Real Part of the Scattering Amplitude and the Behavior of the Total Cross Section at High Energies

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Certain theorems first pointed out by Meiman are used to include the information contained in analyticity in energy and crossing symmetry in the derivation of upper bounds for the forward scattering amplitude. We have obtained theorems relating the asymptotic behavior of the ratio of the real part to the imaginary part of the forward amplitude on the one hand, to that of the total cross section on the other. Except for the special case where $\mathrm{Re}f/\mathrm{Im}f\sim\pi(\ln E)^{-1}$ as the incident laboratory energy $E\to\infty$, we find that the Froissart bound can be improved. We also find that the Greenberg-Low bound which follows from axiomatic field theory can always be improved by a fractional power of E under the physical assumption that the amplitude does *not* become purely real as $E\to\infty$. As a result of our theorems we find that the asymptotic behavior $\mathrm{Re}f\sim cE$ is not allowed. This sheds light on the question of the "elementary" nature of vector mesons. Finally, we give an inequality which, in principle, could lead to an experimental check of analyticity and crossing with data obtained only from a finite range of energy.

I. INTRODUCTION

IN 1961 Froissart[1] obtained upper bounds for the scattering amplitude and the total cross section at high energies based on unitarity and the Mandelstam representation. Later, Martin[2] showed that these upper bounds can be obtained without assuming the validity of the Mandelstam representation. Essentially, all that is needed, other than the properties that follow from the Lehmann-Symanzik-Zimmermann (LSZ) scattering formalism and unitarity, is analyticity in the momentum transfer variable t in a small domain around the origin. The minimum size of this domain should be finite and independent of energy.

In the work of Kinoshita, Loeffel, and Martin,[3] the proof of Ref. 2 was sharpened. It was also shown by construction of counter-examples that the Froissart bound for forward scattering cannot be improved if one does not use information other than analyticity in t, unitarity, and boundedness by a polynomial in the energy variable.

The analyticity in t assumed by Martin has never been proved either in local field theory or in perturbation theory. The best that one can do starting with analyticity in t in the usual Lehmann ellipse[4] is the upper bound obtained by Greenberg and Low[5] which has one more power of laboratory energy than the Froissart bound.

Both Refs. 2 and 5 did not make use of the fact that the scattering amplitude is analytic in the energy variable and satisfies the crossing symmetry relations. It is the purpose of this paper to show that certain theorems first introduced by Meiman[6] provide a very useful tool for including some of the information contained in analyticity in E and crossing in the derivation of upper bounds for the forward scattering amplitude.

We have found that there is a very close relationship between the asymptotic behavior of the ratio $|\mathrm{Re}f|/|\mathrm{Im}f|$ for large energy and that of the total cross section. (For convenience we limit ourselves first to self-crossed forward scattering of a spin-zero particle. Most of our results, however, hold for more general cases, as pointed out below.) We give three theorems in Sec. II which clarify this relationship. We first show that if, as $E\to\infty$, $|\mathrm{Re}f|/|\mathrm{Im}f|>\tan\pi\alpha$, $0<\alpha\leq\frac{1}{2}$, then the total cross section must vanish faster than $E^{-\alpha/2}$. Secondly, we show that, if as $E\to\infty$, $|\mathrm{Re}f|/|\mathrm{Im}f|\geq(\ln E)^{-a}$, with $0<a<1$, the total cross section is bounded by $(\ln E)^{-\lambda}$, where λ is positive and may be chosen as large as we please. Finally, to treat the cases where $|\mathrm{Re}f|/|\mathrm{Im}f|$ goes to zero faster than $(\ln E)^{-1}$, we make some extra assumptions about the smoothness of $\sigma(E)$ as $E\to\infty$. Under these assumptions we show that, if $[|\mathrm{Re}f|/|\mathrm{Im}f|]\ln E\to 0$ as $E\to\infty$, σ is bounded by $(\ln E)^{\epsilon}$, where ϵ is an arbitrarily small positive number. We find that the only case in which the Froissart bound cannot be improved by these methods is the one for which $\mathrm{Re}f/\mathrm{Im}f\sim\pi/\ln E$, as $E\to\infty$. All these results also hold for the amplitude

$$T^{(1)}(E)=\tfrac{1}{2}[f_+(E)+f_-(E)],$$

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[1] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[2] A. Martin, Phys. Rev. **129**, 1432 (1963). See also Ref. 5.
[3] T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963). Also, Phys. Rev. **135**, B1464 (1964).
[4] H. Lehmann, Nuovo Cimento **10**, 579 (1958).
[5] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).
[6] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **43**, 2277 (1962) [English transl.: Soviet Phys.—JETP **16**, 1609 (1963)].

where f_+ and f_- are the forward π^+p and π^-p amplitudes, respectively.

Theorems 1 and 2 do not treat the case where Ref changes sign an infinite number of times as $E\to\infty$. (Such a behavior is not very plausible if the total cross section does not oscillate with large amplitudes for large E.)

All the theorems of Sec. II are obtained by assuming that $f(E)$ satisfies the Froissart bound. In Sec. III we remove this assumption and take as a starting point the Greenberg-Low upper bound. We show that if $|\mathrm{Im}f|/|\mathrm{Re}f|$ does *not* tend to zero as $E\to\infty$, then the Greenberg-Low bound can be improved by a fractional power of E. This result holds even if Ref oscillates around zero an infinite number of times. As a consequence, in any field theory with analyticity in a Lehmann ellipse, the forward dispersion relations require at most two subtractions, provided that the amplitude does not become relatively real as $E\to\infty$. Thus, for example, one does not have to use the Froissart bound to prove that $[f(E)-f(0)]/E$ is a Herglotz function.[7]

Our theorems enable us to make some statements about the kind of contributions that the exchange of "elementary" vector mesons can give to the real part of the forward amplitude. This question will be taken up briefly in Sec. IV.

Recent experiments at Brookhaven[8] and CERN[9] have indicated the existence of a substantial real part in $\pi^\pm p$ and pp scattering amplitudes at high energies. In Sec. V we give an inequality which in principle can give a direct test of analyticity, crossing symmetry, and boundedness by a polynomial. The test depends only on data on Ref and the total cross section obtained for a *finite* range of energies, for example 1–500-BeV incident laboratory energy. If a real part is maintained to such high energies, this inequality could be violated. Such a violation would mean that the forward dispersion relation is invalid. The main attractive feature of this inequality is that, unlike the dispersion relation, it does not contain integrals that extend to infinite energies.

The theorems of Meiman which are the basis of this paper follow immediately from certain inequalities on harmonic measures given in the book of Nevanlinna.[10] For the convenience of the reader we have included in the Appendix a brief summary of definitions and basic results from Nevanlinna's book necessary to understand the origin of Meiman's theorems. For completeness we also include proofs of the two theorems of Meiman. Those interested in the proofs of the results in Secs. II and III should read the Appendix before the paper.

