high-energy large momentum transfer processes, *intrinsic information pertaining to very small distance interactions.*

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APPENDIX

1. Consider the process

$$A+B\to C+D.$$

Let the matrix element for the process with a total isotopic spin I be denoted by a_I. The statistical hypothesis means that

$$\langle a_I a_{I'}^*\rangle=\delta_{II'}a\,, \tag{8}$$

where the average $\langle\cdots\rangle$ is defined in footnote 8. This is essentially the assumption made by Fermi.[9] Using (8), it is clear that the differential cross section on the average is proportional to

$$\textstyle\sum_I|\langle I_3(A),I_3(B)|I\rangle\langle I|I_3(C),I_3(D)\rangle|^2, \tag{9}$$

where $I_3(A)$ etc., are the I_3 component of the isotopic spin of A etc., I is the total isotopic spin, and the $\langle|\rangle$ symbols are the appropriate Clebsch-Gordan coefficients. Application of (9) to pp and πp large-angle scattering yields

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,pp\to pp)&=2\frac{d\sigma}{d\Omega}(\theta,pn\to pn)\\&=2\frac{d\sigma}{d\Omega}(\pi-\theta,pn\to pn)\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\pi^+p\to\pi^+p)&=\frac{9}{5}\frac{d\sigma}{d\Omega}(\theta,\pi^+n\to\pi^+n)\\&=\frac{9}{4}\frac{d\sigma}{d\Omega}(\theta,\pi^+n\to\pi^0p)\,,\\&=\frac{9}{5}\frac{d\sigma}{d\Omega}(\theta,\pi^0p\to\pi^0p)\,.\end{aligned} \tag{11}$$

2. One could discuss the spin dependence in a similar way. For example, consider large-angle $p+n\to p+n$. Denote the spin components of a particle in a direction perpendicular to the scattering plane by u (for up) and d (for down). There are the following possibilities[10] of spin arrangements:

$$uu\to uu\,,\quad uu\to dd\,,\quad ud\to ud\,,\quad ud\to du\,,$$
$$du\to ud\,,\quad du\to du\,,\quad dd\to uu\,,\quad dd\to dd\,.$$

The statistical hypothesis requires that they all have on the average[8] the same amplitude, and random phase differences.

[10] A. Bohr, Nucl. Phys. 10, 486 (1959).

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Uncoupled-Phase Method in the Multichannel N/D Formalism*

PRAN NATH† AND G. L. SHAW

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

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The uncoupled-phase method is a nonperturbative formalism, developed by Ross and Shaw, relating the scattering amplitudes describing n strongly coupled two-body channels to the "uncoupled" amplitudes describing n-1 channels alone. The "uncoupled" scattering amplitudes are defined to be those that would exist if the couplings to the nth channel were switched off while the interactions among the n-1 channels remain unchanged. The uncoupled-phase method, previously based on the potential model, is extended to the relativistic problem by considering a set of n coupled N/D partial-wave dispersion relations. For the situation in which the left-hand cut is approximated by the form $g/(s+m)$ where g is an $n\times n$ matrix of constants and s is square of the total energy in the center-of-mass system, the uncoupled-phase method is exact. The quantitative validity of the uncoupled-phase method for more complicated left-hand singularities is tested by performing a two-channel computer experiment. A full numerical solution of the coupled integral equations for the N functions is obtained by the matrix-inversion technique. We consider the situations in which (a) the left-hand cut is replaced by a set of dipoles and (b) the left-hand cut is assumed to be given by exchange of a scalar particle in the corresponding "crossed" t channel of any given reaction. The coupled-phase method is found to be quantiatively accurate under a wide range of conditions. The range parameter of the coupled-phase method is directly given by a principal-value integral, and an estimate of it can be made *a priori.*

I. INTRODUCTION

THE uncoupled-phase method was developed[1] to confront certain theories of strong interactions with experiment. Consider a physical situation where the scattering is described by a set of n strongly coupled two-body channels. The "uncoupled" scattering ampli-

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† Present address: University of California, Department of Physics, Riverside, California.

