\theta^2\gamma/(4+\gamma^2\theta^4)(1+\gamma^2\theta^2) \tag{2.14}$$
$$\approx 0\,.$$

For convenience of order of magnitude estimate we give some of the values of $d\sigma/d\Omega_k$. and S_z in Table I.

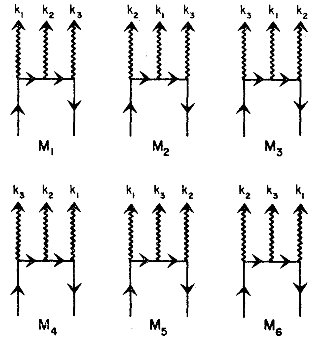

FIG. 3. Feynman diagrams for real radiative corrections.

[4] There is an error in Eqs. (12)–(48) of Jauch and Rohlich's *Theory of Photons and Electrons*, first edition. In the second edition the correct formula was given.

3. RADIATIVE CORRECTIONS

From the experience in the electron scattering from nuclei, we know that the radiative corrections are quite important whenever one deals with interactions involving extremely relativistic charged particles. In our case the radiative corrections will induce a tail to the spike in Fig. 1, a phenomenon very similar to the radiative tails in the electron scatterings. Harris and Brown[5] (HB) have previously calculated the radiative corrections to the pair annihilation, assuming the maximum energy of the third photon emitted, $\omega_3^{\max}$, to be isotropic in angle and $\omega_3^{\max}\ll m$ in the laboratory frame. Their results were obtained by using the substitution rule on the earlier paper by Brown and Feynman[6] (BF) on Compton scattering. The calculation can be divided into two parts: virtual radiative corrections represented by Fig. 2 and real radiative corrections represented by Fig. 3. The expressions for the virtual radiative corrections are independent of experimental conditions. Thus we use the results of (HB) for this part. The real radiative corrections have to be recalculated because the results depend upon the experimental setup. For convenience of calculation we shall use the center-of-mass system. We define $\bar{E}=\bar{E}_1=\bar{E}_2$, $\bar{P}=(\bar{E}^2-m^2)^{1/2}$, and $\bar{\theta}$ to be the angle between $\bar{P}_2$ and $\bar{k}_1$.

TABLE I. θ, $d\sigma/d\Omega_{k1}$, and S_z.

0	$r_0^2\gamma$	0
$1/\gamma$	$\frac{1}{2}r_0^2\gamma$	$1/\gamma$
$(1/\gamma)^{1/2}$	$(5/9)r_0^2$	$8/5\gamma$

According to Harris and Brown, the elastic cross section for the $e^+ + e^- \to 2\gamma$ up to order e^6, averaged over spins and summed over polarizations, can be written as[7]

$$\frac{d\sigma_{2\gamma}}{d\Omega_{k1}}=\frac{d\sigma_0}{d\Omega_{k1}}\left\{1-\frac{\alpha}{\pi}\left[2(1-2x\coth 2x)\ln\frac{\lambda}{m}-4x\coth 2x\left(2g(x)-h(2x)+\frac{\pi^2}{4x}\right)+F(\kappa,\tau)+F(\tau,\kappa)\right]\right\}, \tag{3.1}$$

[5] I. Harris and L. M. Brown, Phys. Rev. **105**, 1656 (1957). Hereafter referred to as (HB).
[6] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1951). Hereafter referred to as (BF).
[7] We give this expression in detail because there are two misprints in Eq. (8), and wrong sign for Eq. (8a) of (HB). The present author did not check their calculation in complete detail. The errors were found by comparing (HB) with the results of (BF). According to Professor Brown, the results in (BF) are more correct.

where λ is the usual fictitious photon mass and

$$F(\kappa,\tau)u=[4x(\kappa\tau)^{-1}\sinh 2x(1+2\cosh^2 x)+2x\tanh x]g(x)+\ln\kappa\Big\{4x\coth 2x\Big[\frac{4}{\kappa^2}-\frac{1}{\kappa}-\frac{\tau}{2\kappa}-1-4(\kappa\tau)^{-1}\sinh^2 x\Big]$$
$$-\frac{2x}{\sinh 2x}\Big(\frac{\kappa-6}{\tau}\Big)+\frac{3\tau}{2\kappa^2}(1+\kappa)+\frac{3}{\tau}+1-\frac{7}{\kappa\tau}+\frac{8}{\kappa}-\frac{8}{\kappa^2}-\frac{\tau^2-2\kappa+\kappa^2\tau}{2\kappa^2\tau(\kappa-1)}-\frac{2\kappa^2+\tau}{2\tau(\kappa-1)^2}\Big\}$$
$$-\frac{\frac{1}{4}\pi^2-x^2}{\cosh^2 x}\Big\{\frac{2}{\kappa}-\frac{7\kappa}{4}-\frac{3\tau^2}{4\kappa}\Big\}-4x\coth x\Big\{\frac{1}{2}-\frac{1}{\kappa}\Big\}+4\Big(\frac{1}{\kappa}+\frac{1}{\tau}\Big)^2-\frac{12}{\kappa}-\frac{3\kappa}{2\tau}-\frac{2\kappa}{\tau^2}+\frac{1}{\kappa-1}\Big(\frac{\kappa}{\tau}+\frac{1}{2}\Big)$$
$$+G_0(\kappa)\Big\{\frac{\kappa^2}{\tau}+\frac{\tau}{\kappa^2}+\frac{\kappa}{\tau}+\kappa+\frac{\tau}{2}+\frac{2}{\kappa}-\frac{3}{\tau}-1\Big\}. \quad (3.2)$$