Throughout this paper we have ignored the possible existence of bound-state poles in the scattering amplitude. However, since these pole contributions are real and behave asymptotically like $1/E$, one can easily subtract them out from the amplitude and deal with the quantity, $f'(E)=f(E)-\Gamma/(E-E_b)$. The theorems in this paper all remain unchanged and hold for f' as well as for f.

II. THE REAL PART OF THE FORWARD AMPLITUDE AND THE IMPROVEMENT OF THE FROISSART BOUND

We shall first consider the forward scattering amplitude, $f(E)$, for the scattering of a neutral scalar particle A by another particle B, where we assume that $\bar{A}\equiv A$. As a function of the energy E of the incident particle A in the laboratory system, $f(E)$ is known to have the following properties:

(i) $f(E)$ is analytic in the cut E plane with cuts on the real axis extending from 1 to $+\infty$ and from $-\infty$ to -1,[11]

(ii) $f(E+i0)=f^*(E-i0)$,

(iii) $f(-E-i0)=f(E+i0)$,

(iv) $f(E)$ is bounded by a polynomial in $|E|$ for large $|E|$ in all directions in the E plane. (For most of the results of this paper it is enough to make the weaker assumption that for large enough $|E|$, $|f| < e^{\epsilon|E|}$, for any arbitrary $\epsilon>0$.)

For some actual scattering processes the properties (i)–(iii) follow rigorously from the LSZ scattering formalism. The property (iv), although never proved, is usually assumed to be a feature of local field theories.

If one further assumes following Martin[2] that, in addition to the properties that result from the LSZ formalism, the nonforward amplitude $f(E,t)$ is analytic in t in a finite neighborhood of $t=0$, whose size is independent of E, one obtains the Froissart bound:

(v) $|f(E)|\leq CE(\ln E)^2$ for sufficiently large E.

In deriving (v), however, analyticity in E and crossing symmetry, i.e., properties (i), (ii), and (iii) were not used.

In this section we shall show that the theorems of Meiman (given in the Appendix) provide us with a useful tool for including the information contained in (i)–(iii). We shall namely show that there exists an intimate relationship between the behavior of the ratio $|\mathrm{Re}f|/|\mathrm{Im}f|$ for $E\to\infty$ and the asymptotic behavior of $f(E)$ itself. For all possible behaviors of $|\mathrm{Re}f|/|\mathrm{Im}f|$, excluding one special case, we find that the Froissart bound (v) can be improved. The main results are contained in the three theorems given below.

[7] If $\phi(z)$ is analytic for Im$z>0$ and if, for Im$z>0$, Im$\phi(z)\geqslant 0$, then ϕ is called a Herglotz function. See J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943), p. 23.

[8] K. J. Foley, R. S. Gilmore, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. H. Willen, R. Yamada, and L. C. L. Yuan, in a paper presented to the 1964 Conference on High Energy Physics, Dubna, USSR (unpublished).

[9] G. Bellettini *et al.*, paper presented to the 1964 Conference on High Energy Physics, Dubna, USSR (unpublished).

[10] R. Nevanlinna, in *Eindeutige Analytische Funktionen* (Springer Verlag, Berlin Gottingen Heidelberg, 1953), 2nd ed.

[11] We set $m_A=1$ throughout this paper. Thus E stands for E/m_A.

Theorem 1: If $f(E)$ satisfies the conditions (i)–(v), and if for sufficiently large real E the inequality

$$\left|\frac{\mathrm{Ref}}{\mathrm{Imf}}\right| \geq \tan\pi\alpha; \quad 0<\alpha\leq\tfrac{1}{2}, \tag{1}$$

holds, then the function $f(E)$ has the upper bound

$$|f(E)| \leq CE^{1-\alpha/2}(\ln E)^2 \tag{2}$$

as $E\to\infty$.

Proof: Consider the function

$$w(E)=\frac{f(E)}{iE[\ln E - i(\pi/2)]^\gamma}, \quad \gamma>2, \quad 0\leq \arg E\leq\pi. \tag{3}$$

By definition $w(E)$ is analytic in the upper half E plane excluding the unit semicircular disk around the origin. Furthermore, $w(E)\to 0$ as $E\to\pm\infty$ above the cuts. Thus, using the Phragmén-Lindelöf principle and (iv) we can conclude that, as $|E|\to\infty$, $w(E)\to 0$ in all directions in the upper half E plane.[12] It also follows from (v) and the dispersion relation for $f(E)$ that $f(E)/E$ has no zeros and that $\mathrm{Im}(f/E)\geq 0$, for $|E|$ greater than some constant. This insures that $w(E)$ has no zeros for sufficiently large $|E|$ in the upper half-plane and that $\mathrm{Re}w\geq 0$ in that region. From unitarity we know that $\mathrm{Im}f(E)\geq 0$ for $E\geq 1$. Hence along the positive real axis $\mathrm{Re}w(E)$ is positive for large E.

The reality condition (ii) and the crossing relation (iii) give us

$$\begin{aligned}\mathrm{Re}w(E+i0)&=\mathrm{Re}w(-E+i0),\\ \mathrm{Im}w(E+i0)&=-\mathrm{Im}w(-E+i0).\end{aligned} \tag{4}$$

For large enough E we have

$$\mathrm{Im}w(E)/\mathrm{Re}w(E)\cong -\mathrm{Re}f/\mathrm{Im}f,$$

and hence

$$|\mathrm{Im}w/\mathrm{Re}w|\geq\tan\pi\alpha. \tag{5}$$

Thus $w(E)$ satisfies the conditions of theorem I of Meiman given in the Appendix, and we have for large enough real E

$$|w(E)/w(E_0)|\leq C(E_0/E)^{\alpha/2}, \tag{6}$$

and the theorem is proved.

As a consequence of Theorem 1 we see that if $|\mathrm{Re}f|/|\mathrm{Im}f|\geq\tan\pi\alpha$ as $E\to\infty$, the total cross section should fall off asymptotically at least as fast as a power of E, or more precisely

$$\sigma(E)\leq CE^{-\alpha'/2}, \quad \text{for} \quad \alpha'<\alpha. \tag{7}$$

Conversely, if the cross section tends to a nonvanishing value or decreases more slowly than any negative power of E as $E\to\infty$, then $|\mathrm{Re}f|/|\mathrm{Im}f|$ must tend to zero in that limit. (For the moment we are excluding the possibility that $\mathrm{Re}f$ changes sign an infinite number of times. We come back to this point at the end of this section.)

It has been known for some time that if $f(E)/E$ has a definite finite limit as $E\to\infty$, then $\mathrm{Re}f/E$ must tend to zero.[13] The theorem above makes this statement much more precise starting with much weaker assumptions. We go even further in the next theorem.