[1] M. Ross and G. Shaw, Ann. Phys. (N. Y.) 9, 391 (1960); G. Shaw and M. Ross, Phys. Rev. 126, 806 (1962).

tudes are defined to be those that would exist if the couplings to the nth channel were to vanish while the interactions among the n-1 channels remain unchanged. The uncoupled-phase method is a nonperturbative formalism that relates the uncoupled amplitudes (for which one may propose theoretical models) to the actual amplitude describing all n channels. For example, a theory may predict uncoupled πY amplitudes $\tilde{\phi}_{\pi Y}$ which describe an idealized πY system in the absence of coupling to the $\bar{K}N$ channel.[2] The uncoupled-phase method can be used to calculate the actual πY amplitudes $\phi_{\pi Y}$ from the $\tilde{\phi}_{\pi Y}$, the complex-scattering length in $\bar{K}N$ channel, the production ratios from the $\bar{K}N$ channel to the πY channels plus the range of forces in the $\bar{K}N$ channel and test it against experiment.[3]

The uncoupled-phase method was based on the model of an $n\times n$ potential matrix coupling the n channels. The interactions were assumed to have a short well-defined range and no hard core.[1] In a previous paper,[4] we found the uncoupled-phase method to be a quantitative procedure over a much wider range of conditions than originally anticipated: we derived the uncoupled-phase method for interactions with hard cores. By performing a two-channel computer experiment, the method was seen to be quantitatively accurate for Yukawa interactions with hard cores, for p- as well as s-wave angular momenta, and when one of the channels was closed as well as when both were open. The uncoupled-phase method was compared with other methods which include the neglected channel as a perturbation and was seen to be superior to these approximations. A review of the general features of the uncoupled phase method is presented in Sec. II.

In this paper, we extend the uncoupled-phase method based previously on the potential model to the relativistic problem in which the scattering in a given partial wave is calculated using an n-channel N/D formalism.[5] We show in Sec. III that for the simple case when the left-hand cut is approximated by the form $g/(s+m)$ where g is an $n\times n$ matrix of constants and s is the square of the total energy in the center-of-mass system, the uncoupled-phase method is exact. The resulting uncoupled-phase relationships are completely analogous to those obtained from the potential model. To test the uncoupled-phase method for more complicated left-hand singularities, we perform a two-channel computer experiment. The calculation, described in Sec. IV, involves a full numerical solution of the coupled integral equations for the N function by the matrix inversion technique. We consider two different situations for the "generalized potential": a set of dipoles and the potential produced by exchange of a scalar particle in the "crossed" t reactions of the scattering processes. The uncoupled phase method is found to be quantitatively accurate over a wide variety of conditions. This is true when one of the channels is closed as well as when both are open. The range parameter L_n of the uncoupled-phase method is given by a principal value integral and hence can be determined *a priori*. It depends on the diagonal singularities in the nth channel alone.

[2] M. Nauenberg, Phys. Rev. Letters **2**, 351 (1959); J. Franklin, Proceedings of Midwest Conference on Theoretical Physics, 1962, p. 82 (unpublished).
[3] M. Ross and G. Shaw, Phys. Rev. **115**, 1772 (1959); Bull. Am. Phys. Soc. **5**, 504 (1960); G. Shaw and M. Ross, Phys. Rev. **126**, 814 (1962).
[4] P. Nath, G. Shaw and C. Iddings, Phys. Rev. **133**, B1085 (1964).
[5] Preliminary results were reported in P. Nath and G. Shaw, Bull. Am. Phys. Soc. **8**, 626 (1963).

II. GENERAL FEATURES OF THE UNCOUPLED-PHASE METHOD

Before we go on to derive the uncoupled-phase method in the N/D formalism, we would like to establish the general form the reaction matrix K for a multichannel problem must have for the uncoupled phase relations to follow. From our discussion of the uncoupled-phase method in the potential model,[4] we expect K to satisfy the $n\times n$ matrix equation for a given partial wave:

$$K(s)=K^0+B(s)+B(s)L^{-1}K(s)\,, \tag{2.1}$$

where B and the diagonal matrices K^0 and L are such that if we switch off the couplings to the nth channel, i.e., set $B_{in}=0$, then B_{ij}, K_{ij}^0 and L_{ii} for $i,\ j\neq n$ remain unchanged. The parameters $L_{ii}\equiv L_i$ are closely related to the range of forces in the ith channel and thus can be estimated *a priori*.