The symbols used in the above equations have the following meaning:

$$\kappa\equiv 2P_1\cdot k_1/m^2=2P_2\cdot k_2/m^2,$$
$$\tau\equiv 2P_1\cdot k_2/m^2=2P_2\cdot k_1/m^2,$$
$$\kappa+\tau=4\bar{E}_2{}^2/m^2\equiv 4\bar{\gamma}^2,$$
$$\cosh x=\bar{\gamma}=(\gamma/2)^{1/2},$$
$$u=4\Big(\frac{1}{\kappa}+\frac{1}{\tau}\Big)^2-4\Big(\frac{1}{\kappa}+\frac{1}{\tau}\Big)-\Big(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\Big),$$
$$g(x)=\frac{1}{x}\int_0^x u\tanh u\,du,$$
$$h(x)=\frac{1}{x}\int_0^x u\coth u\,du,$$

and

$$G_0(x)=\frac{2}{\kappa}\int_1^{1-\kappa}\ln(1-u)\frac{du}{u}.$$

We shall be interested in the energy and angular range such that

$$\tau\gg\bar{\gamma}\gg 1 \quad (3.3a)$$

and

$$\kappa\gg\bar{\gamma}\gg 1. \quad (3.3b)$$

Since the definitions given above by (HB) are unwieldy, we give simplified (but exact) expressions as well as approximate expressions which hold only when (3.3) is satisfied.

$$u\approx-\Big(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\Big),$$
$$x=\tfrac{1}{2}\ln\Big(\frac{\bar{E}+\bar{P}}{\bar{E}-\bar{P}}\Big)\approx\ln(2\bar{\gamma}),$$
$$\sinh x=(\bar{P}/m)\approx\bar{\gamma},$$
$$\cosh 2x=2\bar{\gamma}^2-1\approx 2\bar{\gamma}^2,$$
$$\sinh 2x=2\bar{E}\bar{P}/m^2\approx 2\bar{\gamma}^2,$$

$$g(x)=\ln 2\bar{\gamma}-\frac{x}{2}-\frac{\pi^2}{12x}-\frac{1}{2x}\Phi(-e^{-2x})\approx\frac{x}{2},$$
$$h(x)=\ln(2\sinh x)-\frac{x}{2}-\frac{\pi^2}{12x}-\frac{1}{2x}\Phi(e^{-2x})\approx\frac{x}{2},$$
$$G_0(\kappa)=\frac{-2}{\kappa}[\Phi(1-\kappa)-\Phi(1)]\approx\frac{1}{\kappa}\Big(\ln^2\kappa+\frac{2\pi^2}{3}\Big),$$

where $\Phi(x)$ is the Spence function defined as

$$\Phi(x)=-\int_0^x\frac{\ln|1-u|}{u}du.$$

Substituting the approximate expressions given above into Eq. (3.1) and neglecting terms small compared with 1 inside the bracket [ ], we obtain[8]

$$\frac{d\sigma_{2\gamma}}{d\Omega_{k_1}}\approx\frac{d\sigma_0}{d\Omega}\Big\{1-\frac{\alpha}{\pi}\Big[\Big(1-\frac{\bar{E}^2+\bar{P}^2}{2\bar{E}\bar{P}}\ln\frac{\bar{E}+\bar{P}}{\bar{E}-\bar{P}}\Big)\times\Big(\frac{3}{2}+2\ln\frac{\lambda}{m}\Big)+2\ln^2(2\bar{\gamma})-\frac{\pi^2}{6}+f\Big]\Big\}, \quad (3.4)$$

where

$$fu=\Big(1+\frac{\kappa}{\tau}+\frac{\tau}{2\kappa}\Big)\Big\{\ln^2\Big(\frac{\kappa+\tau}{\kappa}\Big)+\ln\Big(\frac{\kappa}{\kappa+\tau}\Big)\Big\}$$
$$+\Big(1+\frac{\tau}{\kappa}+\frac{\kappa}{2\tau}\Big)\Big\{\ln^2\Big(\frac{\kappa+\tau}{\tau}\Big)+\ln\Big(\frac{\tau}{\kappa+\tau}\Big)\Big\}$$
$$+\Big(\frac{\tau}{\kappa}-\frac{\kappa}{\tau}\Big)\ln\frac{\kappa}{\tau}+\pi^2\Big\{\frac{1}{3}+\frac{1}{12}\Big(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\Big)\Big\}. \quad (3.5)$$

We next calculate the cross section for $e^++e^-\to 3\gamma$. In principle, one could calculate this cross section exactly by going into the center-of-mass system of two undetected particles as was done in other three-particle

[8] This expression is very similar to Eq. (43) of (BF) except for the terms proportional to π^2 in f.

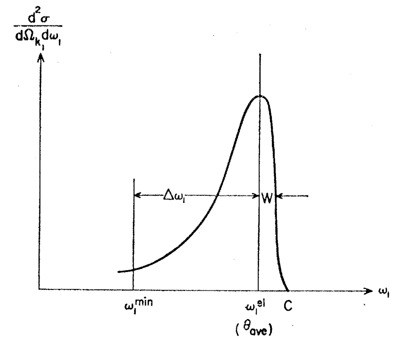

FIG. 4. A typical energy spectrum of the photon at a fixed angle. The point $\omega_1^{el}(\theta_{av})$ is chosen to be the value of ω_1 determined by two-body final-state kinematics at the average angle of the detector. $\Delta\omega_1$ should be chosen such that $\bar{\omega}\ll\Delta\omega_1\ll\omega_1^{el}0.1$.