Theorem 2: If $f(E)$ satisfies the conditions (i)–(v), and if for sufficiently large E

$$|\mathrm{Re}f/\mathrm{Im}f|\geq C/(\ln E)^a, \quad 0<a<1, \tag{8}$$

then the following upper bound holds for large E:

$$|f(E)|\leq C|E|(\ln|E|)^{-\lambda}, \quad \lambda>0. \tag{9}$$

Here λ can be chosen arbitrarily large.

Proof: We again consider the function, $w(E)$, constructed as in (3). To prove our theorem we have to show that $w(E)$ satisfies for large real E the conditions of Theorem II of Meiman (see Appendix), namely

$$|\mathrm{Im}w(E)|\geq C|\mathrm{Re}w(E)|^\nu, \quad \nu>1. \tag{10}$$

For sufficiently large real E we have

$$\mathrm{Im}w(E+i0) \cong\left\{-\mathrm{Re}f(E)+\frac{\pi\gamma\,\mathrm{Im}f(E)}{2\ln|E|}\right\}\frac{1}{|E|(\ln|E|)^\gamma}$$

$$\mathrm{Re}w(E+i0) \cong\left\{\mathrm{Im}f(E)+\frac{\pi\gamma\,\mathrm{Re}f(E)}{2\ln|E|}\right\}\frac{1}{|E|(\ln|E|)^\gamma}. \tag{11}$$

The symmetry relations (4) still hold and give $\mathrm{Re}w\times(-E+i0)$ and $\mathrm{Im}w(-E+i0)$. From (8) and (11) we now get

$$\begin{aligned}|\mathrm{Im}w(E)|&\approx|\mathrm{Re}f|/|E|(\ln|E|)^\gamma\\ &\geq C|\mathrm{Im}f|/|E|(\ln|E|)^{\gamma+a}.\end{aligned} \tag{12}$$

Let us choose ν such that

$$\nu=(\gamma-2+a)/(\gamma-2). \tag{13}$$

Then we obtain

$$|\mathrm{Im}w|\geq C|\mathrm{Im}f|/|E|(\ln|E|)^{(\gamma-2)\nu+2}. \tag{14}$$

Using (v) we get

$$|\mathrm{Im}w|\geq C'|\mathrm{Im}f|\ |\mathrm{Im}f|^{\nu-1}/|E|^\nu(\ln|E|)^{\gamma\nu},$$

and thus by (11)

$$|\mathrm{Im}w|\geq C'|\mathrm{Re}w|^\nu, \quad C'>0. \tag{10'}$$

[12] The Phragmén-Lindelöf theorem is also valid for a region obtained from the half-plane by deforming a finite part of its boundary. (See Ref. 6 footnote 2.)

[13] H. Lehmann, Nucl. Phys. 29, 300 (1962). L. Van Hove, Phys. Letters 5, 252 (1963).

From Theorem II of Meiman we now have

$$|\mathrm{Im}w(E)|\leq C[\ln(E/E_0)]^{\gamma-\nu/(\nu-1)}, \quad (15)$$

where ν is given by (13). It follows that

$$|\mathrm{Re}f|\leq C|E|(\ln|E|)^{\gamma-\nu/(\nu-1)}.$$

Using (13) to eliminate ν we obtain

$$|\mathrm{Re}f|\leq C|E|(\ln|E|)^{-\lambda'}, \quad (16)$$

where

$$\lambda'=[(\gamma-2)(1-a)/a]-1, \quad \gamma>2. \quad (17)$$

Clearly, since $0<a<1$, we can always choose γ large enough to make λ' as large as we please.

The inequality (16) holds only for Ref, but we can get a similar inequality for Imf. Using (10) and (15) we have

$$|\mathrm{Re}w|\leq C(\ln|E|)^{-1/(\nu-1)},$$

and hence

$$|\mathrm{Im}f|\leq C|E|(\ln|E|)^{-\lambda}, \quad (18)$$

where

$$\lambda=[(\gamma-2)(1-a)/a]-2, \quad \gamma>2. \quad (19)$$

Again we can choose γ large enough to make λ arbitrarily large.

Thus we can immediately see from Theorem 2 that, if the total cross section does not go to zero faster than any inverse power of ln$E$, $|\mathrm{Re}f|/|\mathrm{Im}f|$ must go to zero as $E\to\infty$ faster than $(\ln E)^{-a}$, $a<1$.

We now turn to the question of the behavior of the total cross section when $|\mathrm{Re}f|/|\mathrm{Im}f|$ tends to zero faster than $(\ln E)^{-1}$. In proving both Theorems 1 and 2, the inequality (A9) given in the Appendix was essential. The quantity ρ in that inequality is closely related to the ratio of the real and imaginary parts of f. It is easy to see that if ρ is, loosely speaking, smaller than $(\ln E)^{-1}$ the inequality (A9) is always easily satisfied for large E and very little new information can be squeezed out of it. However, under certain assumptions about the smoothness of $\sigma(E)$ for large E it is still possible to improve the Froissart bound from $(\ln E)^2$ to $(\ln E)^\epsilon$ where ϵ is small and positive.

To treat the case where $|\mathrm{Re}f|/|\mathrm{Im}f|\ll 1/\ln E$, we shall supplement the conditions (i)–(v) by the *physical assumption* that the total cross section does not oscillate for large E but has a definite growth, for example,

$$\sigma(E)\sim C(\ln E)^\gamma(\ln\ln E)^\beta(\ln\ln\ln E)^\delta\cdots. \quad (20)$$

From (v) we already know that $\gamma\leq 2$. The exponents β and δ could be any positive or negative numbers. We can cover a much larger class of possibilities than in (20) by assuming that $\sigma(E)$ varies as $E\to\infty$ according to the law

$$\sigma(E)\sim c(\ln E)^\gamma|\psi(E)|, \quad c\neq 0, \quad 0\leq\gamma\leq 2, \quad (21)$$

where $\psi(E)$ is a function satisfying the following properties:

(a) $\psi(E)$ is analytic in the upper half-plane and so is $[\psi(E)]^{-1}$.

(b) For any $\epsilon>0$ and for sufficiently large $|E|\geq E_0(\epsilon)$, we have $(\ln|E|)^{-\epsilon}<|\psi(E)|<(\ln|E|)^{+\epsilon}$.

(c) Along the real axis $[\psi(-E)/\psi(E)]\to 1$ as $E\to\infty$, and $[(\mathrm{Im}\psi/|\psi|)\ln|E|]\to 0$ as $E\to\infty$.

For example, any positive or negative power of ln ln$E$ will satisfy (a), (b), and (c). We can now state our third theorem.

Theorem 3: Assume that $\sigma(E)$ behaves as in (21). If $f(E)$ satisfies (i)–(v), and if as $E\to\infty$

$$|\mathrm{Re}f/\mathrm{Im}f|\ \ln|E|\to 0, \quad (22)$$

then the positive constant γ in (21) must be arbitrarily small.