When the couplings to nth channel are switched off the uncoupled $\tilde{K}$ matrix elements $\tilde{K}_{ij}$ are given by

$$\tilde{K}_{ij}=(K_{ij}^0+B_{ij}+\sum_{k=1}^{n-1}B_{ik}L_k^{-1}\tilde{K}_{kj})(1-\delta_{in})(1-\delta_{jn})\,. \tag{2.2}$$

From (2.1) and (2.2) we get the set of equations

$$\textstyle\sum_{k=1}{}^{n}\beta_{ik}t_{kj}=0\,, \tag{2.3}$$

where

$$\beta_{ik}=\delta_{ik}-B_{ik}/L_k\,, \tag{2.4}$$

and

$$t_{kj}=K_{kj}-\tilde{K}_{kj}(1-\delta_{jn})(1-\delta_{kn})+L_n\delta_{kn}\delta_{jn}\,. \tag{2.5}$$

In order that a solution to (2.3) may exist, all 2×2 submatrices of the matrix t must have zero determinant. We have $(n^2-n)/2$ independent relations of this kind equal to the number of the uncoupled-phase amplitudes $\tilde{K}_{ij}$. We have $(n-1)$ relations

$$\det\begin{pmatrix}K_{ii}-\tilde{K}_{ii} & K_{in}\\ K_{in} & K_{nn}+L_n\end{pmatrix}=0\,,\quad\text{for}\quad i\neq n\,, \tag{2.6a}$$

and $(n-1)(n-2)/2$ relations

$$\det\begin{pmatrix}K_{ii}-\tilde{K}_{ii} & K_{ij}-\tilde{K}_{ij}\\ K_{ij}-\tilde{K}_{ij} & K_{jj}-\tilde{K}_{jj}\end{pmatrix}=0\,,\ \text{for}\ i\neq n,\ j\neq n. \tag{2.6b}$$

For a two-channel problem the uncoupled-phase relationships have the form

$$(K_{11}-\tilde{K}_{11})(K_{22}+L_2)=K_{12}^2. \tag{2.7}$$

The two-channel uncoupled-phase relationship (2.7) also has strikingly similar forms in terms of the scattering matrices M and T which are defined in terms of K by[6]

$$M=K^{-1} \tag{2.8}$$

and

$$T=k^{l+1/2}(M-ik^{2l+1})^{-1}k^{l+1/2}, \tag{2.9}$$

where $(k^{l+1/2})_{ij}=\delta_{ij}k_i^{l_i+1/2}$. The diagonal elements of T have the familiar form $(e^{2i\delta_i}-1)/2i$. The uncoupled phase relationship (2.7) expressed in terms of the matrix M is

$$(M_{11}-\tilde{M}_{11})(M_{22}+L_2^{-1})=M_{12}^2, \tag{2.10}$$

and in terms of the matrix T is

$$(T_{11}-\tilde{T}_{11})(T_{22}+L_0)=T_{12}^2, \tag{2.11}$$

where

$$L_0=((L_2k_2^{2l_2+1})^{-1}+i)^{-1}. \tag{2.12}$$

There are alternative forms of (2.10) and (2.11) which may be more useful in some situations. We define a complex scattering length in channel 2, $a(k)$.