final-state problems.[9] We shall not calculate this cross section exactly here but merely try to find the dominant terms [such as $\ln\gamma$, $\ln\gamma\,\ln(\omega_1/\Delta\omega_1)$, etc.], because these terms can be obtained without much effort and one expects that the resultant formula should be accurate to within one or two percent which is hopefully quite sufficient for the experimenter's needs. We shall calculate everything in the center-of-mass system.[10] To do this we have to specify what we are looking for in the laboratory system and then transform the lab experimental conditions into those of the c.m. system. We are interested in the photon spectrum at a certain angle in the lab system. We anticipate the spectrum will look like what is shown in Fig. 4. (The photons due to ordinary bremsstrahlung have been subtracted already.) Usually the incident positron energy E_2 and the angle θ have certain widths ΔE_2 and $\Delta\theta$ and they cause the width W to the right of the peak in the spectrum. In the vicinity of the peak the shape of the spectrum is dominated by ΔE_2 and $\Delta\theta$ and has very little to do with the radiative corrections. What one can calculate by using the standard method is the area under the curve between $\omega_1^{\rm min}$ and c as a function of $\Delta\omega_1$ as shown in Fig. 4 provided $\Delta\omega_1$ is large compared with W. Even in the ideal case where W is infinitely small, $\Delta\omega_1$ should not be taken too small[11] because of the infrared divergence difficulties inherent in the perturbation method. Since the area as a function of $\Delta\omega_1$ can be calculated, we can obtain the spectrum itself in the region where $\Delta\omega_1\gg W$ by simply differentiating the expression for the area with respect to $\Delta\omega_1$. Thus our experimental condition in the lab frame can be stated as "How many photons can be detected in a small angular range $\Delta\theta$ with energy $\omega_1>\omega_1^{\rm min}$ if the incident positrons have energy E_2?" In the center-of-mass system we have incident positron energy

$$\bar{E}=(\tfrac{1}{2}mE_2)^{1/2}\,,\quad\left(\bar{\gamma}\equiv\frac{\bar{E}}{m}=(\gamma/2)^{1/2}\right),\tag{3.6}$$

the scattering angle $\bar{\theta}$

$$1-\cos\bar{\theta}\approx\frac{\gamma\theta^2}{1+\frac{1}{2}\gamma\theta^2}\,,\tag{3.7}$$

and the energy of the detected photon

$$\bar{\omega}_1=\frac{k_1\cdot(P_1+P_2)}{[(P_1+P_2)^2]^{1/2}}\approx\frac{\omega_1(1+\frac{1}{2}\gamma\theta^2)}{2\bar{\gamma}}\,.\tag{3.8}$$

From the above equations we can construct the phase space for the photon k_1 in the center-of-mass system represented by an area $ABCD$ in Fig. 5. The horizontal line AB is obtained by setting $\bar{\omega}_1=\bar{E}$ where $\bar{E}$ is given by Eq. (3.6). $\bar{\theta}_{\rm min,max}$ is obtained from Eq. (3.7),

$$1-\cos\bar{\theta}_{\rm min,max}=\frac{\gamma\theta^2_{\rm min,max}}{1+\frac{1}{2}\gamma\theta^2_{\rm min,max}}\,.\tag{3.9}$$

The line CD is obtained from Eqs. (3.8) and (3.7)

$$\bar{\omega}_1=\omega_1^{\rm min}\frac{(1+\frac{1}{2}\gamma\theta^2)}{2\bar{\gamma}}=\frac{\omega_1^{\rm min}}{2\bar{\gamma}\cos^2(\bar{\theta}/2)}\,.\tag{3.10}$$

Now we know the infrared divergence occurs near the straight line AB and the main contribution to the cross section comes from the region of the phase space near the line AB. We are thus justified in replacing the line CD by $C'D'$ which is obtained by replacing $\bar{\theta}$ by $\bar{\theta}_{\rm av}$ in

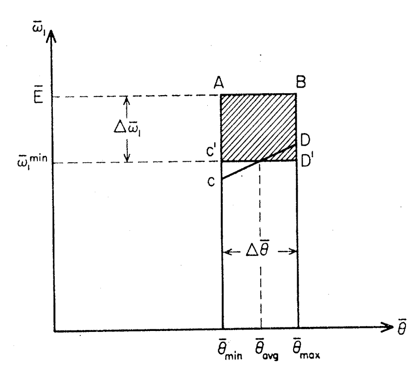

FIG. 5. The phase space for k_1 in the center-of-mass system is represented by the area $ABCD$. We approximate $ABCD$ by $ABC'D'$.

[9] Y. S. Tsai, Phys. Rev. **122**, 1898 (1961).

[10] The advantage of using the center-of-mass system is that it is easier to see under what conditions the k_3 in the numerator of the matrix element can be neglected, and thus obtain an extremely simple expression such as Eq. (3.16). From our considerations it can be concluded that in order to neglect k_3 in the numerator, it is not necessary to assume ω_3 to be much smaller than m in the laboratory system as was done in (BF) and (HB). As will be argued in Sec. 5, in the regions where $\omega_3\ll m$ in the lab system, the perturbation expansion is not valid owing to the existence of terms like $(\alpha/\pi)\ln^2(2\gamma)$ in the radiative corrections δ. A similar consideration was taken into account by the author for the radiative correction to the e-e scattering in the laboratory system. See Sec. 5 of the paper Y. S. Tsai, Phys. Rev. **120**, 269 (1960).

[11] See Ref. 17.

Eq. (3.10). For convenience we define

$$\Delta\bar{\theta} \equiv \bar{\theta}_{\max} - \bar{\theta}_{\min} = 2\gamma\theta\Delta\theta/(1+\tfrac{1}{2}\gamma\theta^2)^2 \sin\bar{\theta} \quad (3.11)$$

and

$$\Delta\bar{\omega}_1 \equiv \bar{E} - \bar{\omega}_1^{\min} = \bar{E}\Delta\omega_1/[\omega_1^{\text{el}}(\theta_{\text{av}})], \quad (3.12)$$

where $\omega_1^{\text{el}}(\theta_{\text{av}})$ is the peak energy of k_1 in the laboratory system as shown in Fig. 4, and corresponds approximately to the value of ω_1 obtained from Eq. (2.8) with average values of θ and γ. Hereafter we assume $\Delta\theta$ to be very small and simply write ω_1^{el} for $\omega_1^{\text{el}}(\theta_{\text{av}})$. Thus we have specified the phase space of k_1 in the center-of-mass system.