Proof: We construct the function

$$w(E)=f(E)/iE[\ln E-i(\pi/2)]^\gamma\psi(E), 0\leq\arg E\leq\pi. \quad (23)$$

By definition $w(E)$ is analytic outside the unit semicircle in the upper half-plane. Furthermore, we have

$$\lim_{|E|\to\infty} w(E)=c, \quad c>0. \quad (24)$$

This limit is approached uniformly in all directions in the upper half-E plane.[12] For sufficiently large E above the real axis we have the symmetry relations

$$\begin{aligned}\mathrm{Re}w(E+i0)&\cong\mathrm{Re}w(-E+i0),\\ \mathrm{Im}w(E+i0)&\cong-\mathrm{Im}w(-E+i0).\end{aligned} \quad (25)$$

As in the case of Meiman's theorems in the Appendix we consider the mapping of a region of the upper half E plane bounded by two semicircles of radius E_0 and $E(E\gg E_0$, both large) onto a domain of the w plane. The upper edge of the real axis from E_0 to E is mapped into a curve in the w plane which lies above a straight line drawn from the origin above the positive real axis making an angle θ_E with it. For sufficiently large E, $|\mathrm{Re}f|/|\mathrm{Im}f|\cong o(1/\ln E)$, and one finds

$$\tan\theta_E\cong\pi\gamma/2\ln|E|. \quad (26)$$

Similarly, the upper edge of the real axis from $-E$ to $-E_0$ is mapped into a curve symmetrical with the one above with respect to the real axis of the w plane [see Figs. 1(a) and (b)].

We set $w=u+iv$ and define u_0 and u_E as in Theorem I of the Appendix. There are two cases to be dealt with separately.

Case 1: $u_0<c$ [see Fig. 1(a)].

In this case the inequality (A9) gives

$$\int_{u_0}^{u_E}\frac{du}{\rho(u)}\geq\tfrac{1}{4}\ln\frac{E}{E_0}. \quad (27)$$

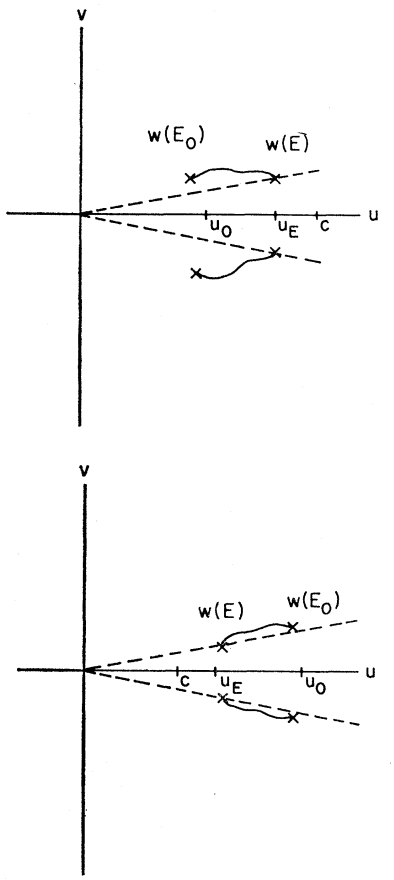

FIG. 1(a). The w plane for large $|E|$ (case 1). (b) The w plane for large $|E|$ (case 2).

From Fig. 1(a), it is clear that

$$\rho(u)\geq u\sin\theta_E,\quad u_0\leq u\leq u_E. \tag{28}$$

One then obtains from (27)

$$\ln|u_E/u_0|\geq\tfrac{1}{4}[\ln(E/E_0)]\sin\theta_E,$$

and hence, using (26),

$$\ln|u_E/u_0|\geq\tfrac{1}{8}\pi\gamma\ln(E/E_0)/\ln E. \tag{29}$$

We now let $E\to\infty$. Then $u_E\to c$ and we get

$$\ln|c/u_0|\geq\tfrac{1}{8}\pi\gamma. \tag{30}$$

The function $w(E)$ approaches c uniformly in all directions in the upper half-E plane. Therefore, given an $\epsilon>0$, we can always choose a finite but large E_0 such that

$$|w(E_0e^{i\phi})-c|<\epsilon,\quad 0\leq\phi\leq\pi. \tag{31}$$

Such a choice of E_0 makes the ratio $c/|u_0|$ close to unity. Thus (30) would be a contradiction unless γ is arbitrarily small.

Case 2: $u_0>c$ [see Fig. 1(b)]

Here we have

$$\int_{u_E}^{u_0}\frac{du}{\rho(u)}\geq\tfrac{1}{4}\ln(E/E_0). \tag{32}$$

Using (28) we obtain

$$\ln|u_0/u_E|\geq\tfrac{1}{8}\pi\gamma\ln(E/E_0)/\ln E,$$

and taking the limit $E\to\infty$, we have

$$\ln(u_0/c)\geq\tfrac{1}{8}\pi\gamma. \tag{33}$$

By an argument similar to case 1, this would be a contradiction unless γ is arbitrarily small. This completes the proof of Theorem 3.

So far we have covered three distinct cases for the asymptotic behavior of the ratio $\mathrm{Re}f/\mathrm{Im}f$:

(1) $|\mathrm{Re}f|/|\mathrm{Im}f|\geq$ constant, as $E\to\infty$,

(2) $|\mathrm{Re}f|/|\mathrm{Im}f|\geq(\ln E)^{-a}$, $0\leq a<1$,

(3) $|\mathrm{Re}f/\mathrm{Im}f|\ln E\to 0$ as $E\to\infty$.

There are two situations not yet covered by (1), (2), and (3). They are

(4) $|\mathrm{Re}f/\mathrm{Im}f|(\ln E)\to\infty$ and

$$|\mathrm{Re}f/\mathrm{Im}f|(\ln E)^{1-\epsilon}\to 0,\ 0<\epsilon<1,\text{ as }E\to\infty, \tag{34}$$

and

$$(5)\ \mathrm{Re}f/\mathrm{Im}f\sim a/\ln E\text{ as }E\to\infty, \tag{35}$$

where a is a constant.

The case (4), which is intermediate between (2) and (3), can be treated by a method almost identical with that of Theorem 3 and leads essentially to the same conclusion. The main difference is that now the angle θ_E is larger than $(\ln E)^{-1}$ but less than $(\ln E)^{-1+\epsilon}$, and one obtains inequalities like (30) with infinity on the right-hand side. This is obviously a contradiction and γ must be zero.

The case (5) can be reduced to that of Theorem 3 if we replace $\frac{1}{2}\pi\gamma$ by $|\frac{1}{2}\pi\gamma-a|$. Thus we find that $|\frac{1}{2}\pi\gamma-a|$ must be arbitrarily small in order that (35) is compatible with (21). For $0<\gamma\leq 2$, this means that $f(E)$ cannot have an asymptotic behavior of the form (21) if $a<0$ or $a>\pi+\delta$ (δ small positive). Only when $0<a<\pi+\delta$ and $a\approx\frac{1}{2}\pi\gamma$, is (35) not in contradiction with (21). Thus, the only case in which the Froissart bound cannot be improved by this method is when $a=\pi$. If $0<a<\pi$, the $(\ln E)^2$ in the Froissart bound can be replaced by $(\ln E)^{(2a)/\pi}$.