$$k_2^{2l_2+1}\cot\delta_2\equiv-1/a. \tag{2.13}$$

Here δ_2 is the complex phase shift in channel 2. Eq. (2.10) may then be written

$$M_{11}=\frac{\tilde{M}_{11}-k_1^{2l_1+1}\mathrm{Im}a^{-1}[\mathrm{Re}a^{-1}-L_2^{-1}]^{-1}}{-k_1^{-(2l_1+1)}\mathrm{Im}a^{-1}[\mathrm{Re}a^{-1}-L_2^{-1}]^{-1}\tilde{M}_{11}+1}, \tag{2.14}$$

while (2.11) has the form

$$T_{11}=\tilde{T}_{11}+\frac{\mathrm{Im}a^{-1}k_2^{-(2l_2+1)}[a^{-1}k_2^{-(2l_2+1)}+i]^{-1}e^{2i\alpha}}{L_0(i+a^{-1}k_2^{-(2l_2+1)})-1}, \tag{2.15}$$

where

$$\alpha=\tan^{-1}(M_{11}^{-1}k_1^{2l_1+1}).$$

Using the above relationships, one is able to estimate all the scattering amplitudes at a given energy from an experimentally determined scattering length a, and theoretical estimates for the range parameter L_2 [see e.g., Eq. (3.18) of Sec. III and Eq. (2.22b) of Ref. 4] and the uncoupled amplitude in channel 1. See Ref. 3 for similar relationships for a three-channel system.

III. EXTENSION OF UNCOUPLED-PHASE METHOD TO N/D FORMALISM

We consider the usual N/D equations for the case where there are n coupled two-body channels and define the invariant partial-wave amplitude A in terms of the S matrix by

$$A\equiv ND^{-1}\equiv\rho^{-1/2}((S-1)/2i)\rho^{-1/2}, \tag{3.1}$$

where the numerator function N has only left-hand cuts and the denominator function D has only right-hand cuts. The diagonal matrix ρ depends on kinematics:

$$\rho_{ij}=\delta_{ij}k_i^{2l_i+1}/\sqrt{s}. \tag{3.2}$$

The N and D equations, in terms of the "generalized potential" $B(s)$[7] which is regular in the physical region, are[8]

$$N(s)=B(s)+\frac{1}{\pi}\int_0^\infty\left[B(s)-\frac{s-s_0}{s-s_0}B(s)\right]\times\theta\rho(s')N(s')\frac{ds'}{s'-s}, \tag{3.3}$$

$$D(s)=1-(s-s_0)\frac{P}{\pi}\int_0^\infty\theta\rho(s')N(s')\times\frac{ds'}{(s'-s)(s'-s_0)}-i\theta\rho(s)N(s), \tag{3.4}$$

where the θ ensures that the right-hand cuts in A^{-1} start at the appropriate thresholds. We note that the solutions A are independent of s_0.[9] Moreover, it is well known that for symmetric input $B(s)$, $A(s)$ is symmetric as required by time reversal invariance.[10] Note that the integrand in (3.3) is well behaved at $s'=s$ so that we do not have a principal value integral in (3.3). The integral equations (3.3) are a set of coupled inhomogeneous Fredholm equations of the second kind,[11] if the kernel $\mathcal{K}(s,s')$ where

$$\mathcal{K}(s,s')=\frac{1}{\pi}\left[B(s')-\frac{s-s_0}{s'-s_0}B(s)\right]\frac{\theta\rho(s')}{(s'-s)} \tag{3.5}$$

is an L^2 kernel.[12] Moreover, if the kernel is degenerate,[13] the integral equations (3.3) may be reduced to a system of linear algebraic equations. If, however, the kernel is nondegenerate the integral equations (3.3) may only be solved numerically.

We shall now show that for the situation in which the left-hand cut may be approximated by the simple form $g/(s+m)$, where g is an $n\times n$ matrix of constants, the uncoupled-phase method is exact. Let

$$B(s)=g/(s+m). \tag{3.6}$$

Now choose the subtraction point $s_0=-m$. Then

[6] We use units $\hbar=c=m_\pi=1$.

[7] G. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).

[8] J. Uretsky, Phys. Rev. **123**, 1459 (1961); D. Y. Wong, Phys. Rev. **126**, 1220 (1962).

[9] A. W. Martin, Phys. Rev. **135**, B967 (1964).

[10] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

[11] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part I, Chap. 8.