We next determine the phase space of k_2 and k_3. From energy and momentum conservation

$$\bar{\omega}_1 + \bar{\omega}_2 + \bar{\omega}_3 = 2\bar{E}, \quad (3.13)$$

and

$$\bar{\mathbf{k}}_1 + \bar{\mathbf{k}}_2 + \bar{\mathbf{k}}_3 = 0, \quad (3.14)$$

we can show that for each value of $\bar{\mathbf{k}}_1$ the allowed values of $\bar{\mathbf{k}}_2$ and $\bar{\mathbf{k}}_3$ must be on the surface of an ellipsoid with the $\bar{\mathbf{k}}_1$ vector connecting the two foci as shown in Fig. 6. The ellipsoid shrinks to a line when $\bar{\omega}_1 = \bar{E}$ and has a maximum extension when $\bar{\omega}_1 = \bar{\omega}_1^{\min}$. Thus the phase space of k_2 and k_3 can be represented by all the points inside the ellipsoid obtained by setting $\bar{\omega}_1 = \bar{\omega}_1^{\min}$. In principle, one has to treat k_2 and k_3 symmetrically since they are identical particles and both are undetected. However, as is shown in the Appendix we need to consider only half of the phase space, Fig. 7, due to the symmetry between k_2 and k_3. In this phase space only k_3 but not k_2 can become infrared. The distance from the point O to any point on the *surface* of this semiellipsoid gives the maximum value of ω_3, and is a function of angle ($\bar{\theta}_{13}$) between k_1 and k_3 in the center-of-mass system. Explicitly one obtains

$$\bar{\omega}_3^{\max} = \frac{2\bar{E}\Delta\bar{\omega}_1}{\bar{E} + \Delta\bar{\omega}_1 + (\bar{E} - \Delta\bar{\omega}_1)\cos\bar{\theta}_{13}}, \quad (3.15\text{a})$$

for

$$\cos\bar{\theta}_{13} \geq -\frac{\bar{E} - \Delta\bar{\omega}_1}{\bar{E} + \Delta\bar{\omega}_1} \equiv -\cos\beta = -\frac{1-(\Delta\omega_1/\omega_1^{\text{el}})}{1+(\Delta\omega_1/\omega_1^{\text{el}})};$$

and

$$\bar{\omega}_3^{\max} = -(\bar{E} - \Delta\bar{\omega}_1)/2\cos\bar{\theta}_{13}, \quad (3.15\text{b})$$

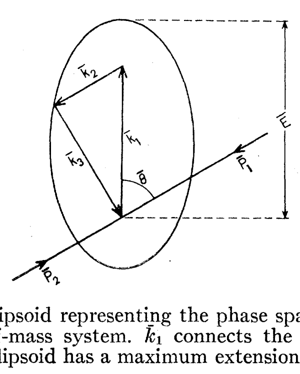

FIG. 6. An ellipsoid representing the phase space for k_2 and k_3 in the center-of-mass system. $\bar{k}_1$ connects the two foci of the ellipsoid. The ellipsoid has a maximum extension when $\bar{\omega}_1 = \bar{\omega}_1^{\min}$.

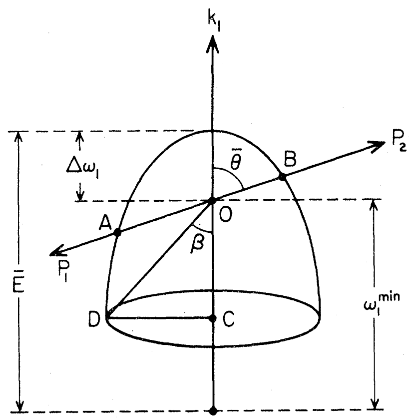

FIG. 7. The semiellipsoid to be used to determine $\bar{\omega}_3^{\max}$ as a function of $\bar{\theta}_{13}$ (angle between $\bar{k}_1$ and $\bar{k}_3$).

for

$$\cos\bar{\theta}_{13} \leq -(\bar{E} - \Delta\bar{\omega}_1)/(\bar{E} + \Delta\bar{\omega}_1).$$

Equation (3.15a) represents the equation for the ellipsoid and Eq. (3.15b) represents the bottom flat surface in Fig. 7. Some simplification of the phase space of k_3 can be obtained by explicitly considering the nature of the matrix elements.

There are 6 diagrams which contribute to the 3γ annihilation process as shown in Fig. 3. k_3 is not connected to the external charged lines in M_5 and M_6 and thus M_5 and M_6 are not infrared divergent as can be verified by explicit calculations. For simplicity of the calculation we shall ignore M_5 and M_6 and also the terms proportional to k_3 in the numerator.[12] Then the cross section for 3γ annihilation can be written as

$$\frac{d\sigma_{3\gamma}}{d\Omega} = \frac{d\sigma_0}{d\Omega}\frac{\alpha}{4\pi^2}\int \frac{\bar{k}_3^2 d\bar{k}_3 d\bar{\Omega}_3}{(\bar{k}_3^2+\lambda^2)^{1/2}} \times\left(\frac{m^2}{(P_2\cdot k_3)^2} - \frac{2P_1P_2}{(P_1\cdot k_3)(P_2\cdot k_3)} + \frac{m^2}{(P_1\cdot k_3)^2}\right). \quad (3.16)$$

In the center-of-mass system both P_1 and P_2 are extremely relativistic and the denominators in the