So far we have only treated the forward scattering for the self-crossed case. However, a generalization of the results of this section to other cases is easy. For example, we can consider the amplitudes $T^{(1)}(E)$ defined in the usual way,

$$T^{(1)}(E)=\tfrac{1}{2}[f_+(E)+f_-(E)], \tag{36}$$

where f_+ and f_- are the π^+p and π^-p forward amplitudes. This amplitude satisfies the same crossing relation (iii) as the self-crossed amplitude and Theorems 1, 2, and 3 hold also for $T^{(1)}$.

Finally, we would like to remark on the possibility that, as $E\to\infty$, $\mathrm{Re}f$ changes sign an infinite number of times. Although such a behavior can be tolerated in Theorem 3 as far as $|\mathrm{Re}f/\mathrm{Im}f|\ \ln E \to 0$, it is explicitly excluded in Theorems 1 and 2. Therefore, from a strictly mathematical point of view we have not covered all possibilities for the asymptotic behavior of $\mathrm{Re}f/\mathrm{Im}f$. However, we feel that such oscillations are highly unphysical for it would mean that for very high energies the interaction changes from being repulsive to being attractive and vice versa an infinite number of times.

III. IMPROVEMENT OF THE GREENBERG-LOW UPPER BOUND

The analyticity in the t plane needed to prove the Froissart bound (v) has never been established rigorously either in axiomatic field theory or in perturbation theory. The best that one has been able to do within the LSZ formalism is the much weaker bound obtained by Greenberg and Low,[5] namely

$$|f(E)| \leq CE^2(\ln E)^2. \qquad \text{(v}')$$

This inequality follows from unitarity and analyticity in the Lehmann ellipse in the t plane. At this point it will be natural to ask whether it can be improved by the methods used in the previous section. The following theorem analogous to Theorem 1 can easily be proved.

Theorem 4: If $f(E)$ satisfied (i)–(iv) and (v′), and if for sufficiently large E

$$|\mathrm{Im}f/\mathrm{Re}f| \geq \tan\pi\alpha, \quad 0<\alpha\leq\tfrac{1}{2}, \qquad (37)$$

then as $E\to\infty$

$$|f(E)| \leq CE^{(2-\alpha/2)}(\ln E)^2. \qquad (38)$$

To prove this theorem one only has to apply Theorem I of the Appendix to the function,

$$w(E)=f(E)/E^2[\ln E-i(\pi/2)]^\gamma, \quad \gamma>2, \quad 0\leq\arg E\leq\pi. \qquad (39)$$

We note that this theorem is much stronger than those discussed in the previous section. First, (37) can hold even if $\mathrm{Re}f$ oscillates through zero an infinite number of times. Secondly, one must notice that the only condition outside the LSZ formalism which we impose here is the assumption that $\mathrm{Im}f/\mathrm{Re}f$ does *not* tend to zero as $E\to\infty$.[14] This is an extremely reasonable physical assumption for, in a theory with many inelastic channels, the scattering amplitude will not become purely real at high energies.

Thus, without using the Froissart bound, one can conclude that any forward-scattering amplitude which does not become purely real at infinite energies (i.e., $\mathrm{Im}f/\mathrm{Re}f \nrightarrow 0$ as $E\to\infty$) needs at most two subtractions in the dispersion relations. From this statement one can easily show that the function

$$g(E)\equiv[f(E)-f(0)]/E$$

is a Herglotz function so that $\mathrm{Im}g(E)\geq0$ for $\mathrm{Im}E>0$. Various useful results may be derived from this property making use of the methods of Jin and Martin.[15]

If $\mathrm{Im}f/\mathrm{Re}f$ goes to zero as $E\to\infty$ but not faster than $(\ln E)^{-a}$, $0<a<1$, we obtain a theorem similar to Theorem 2. We find that

$$|f|<CE^2(\ln E)^{-\lambda}, \quad E\to\infty, \quad \lambda>0,$$

where λ can be chosen arbitrarily large. Thus again in this case $f(E)$ will satisfy dispersion relation with two subtractions.

IV. REMARK ON VECTOR MESONS

We would like to remark here on the relation between the theorems of Sec. II and the contribution of an "elementary" vector meson to $f(E)$ at high energies. It has been conjectured on the basis of perturbation theory that, if an "elementary" particle of spin J is exchanged, then, as $E\to\infty$, $\mathrm{Re}f\sim CE^J$. For $J\geq\frac{3}{2}$, this is already in contradiction with the Froissart bound. One usually concludes from this that either all particles of spin $J\geq\frac{3}{2}$ are Reggeized, or there exist some mechanism, other than Reggeization, which damps $\mathrm{Re}f$ and gives it an asymptotic behavior quite different from what one expects by looking at perturbation theory.

For vector mesons, however, there have been no direct arguments to show whether it could be elementary in the above sense or not. We point out here that the behavior $\mathrm{Re}f\sim CE$ is *not* allowed by the theorems of Sec. II. For example if $\sigma(E)\to$ const or vanishes as $E\to\infty$, then $|\mathrm{Re}f|/|\mathrm{Im}f|$ is asymptotically larger than a constant when $\mathrm{Re}f\sim CE$. By Theorem 1, however, we obtain an upper bound for $|f(E)/E|$ which decreases faster than a negative power of E, and hence a contradiction. Similarly, $C'<\sigma(E)\leq C(\ln E)^a$, $0<a<1$, leads us to a contradiction by Theorem 2. Finally, if $C'(\ln E)<\sigma(E)\leq C''(\ln E)^a$, $1<a\leq2$, then Theorem 3 will give a contradiction again. So except for one special case where $[\mathrm{Re}f/\mathrm{Im}f\sim a(\ln E)^{-1}, \pi/2\leq a\leq\pi$, as $E\to\infty]$ [see argument below (35)], we are forced to conclude that $\mathrm{Re}f$ cannot behave like $\sim CE$. Thus the possibility for the existence of a vector meson, "elementary" in the sense discussed above, is ruled out in almost all cases.[16]

V. POSSIBLE EXPERIMENTAL TEST OF ANALYTICITY AND CROSSING

At present, analyticity of scattering amplitudes and crossing symmetry play a central role in strong interaction physics. It is of fundamental importance to examine whether such properties are consistent with experimental data. Of course, the ordinary dispersion

[14] One should notice that here we are dealing with the inverse of the ratio used in Sec. II.

[15] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[16] Conclusions similar to those reached in this section were obtained under stronger assumptions and by using unitarity by P. G. O. Freund and R. Oehme, Phys. Rev. Letters **10**, 199 (1963) and also K. Yamamoto, Phys. Letters **5**, 355 (1963).