[12] $\mathcal{K}(s,s')$ is an L^2 kernel if $\int_0^\infty ds\int_0^\infty ds'|\mathcal{K}(s,s')|^2<\infty$. See F. Smithies, *Integral Equations* (Cambridge University Press, New York, 1958).

[13] $\mathcal{K}(s,s')$ is called degenerate if it can be written as $\mathcal{K}(s,s')=\sum_{i=1}^n g_i(s)h_i(s')$ where n is finite.

$\mathcal{K}(s,s')=0$ so

$$N(s)=g/(s+m)=B(s) \tag{3.7}$$

and

$$A(s)=\left(B^{-1}(s)-\frac{(s+m)^2}{\pi}\times P\int_0^\infty \frac{\theta\rho(s')ds'}{(s'+m)^2(s'-s)}-i\theta\rho(s)\right)^{-1}. \tag{3.8}$$

Thus

$$K(s)=(B(s)/\sqrt{s})+(B(s)/\sqrt{s})L^{-1}K(s) \tag{3.9}$$

where

$$L=\left(\frac{(\sqrt{s})(s+m)^2}{\pi}P\int_0^\infty \frac{\theta\rho(s')ds'}{(s'+m)^2(s'-s)}\right)^{-1}. \tag{3.10}$$

Equation (3.9) satisfies the conditions of (2.1) so that the uncoupled phase relations (2.6) follow immediately. The parameter L_n of the uncoupled-phase method is given directly in terms of a principal value integral from (3.10).

In the simple situation for the left-hand cut considered above, the uncoupled-phase method is exact. In general, we would need some approximations (as in the case of the potential model) to derive the uncoupled-phase relations. Let us consider, for example, a somewhat more general form for $B(s)$. Let

$$B(s)=g/f(s) \tag{3.11}$$

where again g is an $n\times n$ matrix of coefficients. We now use the Fulton[14]-Shaw[15] approximation to solve for the scattering matrix A. This approximation has the exact same degree of simplicity as Baker's determinantal method but has the added advantages that (a) the scattering matrix A does not depend on the subtraction point s_0 and (b) the scattering matrix is symmetric. We define

$$N(s)=B(s)C(s). \tag{3.12}$$

We use (3.12) in (3.3) and (3.4) and replace $C(s')$ by $C(s)$ in all integrals over $N(s)$. Therefore,

$$A(s)=B(s)\left(C^{-1}(s)-\frac{s-s_0}{\pi}\times P\int_0^\infty \frac{\theta\rho(s')B(s')ds'}{(s'-s)(s'-s_0)}-i\theta\rho(s)B(s)\right)^{-1} \tag{3.13}$$

and

$$C(s)^{-1}=1-B^{-1}(s)\frac{P}{\pi}\int_0^\infty B(s')\theta\rho(s')B(s')\frac{ds'}{s'-s}+\frac{s-s_0}{\pi}P\int_0^\infty \frac{\theta\rho(s')B(s')ds'}{(s'-s)(s'-s_0)}. \tag{3.14}$$

Substituting (3.14) in (3.13), we get

$$A(s)=\left(B^{-1}(s)-\frac{P}{\pi}\int_0^\infty \frac{f^2(s)\theta\rho(s')ds'}{f^2(s')(s'-s)}-i\theta\rho(s)\right)^{-1}. \tag{3.15}$$

Using (3.1), (2.9), and (2.8), we can calculate $K(s)$. We have

$$K(s)=B(s)/\sqrt{s}+(B(s)/\sqrt{s})L^{-1}K(s), \tag{3.16}$$

where

$$L=\left(\frac{\sqrt{s}}{\pi}P\int_0^\infty \frac{f^2(s)\theta\rho(s')}{f^2(s')(s'-s)}\right)^{-1}. \tag{3.17}$$

The uncoupled-phase relations (2.6) follow immediately.