[12] The approximations made here are certainly the most serious ones and impose severe limitations on the range of applicability of our result, represented by the first inequality in (3.30) and inequalities (3.31). The problem one has to investigate is whether the curve in Fig. 4 goes down monotonically to the low-energy end or whether it goes up again and gives some additional (unwelcome) low-energy photons. The author tends to believe the latter because a positron can first emit a high-energy photon and then annihilate at a very low energy at which the cross section is very large. (A similar phenomenon is well known in the bremsstrahlung produced by Coulomb scattering.) Fortunately the exact calculation of the low-energy end of the spectrum can be done without much difficulty with the help of a computer. A computer can take traces, sum the tensor indices, do necessary cancellations, sort out terms in a convenient order, and obtain an analytical expression in less than one hour for a job which takes about six months for a physicist to do by paper and pencil. Two working computer procedures are available at Stanford, one by S. Swanson and another by A. Hearn, both of the Physics Department, Stanford University.

integrand of Eq. (3.16) tell us that practically all k_3 are emitted either along $\bar{P}_1$ or $\bar{P}_2$. Thus we expect the result of the integration to be quite insensitive to the detailed shape of the phase space except in the vicinity of $\bar{k}_3\|\bar{P}_1$ and $\bar{k}_3\|\bar{P}_2$. The maximum values of $\bar{\omega}_3$ along P_1 and P_2 can be obtained from Eqs. (3.15a, b) by setting $\bar{\theta}_{13}=\pi-\bar{\theta}$ and $\bar{\theta}_{13}=\bar{\theta}$, respectively, and they are to be denoted as $\omega_3^{\max}(\bar{k}\|\bar{P}_1)$ and $\bar{\omega}_3^{\max}(\bar{k}_3\|\bar{P}_2)$, respectively. We have tacitly assumed that $\bar{\omega}_3$ is much smaller compared with $\bar{E}$ in order to obtain a very simple expression [Eq. (3.16)] for the 3γ annihilation cross section. But from Fig. 7 we see that even for small $\Delta\bar{\omega}_1$, $\bar{\omega}_3$ can be as large as $\sim(\bar{E}/2)$. However, as can be seen from Eq. (3.15) and Fig. 7, if

$$\pi-\beta\gg\bar{\theta}\gg\beta\,, \tag{3.17}$$

both $\omega_3^{\max}(\bar{k}_3\|\bar{P}_1)$ and $\omega_3^{\max}(\bar{k}_3\|\bar{P}_2)$ are always small compared with $\bar{E}_2$ if $\Delta\bar{\omega}_1$ is small. Hence we shall assume that inequalities (3.17) are satisfied.[13] The integration (3.16) is a familiar one occurring in every bremsstrahlung calculation. Past experiences[14] show that the phase space shown in Fig. 7 can be replaced by a sphere with a radius $K_3^{\max}$ defined by a geometrical mean

$$\begin{aligned} K_3^{\max}&\equiv[\bar{\omega}_3^{\max}(\bar{k}_3\|\bar{P}_1)\times\bar{\omega}_3^{\max}(\bar{k}_3\|\bar{P}_2)]^{1/2}\\ &=2\bar{E}\Delta\bar{\omega}_1[(\bar{E}+\Delta\bar{\omega}_1)^2-(\bar{E}-\Delta\bar{\omega}_1)^2\cos^2\theta]^{1/2}. \end{aligned} \tag{3.18}$$

Carrying out the integration in this spherical phase space, and neglecting nonlogarithmic terms, we obtain[15]

$$\frac{d\sigma_{3\gamma}}{d\Omega_{k_1}}=-\frac{d\sigma_0}{d\Omega}\frac{\alpha}{\pi}\left[2\ln\frac{\bar{E}}{K_3^{\max}}\left(\ln\frac{4\bar{E}^2}{m^2}-1\right)+\tfrac{1}{2}K(P_1,P_1)+\tfrac{1}{2}K(P_2,P_2)-K(P_1,P_2)\right], \tag{3.19}$$

where $K(P_i,P_j)$ are the infrared terms defined by

$$K(P_i,P_j)=(P_iP_j)\int_0^1\frac{\ln(P_x^2/\lambda^2)}{P_x^2}dx\,, \tag{3.20}$$

with $P_x=P_iX+P_j(1-X)$. It was emphasized by the author in previous works[9,10] that it is a good idea not to integrate terms like $K(P_i,P_j)$ explicitly because they always cancel between elastic and inelastic radiative corrections. However, since we are going to use Harris and Brown's results on the elastic part, we give the explicit expressions for $K(P_i,P_j)$ in the center-of-mass system. (Of course the expression can be made covariant easily.)

$$K(P_1,P_1)=K(P_2,P_2)=\ln(m/\lambda)\,. \tag{3.21}$$

$$\begin{aligned} K(P_1,P_2)=\frac{\bar{E}^2+\bar{P}^2}{2\bar{P}\bar{E}}\Bigg[&\ln\frac{4\bar{P}^2}{\lambda^2}\ln\frac{\bar{E}+\bar{P}}{\bar{E}-\bar{P}}+\frac{1}{2}\ln^2\frac{\bar{E}+\bar{P}}{2\bar{P}}\\ &-\frac{1}{2}\ln^2\left(\frac{\bar{E}-\bar{P}}{2\bar{P}}\right)+\ln\frac{\bar{E}}{\bar{P}}\ln\frac{\bar{E}+\bar{P}}{\bar{E}-\bar{P}}\\ &-2\Phi\left(\frac{\bar{E}+\bar{P}}{2\bar{E}}\right)+2\Phi\left(\frac{\bar{E}-\bar{P}}{2\bar{E}}\right)\Bigg]. \end{aligned} \tag{3.22}$$

$$\approx\frac{\bar{E}^2+\bar{P}^2}{2\bar{P}\bar{E}}\left[\ln\frac{4\bar{P}^2}{\lambda^2}\ln\frac{\bar{E}+\bar{P}}{\bar{E}-\bar{P}}-\frac{1}{2}\ln^2\left(\frac{4\bar{P}^2}{m^2}\right)-\frac{\pi^2}{6}\right]. \tag{3.23}$$