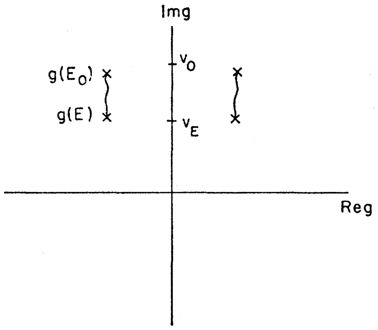

FIG. 2. A schematic plot of the function $g(E)$.

relation for the forward scattering amplitude $f(E)$ may be used for this purpose, if data on $\text{Im}f(E)$ and $\text{Re}f(E)$ are available at all energies. This approach will not be fully conclusive, however, since it requires data for $E\to\infty$ which cannot be obtained experimentally.

We wish to point out here that experimental test of analyticity and crossing symmetry may also be made starting from the inequality (A6) or somewhat improved version (41). This method has a difficulty similar to that of dispersion relations in the sense that it requires information of $f(E)$ for complex E. However, if we can make a reasonable estimate of $f(E)$ for some complex E, this approach may have some practical advantage over dispersion relations because it requires only data at finite energies and, furthermore, it utilizes the experimental information on $\text{Re}f(E)$ and $\text{Im}f(E)$ in a more direct fashion.

Let us consider the function $g(E)$ defined by

$$g(E)=T^{(1)}(E)/E, \tag{40}$$

where $T^{(1)}$ is the average of π^+p and π^-p forward amplitudes defined by (36). Let us assume that experimental data for $\text{Re}g(E)$ and $\text{Im}g(E)$ are given for the energy range $E_0-E(E>E_0)$. If the present trend of the data continues up to higher energies, the plot of $g(E')$ as E' varies between E_0 and E would look somewhat like the curve in Fig. 2. By crossing symmetry the curve as we go from $-E_0$ to $-E$ will lie symmetrically on the other side of the $\text{Im}g$ axis. Let v_E be the farthest intersection of the curve $g(Ee^{i\varphi})$ with the positive imaginary axis as φ varies between 0 and π. Similarly, let v_0 be the nearest intersection of the curve $g(E_0{}^{i\varphi})$ with the positive imaginary axis.

Following the arguments given in the Appendix, the conditions (i)–(iv) on $T^{(1)}(E)$ lead us to the inequality (A6), or the somewhat more accurate formula (See Ref. 10, p. 84)

$$\int_{v_E}^{v_0}\frac{dv}{\rho(v)}\geq-\frac{1}{b}\ln\{[1-(4/\pi)\tan^{-1}(\sqrt{k})]/a\}$$

$$\approx\frac{1}{2b}\ln(E/E_0)-\frac{1}{b}\ln(8/\pi a), \tag{41}$$

$$a=(2/\pi)\tan^{-1}(1/\pi),\quad b=(1+e)/2=1.859,$$

where we integrate along the imaginary axis in the g plane and $\rho(v)$ is the shortest distance from the point v on the imaginary axis to the curve representing the data.

Of course v_E and v_0 are not directly given by any measurement. Thus, before we can use (41), we have to make an estimate of both v_E and v_0. Since the usefulness of (41) for our purpose depends on this information, we are at present looking for a method for estimating v_E and v_0. It might turn out, however, that we need only a rough estimate of v_E and v_0. For the moment we only note that $v_E\geq0$ for $|E|$ larger than some constant. If E/E_0 is too small, the right-hand side of (41) is negative because of the second term, and thus the inequality is trivially satisfied. The right-hand side of (41) becomes positive when the ratio E/E_0 passes about 170. Thus for our purpose we need the energy range in which E/E_0 is substantially greater than 170.

As we make E/E_0 larger, the right-hand side of (41) grows logarithmically. Thus, unless ρ decreases [e.g., $\text{Re}g(E)$ decreases], we would eventually reach an energy where the inequality (41) is violated. In this manner, it will therefore be possible to say whether a certain set of data for a finite energy range is consistent with analyticity and crossing symmetry.

We should choose as E_0 the smallest energy beyond which $\text{Re}T^{(1)}(E)$ is of definite sign (repulsive according to recent experiments[8,9]). Since E_0 will be around 1 BeV, we will then need E of several hundred BeV. For a crucial test of analyticity and crossing by means of the formula (41), we may therefore have to wait until an accelerator of 300 BeV or more becomes available. Meanwhile, we might try to improve (41) further because it is not the best possible inequality. A more detailed discussion of the problem of this section will be published separately.

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APPENDIX

For the convenience of the reader we give in this Appendix a few definitions and theorems on harmonic measures which are necessary to understand the origin of Meiman's theorems.[6] Most of these results are found scattered in Ref. 10. We also state and prove the two theorems of Meiman used in this paper. There are no new results in this Appendix.

Let D be a domain in the z plane whose boundary Γ consists of a finite number of Jordan curves. Let Γ be divided into two subsets α and β. Then the harmonic measure $\omega(z,\alpha,D)$ of α with respect to D at the point z

in D is defined uniquely by the conditions:

(i) $\omega(z,\alpha,D)$ is harmonic and bounded for all z in D,
(ii) $\omega=1$ on α and $\omega=0$ on β.

For a discussion of properties of harmonic measure see Ref. 10. We note here in particular that it is invariant under the one-to-one conformal mapping $z\to z'$, $\alpha\to\alpha'$, $D\to D'$:[17]

$$\omega(z,\alpha,D)=\omega(z',\alpha',D'). \tag{A1}$$

If the mapping is not one-to-one, (A1) is no longer valid. However the harmonic measure still satisfies a useful inequality: Let $w(z)$ be regular in the domain D bounded by Jordan curves Γ_z. Assume that

(i) the value $w=w(z)$, z in D, falls in a domain G whose boundary Γ_w consists of a finite number of Jordan curves (note that Γ_w is not necessarily the map of Γ_z),

(ii) at each point of a given subset α_z of Γ_z, $w(z)$ is continuous and takes a value which lies in a subset A, bounded by a finite number of Jordan curves α_w, of G. Let G^* be a domain obtained from G by removing A. Obviously G^* is bounded by some parts of α_w and Γ_w. Then, at each z in D such that $w(z)$ is in G^*, the harmonic measure satisfies the inequality[18]

$$\omega(z,\alpha_z,D)\leq\omega(w(z),\alpha_w,G^*). \tag{A2}$$

In the following we shall specialize D to a rectangular domain bounded by the straight-line segments $\mathrm{Re}z=x_1$, $y_1\leq\mathrm{Im}z\leq y_2$), ($\mathrm{Re}z=x_2$, $y_1\leq\mathrm{Im}z\leq y_2$), ($\mathrm{Im}z=y_1$, $x_1\leq\mathrm{Re}z\leq x_2$), and ($\mathrm{Im}z=y_2$, $x_1\leq\mathrm{Re}z\leq x_2$).[19] The last two lines, parallel to the real axis, will be called L_1 and L_2. The function $w(z)$ is assumed to be regular in D and continuous on the boundary. We further assume that the values of $w(z)$ on L_1 and L_2 fall in some given subsets A_1 and A_2 of G, respectively, where A_1 and A_2 are a positive distance apart. We let α_1 and α_2 be the sets of Jordan curves which form the boundaries of A_1 and A_2, respectively. The boundary Γ_w of G will contain some curves belonging to $\alpha=\alpha_1+\alpha_2$ as well as a complementary part not belonging to α which we denote by β. We denote by α' the part of α_1 which is not in Γ_w and α'' the part of α_2 not in Γ_w. We also let β' and β'' be the two parts of β joining α_1 to α_2, $\beta=\beta'+\beta''$. Thus the connected domain G^*, which is obtained from G by removing A_1 and A_2, is bounded by the curves $\alpha'+\beta''+\alpha''+\beta'$ (see Fig. 3).