In a general situation where one has more complicated left-hand singularities, the uncoupled-phase method can be derived under more restrictive approximations. We would not go further in this direction but would only summarize the essence of such calculation.[16] For a general left-hand cut, Eq. (3.16) can be derived under the restrictive approximations. However, the parameter L_n of the uncoupled phase method is now given by

$$L_n=\left(\frac{\sqrt{s}}{\pi}P\int_{s_n}^\infty \frac{B_{nn}^2(s')\rho_n(s')ds'}{B_{nn}^2(s)(s'-s)}\right)^{-1}, \tag{3.18}$$

where s_n is the threshold for the nth channel. We note that the parameter L_n depends only on the diagonal interaction in the nth channel. The estimate L_n given by (3.18) is expected to be particularly good at low energies. However, due to the nature of the approximations made in obtaining (3.18), the estimate is not expected to be good in regions of energy several orders of magnitude higher than the threshold energies. Note that in the potential model we estimated L_n directly from the range of interaction in the nth channel [see Eq. (2.22b) of Ref. 4]. In the above relativistic model, Eq. (3.18) allows us to make a somewhat more quantitative estimate of L_n.

IV. TWO-CHANNEL COMPUTER EXPERIMENT

We have shown in Sec. III that the uncoupled-phase method can be derived in the multichannel N/D formalism and the resulting uncoupled-phase relationships are exactly analogous to those obtained in the case of the potential model. Furthermore, an estimate of the parameter L_n of the uncoupled-phase method can be made *a priori* in terms of a principal value integral which depends only on the diagonal interaction in the nth channel. In this section we shall demonstrate that these relations are quantitatively valid for a variety of conditions by solving numerically a set of two coupled ND^{-1} integral equations. The present calculations consider p-wave scattering, and two different situations for the left-hand cut are investigated. In the first situation the left-hand cut is replaced by a set of dipoles,

[14] T. Fulton, *Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. 1, p. 55.

[15] G. Shaw, Phys. Rev. Letter **12**, 345 (1964).

[16] P. Nath, Ph.D. thesis, Stanford University, 1964 (unpublished).

while in the second situation the left-hand cut is assumed to be produced by the exchange of a scalar particle in the "crossed" t reaction of the corresponding channels. The two-channel uncoupled phase relation tested in each case is

$$(K_{11}-\tilde{K}_{11})(K_{22}-L_2)=K_{12}^2\,, \tag{4.1}$$

where $\tilde{K}_{11}$ describes the scattering in channel 1 when we set $B_{12}=0$ (with B_{11} remaining unchanged). The theoretical estimate of L_2 is given by

$$L_2=\left(\frac{\sqrt{s}}{\pi}P\int_{s_2}^{\infty}\frac{B_{22}^2(s')k_2^3(s')ds'}{B_{22}^2(s)(s'-s)\sqrt{s'}}\right)^{-1}. \tag{4.2}$$

The coupled N/D equations we solve for a two-channel problem are

$$N_{ij}(s)=B_{ij}(s)+\sum_{k=1}^{2}\int_{sk}^{\infty}\left[B_{ik}(s')-\frac{s-s_0}{s'-s_0}B_{ik}(s)\right] \times\rho_k(s')N_{kj}(s')\frac{ds'}{s'-s}\,, \tag{4.3a}$$

$$\mathrm{Re}D_{ij}(s)=\delta_{ij}-\frac{s-s_0}{\pi}P\int_{s_i}^{\infty}\frac{\rho_i(s')N_{ij}(s')ds'}{(s'-s_0)(s'-s)}\,, \tag{4.3b}$$

$$\rho_i(s)=\frac{k_i^3}{\sqrt{s}}\,.$$

In order to solve the coupled integral equations (4.3) numerically, we choose a certain distribution of energy points $s(1)$, $s(2)$, $\cdots$, $s(N)$ in the interval s_1 to ∞ and reduce the integral equations (4.3) into a set of matrix equations over these points. Calculations involve inversion of a number of $N\times N$ matrices. A number of checks are used to test the accuracy of our solution. First the subtraction point s_0 is moved around to see if the solution is independent of s_0. The symmetry of the reaction matrix (i.e., $K_{12}=K_{21}$) provides another check on our solution. We also change the size of mesh to check the stability of the results. A mesh of 35 points gives excellent stability (solutions are stable to within 1%). Finally we test the accuracy of the solutions by substituting the values of N and D thus calculated in the equation

$$N(s)=B(s)\mathrm{Re}D(s)+\frac{P}{\pi}\int_0^{\infty}B(s')\theta\rho(s')N(s')\frac{ds'}{s'-s}\,. \tag{4.4}$$