Experimentally meaningful cross sections can be obtained by adding Eq. (3.19) to Eq. (3.4) and we obtain finally

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_{2\gamma}}{d\Omega}+\frac{d\sigma_{3\gamma}}{d\Omega}=\frac{d\sigma_0}{d\Omega}(1+\delta)\,, \tag{3.24}$$

with

$$\delta=\frac{-2\alpha}{\pi}\left\{\left(\ln\frac{\bar{E}}{K_3^{\max}}-\frac{3}{4}\right)\left(\ln\frac{4\bar{E}^2}{m^2}-1\right)+\frac{f}{2}\right\}\,, \tag{3.25}$$

where f is the function defined in Eq. (3.5), and is numerically very small in the range where inequalities (3.17) hold.[16]

The radiative correction δ in Eq. (3.24) can be written explicitly in terms of laboratory quantities by the following substitutions:

$$4\bar{E}^2/m^2=2E_2/m=2\gamma\,, \tag{3.26}$$

$$\frac{\bar{E}}{K_3^{\max}}\approx\frac{\omega_1^{\mathrm{el}}}{\Delta\omega_1}\frac{(\gamma\theta^2)^{1/2}}{(1+\frac{1}{2}\gamma\theta^2)\sqrt{2}}\,, \tag{3.27}$$

$$\kappa=\frac{2\omega_1^{\mathrm{el}}}{m}\approx\frac{2\gamma}{1+\frac{1}{2}\gamma\theta^2}\,, \tag{3.28}$$

and

$$\tau=\frac{2\omega_2^{\mathrm{el}}}{m}\approx\frac{\gamma^2\theta^2}{1+\frac{1}{2}\gamma\theta^2}\,. \tag{3.29}$$

In summary, Eq. (3.24) gives the area under the curve of Fig. 4 from $\omega_1^{\min}$ to c as a function of $\Delta\omega_1$ in the range

$$\omega_1^{\mathrm{el}}0.1>\Delta\omega_1>W\,, \tag{3.30}$$

provided the angle θ is such that

$$\omega_1/\Delta\omega_1\gg\gamma\theta^2/2\gg\Delta\omega_1/\omega_1\,. \tag{3.31}$$

---

[13] Inequalities (3.17) reduce to (3.31) in the laboratory system. Fortunately the angular range most convenient for experimenters (Ref. 1) is in the vicinity of $\frac{1}{2}\gamma\theta^2\sim\frac{1}{4}$ to 4 and thus these inequalities are easily satisfied.

[14] See the discussion concerning Eq. (1.4) of Ref. 9 at the end of Sec. III. Notice that the difference between the exact result and the approximation is a Spence function $\Phi((E_3-E_1)/E_1)$ in this reference. This term vanishes in our case because we are in the c.m. system.

[15] See D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961), Appendix C.

[16] See Sec. 4 for numerical examples, $(-\alpha/\pi f=Y)$.

The lower limit of $\Delta\omega_1$ in (3.30) is necessary because near the peak in Fig. 4 the shape of the spectra is mainly determined by ΔE_2 and $\Delta\theta$, and further even if ΔE_2 and $\Delta\theta$ were zero (i.e., $W=0$), $\Delta\omega_1$ should not be taken too small because of infrared divergence difficulties, i.e., Eq. (3.24) diverges as $\Delta\omega_1 \to 0$. A proper criterion to ascertain that we are not in the infrared region is that $\Delta\omega_1$ should be taken large enough such that[17] $-\delta \lesssim 0.2$. On the other hand, the upper limit of $\Delta\omega_1$ and inequalities (3.31) were imposed because we had assumed $\bar{\omega}_3$ to be small compared with $\bar{E}$ in order to obtain Eq. (3.16). Inequalities (3.31) are merely the re-expression of the c.m. condition (3.17) in terms of lab quantities.

Differentiating Eq. (3.24) with respect to $\Delta\omega_1$ we obtain the spectral distribution of ω_1 at angle θ in the lab system

$$\frac{d^2\sigma_{3\gamma}}{d\Omega_{k_1}d\omega_1}=\frac{\partial}{\partial\Delta\omega_1}\left(\frac{d\sigma}{d\Omega_{k_1}}\right)=\frac{d\sigma_0}{d\Omega_{k_1}}\frac{2\alpha}{\pi}\left(\ln\frac{2E_2}{m}-1\right)\times\frac{1}{\omega_1^{\mathrm{el}}-\omega_1}, \quad (3.32)$$

which is correct only if $\Delta\omega_1=\omega_1^{\mathrm{el}}-\omega_1$ and θ are such that (3.30) and (3.31) are satisfied.

4. NUMERICAL EXAMPLES

In order to facilitate numerical computations we shall rewrite all our formulas in more compact form. We are mainly interested in the regions of θ and $\Delta\omega_1$ specified by inequalities (2.11), (3.30), and (3.31). Let us introduce two new symbols

$$z=\gamma\theta^2/2, \quad (4.1)$$

$$R=\omega_1^{\mathrm{el}}/\Delta\omega_1. \quad (4.2)$$

The lowest order cross section for $e^+ + e^- \to 2\gamma$ can be written as [see Eq. (2.13)]

$$\frac{d\sigma_0}{d\Omega_k}\approx\frac{r_0^2}{2}\frac{1}{(1+z)^2}\left(\frac{1}{z}+z\right). \quad (4.3)$$

The radiative corrections δ can be written as [see Eq. (3.25)]

$$\delta=-\frac{2\alpha}{\pi}\left\{\ln\left(R\frac{\sqrt{z}}{1+z}\right)-\frac{3}{4}\right\}(\ln 2\gamma-1)+Y, \quad (4.4)$$

where

$$Y=-\frac{\alpha}{\pi}\frac{1}{[z+(1/z)]}\left\{\left(1+z+\frac{1}{2z}\right)\right\}\times\left\{\ln^2\left(1+\frac{1}{z}\right)-\ln\left(1+\frac{1}{z}\right)\right\}+\left(1+\frac{1}{z}+\frac{z}{2}\right)\{\ln^2(1+z)-\ln(1+z)\}+\left(\frac{1}{z}-z\right)\ln z+\pi^2\left\{\frac{1}{3}+\frac{1}{12}\left(z+\frac{1}{z}\right)\right\}. \quad (4.5)$$