Let us consider a harmonic measure $\omega(z,\alpha'+\alpha'',G^*)$ and denote by $m(\lambda)$ its minimum on a curve λ which connects α' and α'' inside G^*. We define further

$$m_w=\limsup m(\lambda),$$

where lim sup is taken with respect to all possible paths λ. Similarly, let $m(x)$ be the minimum of the harmonic measure $\omega(z,L_1+L_2,D)$ on the line $x(x_1\leq x\leq x_2)$, and

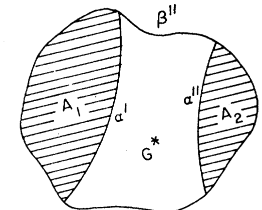

FIG. 3. The domain G^* in the w plane.

let

$$m_z=\lim_{x_1\leqslant x\leqslant x_2}\sup m(x).$$

Then we can derive from (A2) the inequality[20]

$$m_z\leq m_w. \tag{A3}$$

It is obvious from the symmetry that m_z is equal to the value of the harmonic measure $\omega(z,L_1+L_2,D)$ at the center $(x_1+x_2)/2+i(y_1+y_2)/2$ of the rectangle. This can be evaluated easily by mapping the rectangle on the upper half-plane by means of an elliptic function.[21] The result is

$$m_z=(4/\pi)\tan^{-1}(\sqrt{k}), \tag{A4}$$

where k is the modulus of the complete elliptic integral of the first kind $K(k)$, $0<k<1$. It is determined by the relation $2K/K'=(x_2-x_1)/(y_2-y_1)$ where K' is the associated complete elliptic integral. On the other hand, m_w is found to have an upper bound[22]

$$m_w\leq1-\frac{1}{2\pi}\exp\left(-2\int_{s_1}^{s_2}\frac{ds}{\rho(s)}\right), \tag{A5}$$

where the integration is carried out along a curve l which connects the boundary curves β' and β'' inside G^*. Here, s is the parameter describing the length of this curve l, taking the value s_1 on β' and s_2 on β'', and $\rho(s)$ is the shortest distance from the point w_s on l, corresponding to the value s of the parameter, to the boundary curve $\alpha'+\alpha''$.

From (A3), (A4), and (A5) one obtains easily the inequality

$$\int_{s_1}^{s_2}\frac{ds}{\rho(s)}\geq-\tfrac{1}{2}\ln[2\pi-8\tan^{-1}(\sqrt{k})]. \tag{A6}$$

Meiman's theorems are derived by applying this inequality to a function related in a certain way to the forward-scattering amplitude. Before showing how this is done, let us recall some specific features of mapping by such a function.

We are interested in a function $g(E)$ which is regular and bounded by a polynomial of E in the region $\mathrm{Im}E>0$ of the complex E plane. Let us assume for simplicity

[17] See Ref. 10, p. 38.
[18] See Ref. 10, p. 39.
[19] See Ref. 10, p. 79–86 for a treatment of a much more general case.
[20] See Ref. 10, p. 65.
[21] H. Kober, *Dictionary of Conformal Representations* (Dover Publications, Inc., New York, 1957), 2nd ed., p. 172.
[22] See Ref. 10, p. 85.

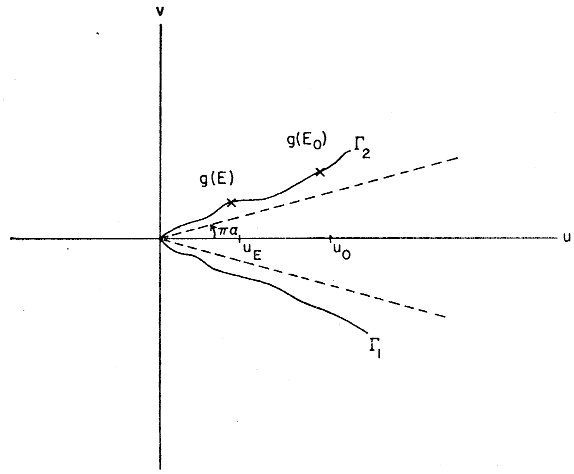

FIG. 4. The g plane for large $|E|$.

that it has the symmetry property

$$g(-E+i0)=g^*(E+i0) \qquad (A7)$$

on the real axis and that

$$\lim_{E\to+\infty} g(E+i0)=0. \qquad (A8)$$

Then the function $g(E)$ maps the upper half E plane into a certain domain of g plane. In particular the upper edges of the semi real axis $(0,-\infty)$ and $(0,+\infty)$ are mapped onto the curves Γ_1 and Γ_2, symmetrically located with respect to the real axis of g plane. In this mapping a sufficiently distant upper-half neighborhood of the point $E=\infty$, which is the region we are primarily interested in, is mapped into a certain neighborhood (perhaps many sheeted) of the point $g=0$. Let us assume that for $|E|>E_1$, E_1 being a large positive constant, Γ_1 and Γ_2 have no common point except the endpoint $g=0$.

To apply the formula (A6) to our problem it is convenient to consider the function $g(E)$ not in the E plane but rather in the z plane, where $z=\ln E$. Accordingly, we define the function $w(z)$ by $w(z)=g(e^z)$ and consider the mapping $z\to w$. As the domain D we choose the rectangle enclosed by the straight lines $\mathrm{Re}z=x_0$, $\mathrm{Re}z=x$, $\mathrm{Im}z=0$, and $\mathrm{Im}z=\pi$, where $x>x_0\geq\ln E_1$. Obviously this is a map by $z=\ln E$ of a region of upper half-E plane bounded by two semicircles of radius $E_0=e^{x_0}$ and $E=e^x$. Let us denote by G the map of D by $w(z)$ and denote by G^* the subset of G bounded in part by Γ_1 and Γ_2. For definiteness we assume that G^* is bisected by the positive real axis of the w plane. This can always be achieve by a proper choice of the sign of $g(E)$.