We first consider the situation where $B(s)$ is 2×2 matrix of dipoles:

$$B(s)=\begin{bmatrix}\dfrac{g_{11}}{(s+m_{11})^2} & \dfrac{g_{12}}{(s+m_{12})^2}\\[2ex] \dfrac{g_{12}}{(s+m_{21})^2} & \dfrac{g_{22}}{(s+m_{22})^2}\end{bmatrix}. \tag{4.5}$$

TABLE I. P-wave case in which both the channels are open. Masses in channel 1 are 4.0 and 4.5 while in channel 2 they are 4.0 and 5.0 (Ref. 6). Total energy is 9.2. The position of the dipoles for the top 16 cases is 5.0 in all channels. The last seven entries show the results when the dipole in channel 2 is kept fixed at 5.0 while the dipole in channel 1 is moved to 3.0 and the dipole in the inelastic channel is moved to 4.0. Theoretical estimate of L_2 obtained from the principal value integral$=0.13$.

L_2	K_{11}	$\tilde{K}_{11}$	K_{22}	K_{12}
0.133	0.080	0.022	0.119	0.121
0.137	0.107	0.038	0.105	0.129
0.140	0.027	0.022	0.041	0.029
0.147	−0.416	0.124	−0.046	−0.408
0.128	0.012	0.009	0.012	0.021
0.129	0.010	0.009	0.009	0.010
0.126	0.006	0.005	0.006	0.010
0.123	0.170	0.010	0.370	0.281
0.138	0.536	0.086	0.395	0.489
0.136	0.493	0.086	0.342	0.440
0.151	−0.341	0.278	−0.281	−0.283
0.146	0.066	0.059	0.029	0.035
0.145	0.091	0.059	0.056	0.081
0.136	0.359	0.059	0.341	0.378
0.152	0.124	0.059	0.270	0.165
0.125	−0.150	0.125	−0.217	−0.159
0.121	0.287	0.014	0.550	0.431
0.160	−0.352	0.207	−0.336	−0.313
0.154	0.038	0.032	0.029	0.033
0.152	2.664	0.135	1.794	2.218
0.147	0.120	0.032	0.153	0.163
0.152	0.168	0.056	0.144	0.182
0.145	0.015	0.014	0.011	0.013

Using the $B(s)$ given by (4.5), we solve the Eq. (4.3) to determine the elements K. Then g_{12} is set equal to zero and the one-channel equations are solved for $\tilde{K}_{11}$. First we fix the "kinematical conditions," i.e., the masses of the particles in channels 1 and 2 and the orbital momenta (we consider $l_1=l_2=l=1$), and in addition we fix the positions of the dipoles, i.e., m_{11}, m_{22}, and $m_{12}(=m_{21})$. Then (4.3) are solved for many different sets of g_{ij},[17] and K_{ij} and $\tilde{K}_{11}$ are determined for each set; the quantity L_2 is calculated using (4.1).

Typical results are shown in Table I. This is the case when both the channels are open. The first 16 rows show the results when we assume $m_{11}=m_{22}=m_{12}$. We find that over a wide range of coupling strengths, L_2 is found to be a constant[18] as well as equal to its theoretical estimate given by the principal value integral

$$L_2=\left(\frac{(\sqrt{s})(s+m_{22})^4}{\pi}P\int_{s_2}^{\infty}\frac{\rho_2(s')ds'}{(s'+m_{22})^4(s'-s)}\right)^{-1}. \tag{4.6}$$

The last seven entries in Table I substantiate our conjecture that L_2 is dependent only on the diagonal singularities in the nth channel (in this case the second channel) since L_2 for these entries is unchanged when m_{11} and m_{12} are given values different from m_{22} (which is kept fixed). Calculations similar to those presented in Table I were performed with different values of the

[17] The ratios of the diagonal to nondiagonal elements g_{11}/g_{12}, g_{22}/g_{12} have been allowed to vary over the range ~ 0 to 10.