The spectra of the photon for $e^+ + e^- \to 3\gamma$ can be written as [see Eq. (3.32)]

$$\frac{d^2\sigma_{3\gamma}}{d\Omega_{k_1}d\omega_1}=\frac{\alpha r_0^2}{\pi}\frac{1}{(1+z)}\left(z+\frac{1}{z}\right)(\ln 2\gamma-1)\frac{R}{E_2}. \quad (4.6)$$

The quantity $Y(z)$ has a rather complicated expression but numerically it is very small. For example

$$Y(10)=Y(0.1)=0.0024,$$
$$Y(1)=0.004.$$

The numerical values for δ are given for $\gamma=3\times10^4$ (i.e., $E_2=15$ BeV) and $z=1$:

$R=100$, $\delta=-0.207$;
$R=50$, $\delta=-0.175$;
$R=25$, $\delta=-0.142$;
$R=10$, $\delta=-0.10$;
$R=5$, $\delta=-0.068$.

5. CONCLUDING REMARKS

A. The purpose of writing this paper is threefold.

(1) To obtain useful formulas concerning the use of γ rays from $e^+ + e^- \to 2\gamma$ as a source for high-energy γ rays. It is hoped that the various formulas and considerations given here may be of some help to experimenters in designing experiments.

(2) To show how this type of calculation can be done in order to take into account the realistic experimental requirements.

(3) As a byproduct of this calculation, we observe that there is no term such as $(\alpha/\pi)\ln^2(4\bar{E}^2/m^2)$ in our expression for δ [see Eq. (3.25)]. Such a term occurs only in the infrared term $K(P_1,P_2)$ which is cancelled completely after addition of elastic and inelastic cross sections as previously observed.[10] Thus we conclude that the appearance of such a term in Brown and Feynman's and in Harris and Brown's results is due purely to their choice of phase space for k_3, namely, ω_3 isotropic and $\ll m$ in the lab frame. As emphasized

[17] If we believe in Schwinger's conjecture that the radiative corrections to all order can probably be written as $e^\delta=1+\delta+\delta^2/2!+\cdots$, then for $-\delta\lesssim0.2$, the next-order correction will be expected to be less than 2%.

previously, this kind of term is very undesirable. If, irrespective of how one chooses the phase space for k_3, terms such as $(\alpha/\pi)\ln^2(4\bar{E}^2/m^2)$ occur in δ, it means that quite independent of infrared catastrophies the perturbation expansion is not valid at energies higher than the value at which $(\alpha/\pi)\ln^2(4\bar{E}^2/m^2)\sim 1$. (This will occur at $\bar{E}\sim$BeV.) In other words, the existence of such terms in (BF) and (HB) and the disappearance of such terms in our expression simply indicate that such terms are closely related to the infrared catastrophe of the perturbation expansion and can be eliminated if one chooses the phase for ω_3 to be such that $\omega_3^{\max}\gg m$ in the center-of-mass system.

B. It is interesting to notice the similarity between our expression for δ [see Eq. (3.25)] and Schwinger's radiative corrections to the potential scattering,

$$\delta_{\text{Schwinger}}\approx\frac{-2\alpha}{\pi}\left\{\left(\ln\frac{E}{\Delta E}-\frac{13}{12}\right)\times\left[\ln\left(\frac{-q^2}{m^2}\right)-1\right]+\frac{17}{36}\right\}. \quad (5.1)$$

The similarity can be made more striking if we decompose $\delta_{\text{Schwinger}}$ into the contributions from the vacuum polarization, the vertex part and the bremsstrahlung:

$$\delta_{\text{Schwinger}}=\delta_{\text{vac}}+\delta_{\text{vertex}}+\delta_{\text{brem}}, \quad (5.2)$$

where

$$\delta_{\text{vac}}=\frac{2\alpha}{\pi}\left[\frac{-5}{9}+\frac{1}{3}\ln\left(\frac{-q^2}{m^2}\right)\right], \quad (5.3)$$

$$\delta_{\text{vertex}}=\frac{-2\alpha}{\pi}\left[\frac{1}{2}K(P_1,P_3)-\frac{1}{2}K(P_1,P_1)-\frac{3}{4}\ln\left(\frac{-q^2}{m^2}\right)+1\right], \quad (5.4)$$

$$\delta_{\text{brem}}=\frac{-2\alpha}{\pi}\left[\frac{-1}{2}K(P_1,P_3)+\frac{1}{2}K(P_1,P_1)+\ln\frac{E}{\Delta E}\left(\ln\frac{-q^2}{m^2}-1\right)\right]. \quad (5.5)$$

Now if we omit δ_{vac} and consider

$$\delta_{\text{vertex}}+\delta_{\text{brem}}=\frac{-2\alpha}{\pi}\left[\left(\ln\frac{E}{\Delta E}-\frac{3}{4}\right)\times\left(\ln\frac{-q^2}{m^2}-1\right)+\frac{1}{4}\right], \quad (5.6)$$

we arrived at a formula almost identical to Eq. (3.25) if we make the following substitutions:

$$\frac{E}{\Delta E}\to\frac{\bar{E}}{K_3^{\max}}, \quad (5.7a)$$

$$-q^2\to 4\bar{E}^2 \quad \text{(in the more usual notation } -t\to s). \quad (5.7b)$$