Let $w(x_0)$ and $w(x)$ be two points on Γ_2 and let $\Gamma(x,x_0)$ be the portion of Γ_2 connecting these points. Let $\rho(u)$ be the shortest distance from the real point $w=(u,0)$ to $\Gamma(x,x_0)$. Let u_0 be the nearest point of intersection of the map of $\mathrm{Re}z=x_0$ (hence the map of semicircle of radius $E_0=e^{x_0}$ of the E plane) with the positive semi axis $u\geq 0$ $(w=u+iv)$, and let u_E be the farthest point of intersection of the map of $\mathrm{Re}z=x$ with the same semi axis. We can now apply the inequality (A6) to this problem. As is easily seen, when $E\gg E_0$, it can be reduced to the form

$$\int_{u_E}^{u_0}\frac{du}{\rho(u)}\geq\tfrac{1}{4}\ln(E/E_0). \qquad (A9)$$

This is the basic inequality of Meiman. Applying it to special cases he obtained the theorems discussed below.

The function $g(E)$ we are considering is regular and bounded by a polynomial of $|E|$ in the region $\mathrm{Im}E>0$. It is continuous on the real axis and it satisfies (A7) and (A8). We further assume that $g(E)$ has no zeros in the finite upper half E plane when $|E|$ is greater than some constant. For such a function $g(E)$ one has the following two theorems.[23]

Theorem I: If $g(E)$, in addition to the properties above, satisfies for sufficiently large real E the inequality,

$$|\mathrm{Im}g(E)/\mathrm{Re}g(E)|\geq\tan\pi\alpha,\quad 0<\alpha\leq\tfrac{1}{2}, \qquad (A10)$$

then, starting with some E_0,

$$|g(E)/g(E_0)|\leq C(E_0/E)^{\alpha/2}. \qquad (A11)$$

Proof: Although this theorem is proved in Ref. 10 we shall repeat the proof here for the convenience of the reader. In this case the straight lines through the origin of slope $\tan\pi\alpha$ and $-\tan\pi\alpha$ can be considered as part of the boundary of the region G^*, corresponding to α' and α'' above (see Fig. 4).

It suffices to consider the case $\alpha=\frac{1}{2}$ since other cases can be reduced to it by considering the function $g'=g^{1/2\alpha}$. Now, for $\alpha=\frac{1}{2}$, we have $\rho(u)\geq(u^2+|g(E)|^2)^{1/2}$, $u_E\leq u\leq u_0$, as is obvious from Fig. 4 when the angles are opened up to 90°. Thus we have

$$\int_{u_E}^{u_0}\frac{du}{\rho(u)}\leq\int_0^{u_0}\frac{du}{(u^2+|g(E)|^2)^{1/2}} =\ln\left[\frac{u_0+(u_0^2+|g(E)|^2)^{1/2}}{|g(E)|}\right]. \qquad (A12)$$

Combining this with (A9) one obtains

$$\frac{|g(E)|}{u_0+(u_0^2+|g(E)|^2)^{1/2}}\leq C(E_0/E)^{1/4}, \qquad (A13)$$

and hence

$$|g(E)|<C'(E_0/E)^{1/4}. \qquad (A14)$$

[23] *Note added in proof.* From a strictly mathematical point of view, the proof of Theorems I and II given below requires a further restriction on $g(E)$. The proof does not hold when $|g(E)|$ as $E\to\infty$ is allowed to oscillate an infinite number of times with an amplitude whose ratio to the minimum magnitude in a certain interval becomes arbitrarily large as $E\to\infty$. We limit ourselves in this paper to functions $g(E)$ that do not have such a physically pathological behavior. We would like to thank Professor W. Fuchs for critical remarks on this point.

This completes Meiman's proof except for the following special case. Even though we always choose $E \gg E_0$ and E_0 large but finite, it is still possible that for some real E', $E_0 < E' < E$, $|g(E')| < |g(E)|$. In that case the inequality for ρ reads $\rho(u) \geq (u^2 + |g(E')|^2)^{1/2}$. One then gets, instead of (A14), the inequality

$$|g(E') \leq C(E_0/E)^{1/4}.$$

However, $E' < E$, and we have finally

$$|g(E')| \leq C(E_0/E')^{1/4}.$$

Theorem II: If α vanishes but $g(E)$ satisfies the relation

$$|\operatorname{Im} g(E)| \geq C |\operatorname{Re} g(E)|^{\nu}, \quad C > 0, \quad \nu > 1, \tag{A15}$$

for large real E, then, starting with some E_0,

$$|\operatorname{Im} g(E)/\operatorname{Im} g(E_0)| \leq (1 + \tfrac{1}{4} C'(\nu - 1) \ln(E/E_0))^{-\nu/(\nu-1)}, \tag{A16}$$

where C' is a constant.

Proof: Put $g = u + iv$. According to the assumption (A15), the two curves defined by

$$v = \pm a (u/b)^{\nu}, \tag{A17}$$

where a, b are properly chosen positive constants, can be regarded as the boundary curves of the domain G^*. Let us take a point (u', v') on the upper curve and draw a normal to it at this point. Let $(u_1, 0)$ be the intersection of this normal with the u axis. Then it is easily seen that

$$\rho(u_1) = [(u' - u_1)^2 + v'^2]^{1/2} > v', \tag{A18}$$

with

$$\begin{aligned} u' &= b(v'/a)^{1/\nu}, \\ u_1 &= u' + \nu(a^2/b)(v'/a)^{(2\nu-1)/\nu}. \end{aligned} \tag{A19}$$

Since $\nu > 1$, we obtain

$$u_1 \simeq b(v'/a)^{1/\nu}, \quad du_1/dv' \simeq (1/\nu)(b/a)(v'/a)^{(1-\nu)/\nu} \tag{A20}$$

in a sufficiently small neighborhood of $g = 0$.

Making use of (A18) and (A19), we find that

$$\begin{aligned} \int_{u_E}^{u_0} \frac{du_1}{\rho(u_1)} &< \int_{u_E}^{u_0} \frac{du_1}{v'} \simeq \frac{1}{\nu} \frac{b}{a} \int_{\operatorname{Im} g(E)/a}^{\operatorname{Im} g(E_0)/a} x^{(1-2\nu)/\nu} dx \\ &= \frac{1}{\nu - 1} \frac{b}{a} \{ [a/\operatorname{Im} g(E)]^{(\nu-1)/\nu} - [a/\operatorname{Im} g(E_0)]^{(\nu-1)/\nu} \}. \end{aligned} \tag{A21}$$

From (A9) and (A21) we obtain the inequality

$$|\operatorname{Im} g(E)/\operatorname{Im} g(E_0)| < \left[1 + \frac{a}{b} [\operatorname{Im} g(E_0)/a]^{(\nu-1)/\nu} \left(\frac{\nu - 1}{4} \right) \ln\left(\frac{E}{E_0} \right) \right]^{-\nu/(\nu-1)} \tag{A22}$$

which reduces to (A16) for an appropriate choice of C', Q.E.D.