It is obvious why one should omit δ_{vac}, because there is no vacuum polarization in our problem. It is also quite obvious why by making substitutions (5.7a), (5.7b) on Eq. (5.5) we can obtain our Eq. (3.19). However, it is not quite obvious that by making a substitution $-t\to s$ into the ordinary vertex correction one obtains the bulk of the virtual radiative corrections represented by Eq. (3.4).[18]

C. The present calculation can be easily adapted to the need of the colliding beam experiment proposed at Stanford. The purpose of investigating the reaction $e^+ + e^- \to 2\gamma$ in the colliding experiment is to test the validity of quantum electrodynamics at high energies. Owing to the background bremsstrahlung, the experiment will probably be done by detecting two photons in coincidence, each photon detector having fairly good energy and angular resolutions. The lowest order cross section can be obtained from Eq. (2.4) by making the expression inside the bracket covariant and recalculating the function A which represents the phase space and flux density. The result is

$$\frac{d\sigma_0}{d\Omega_{k_1}}=\frac{r_0^2m^2}{8\bar{P}\bar{E}}\left\{\frac{\bar{E}+\bar{P}\cos\bar{\theta}}{\bar{E}-\bar{P}\cos\bar{\theta}}+\frac{\bar{E}-\bar{P}\cos\bar{\theta}}{\bar{E}+\bar{P}\cos\bar{\theta}}+4m^2\left(\frac{1}{\bar{E}^2-\bar{P}^2\cos^2\bar{\theta}}-\frac{m^2}{(\bar{E}^2-\bar{P}^2\cos\bar{\theta})^2}\right)\right\}$$
$$\approx(r_0^2m^2/8\bar{E}^2)\{\cot^2(\bar{\theta}/2)+\tan^2(\bar{\theta}/2)\}. \quad (5.8)$$

The second equation holds only if $\bar{\theta}$ and $\bar{E}_2$ satisfy

$$\sin\bar{\theta}\gg 1/\bar{\gamma}\ll 1. \quad (5.9)$$

The radiative correction can be calculated by using Eq. (3.25). The only thing one has to do is to relate $K_3^{\max}$ to the energy and angular resolutions of the detecting system. If $\Delta\theta$ is negligible, and further, if $\Delta\bar{\omega}_1$ is the smaller one of the energy resolutions of the two detectors, then Eq. (3.18) can be used as $K_3^{\max}$.

ACKNOWLEDGMENTS

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[18] Brown and Feynman's results (Ref. 6) show that after renormalization and infrared cancellation, only the matrix element represented by J in Fig. 5 contains logarithmic terms when inequalities (3.3a), (3.3b) are satisfied. Thus our observation can be stated simply that the bulk of the contribution from the matrix element J to the radiative corrections can be obtained from the Schwinger vertex correction by a substitution $-t\to s$.

Brown and Feynman and by Harris and Brown. He also wishes to thank Dr. Z. G. T. Guiragossián for conversations concerning experimental matters.

APPENDIX

In applying the Feynman rules to calculate the transition probabilities involving identical particles in the final states, considerable care must be exercised. Take the example of reaction $e^+ + e^- \to k_1 + k_2 + k_3$ in the c.m. system. Because of energy-momentum conservation, only momenta of two particles (say $\mathbf{k}_1$ and $\mathbf{k}_2$) are needed to determine the kinematics completely. Suppose we have two detectors A and B. We may arbitrarily assign the photon detected by A to be k_1 and that detected by B to be k_2, and calculate the cross section according to the standard rule

$$d\sigma = (2\pi)^4 \frac{1}{4[(P_1 \cdot P_1)^2 - m_1^2 m_2^2]^{1/2}} \frac{1}{(2\pi)^{3\times 3}} \times \int \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \frac{d^3k_3}{2\omega_3} \delta^4(p_f - p_i) S \left| \sum_{i=1}^{6} M_i \right|^2, \quad \text{(A1)}$$

where S is averaged over initial spin and summed over the final polarizations, and the M_i's are the matrix elements written according to the standard rule. [We normalize to one particle per $(2E)^{-1}$ cm^3 for each particle.] The integral is to be taken in the range specified by detectors A and B. Now suppose the detector B is removed; do we get the right result by just integrating over all d^3k_2? The answer is no. The expression (A1) must be divided by 2 and then integrated over all possible d^3k_2. To show this, consider an event where one of three photons goes into A and another into B. The probability of this event is independent of whether or not there is a detector at B. Now corresponding to this event, there are two possibilities, either k_2 goes into B or k_3 goes into B. Both of these possibilities occur when there is no detector at B. But now as soon as we put a detector in B k_3 can not go into B because we said the photon detected by B is called k_2. This proves our assertion. The result can be generalized to n identical particles in the final states. We need $n-1$ detectors to determine the kinematics of the problem. Thus if there are $n-1$ detectors we use the standard rule. If we have $n-2$ detectors we divide the whole expression by 2. If there are $n-d$ detectors we divide the standard expression by $d!$. Specifically if there is no detector (i.e., total cross section) we have to divide the Eq. (A1) by $n!$.

The arguments given above can be used in our calculation of spectra of k_1 due to $e^+ + e^- \to 3\gamma$. The radiative corrections calculated by Brown and Feynman and by Harris and Brown apply only to the coincidence experiments. Since we are not measuring k_2 and k_3 we have to integrate over all the phase space of k_2 and k_3 and divide it by 2! according to our prescription. The phase space of k_2 and k_3 can be represented as an ellipsoid as shown in Fig. 6. Now the matrix element $|M|^2$ is symmetric with respect to interchange of $k_2 \leftrightarrow k_3$. Thus instead of integrating k_2 and k_3 in the whole phase space and dividing it by 2!, we may equivalently integrate over half of the phase space as shown in Fig. 7. Now in this modified phase space k_2 is never small and thus an infrared divergence occurs only when $k_3 \to 0$, although both k_2 and k_3 can become infrared in the complete phase space.