[18] Total variation in L_2 is seen to be $\lesssim\pm10\%$.

TABLE II. Results for scalar particle exchange potentials (in the "crossed" t reactions). Threshold energy for channel 1 is 2.0 while for channel 2 it is 2.1 (Ref. 6). Masses of the exchanged particles are $m_{11}=4.5$, $m_{22}=7.1$ and $m_{12}=7.1$. Total energy in the center-of-mass system $=2.6$. Theoretical estimate of $L_2=0.23$.

L_2	K_{11}	$\tilde{K}_{11}$	K_{22}	K_{21}
0.249	−7.343	9.531	−0.517	−2.125
0.250	−1.279	−2.819	−0.055	−0.549
0.250	−0.740	−1.450	0.044	−0.457
0.254	−0.606	2.956	−0.355	−0.601
0.253	−0.602	0.253	−0.875	−0.732
0.252	−0.606	−1.827	−0.041	−0.508
0.245	0.385	0.145	0.526	0.430
0.248	0.829	0.145	1.310	1.031
0.242	0.237	0.145	0.204	0.202
0.238	2.087	0.145	2.234	2.191
0.249	−1.590	−1.827	0.106	−0.290
0.250	5.834	2.956	0.450	1.420
0.253	−36.230	2.956	−5.718	−14.636
0.253	50.233	4.662	4.745	15.095
0.253	3.422	2.087	0.456	0.973
0.251	62.557	2.956	7.987	22.162
0.252	6.624	2.956	0.649	1.818
0.249	−1.256	−1.827	0.061	−0.421
0.253	−1.414	−12.458	−0.198	−0.781
0.252	−1.357	−1.827	0.152	−0.436
0.248	−1.151	−12.458	−0.216	−0.595

masses and positions of the dipoles and entirely similar conclusions were drawn.

Let us now consider the situation where the left-hand cut corresponding to a particular channel is produced by exchange of a scalar particle in its "crossed" t reaction. For the sake of simplicity let us assume that the masses in channel 1 are equal and have the value m_1, while in channel 2 they have the value m_2. The elements of the 2×2 matrix $B(s)$ for scattering in the lth partial wave are given by

$$B_{ij}(s)=\frac{g_{ij}}{2}k_i^{-l_i}k_j^{-l_j}\int_{-1}^{+1}\frac{P_l(z)dz}{t_{ij}-m_{ij}^2}, \tag{4.7}$$

where t_{ij} is the square of the four-momentum transfer for scattering from channel i to channel j and z is the cosine of the scattering angle in the center-of-mass system,

$$t_{ij}=m_i^2+m_j^2-s/2+2k_ik_jz, \tag{4.8}$$

m_{ij} is the mass of the scalar particle exchanged in the "crossed" t reaction of this process.

For p-wave scattering we get

$$B_{ij}=\tfrac{1}{2}g_{ij}k_i^{-2}k_j^{-2}Q_1\left(\frac{s/2-m_i^2-m_j^2+m_{ij}^2}{2k_ik_j}\right), \tag{4.9}$$

where for α real

$$Q_1(\alpha)=\tfrac{1}{2}\alpha\ln|(\alpha+1)/(\alpha-1)|-1 \tag{4.10}$$

and

$$Q_1(i\alpha)=\alpha\tan^{-1}(1/\alpha)-1, \tag{4.11}$$

we use (4.9), (4.10), and (4.11) in (4.3) to solve for K and $\tilde{K}_{11}$ as before. As usual we first fix the "kinematical conditions" and in addition we fix the masses of the exchanged particles; m_{11}, m_{22}, and $m_{12}(=m_{21})$. Calculations are then done for many different sets of coupling strengths. Some typical results are shown in Table II for a fairly large variation of coupling strengths. Even for large modifications of K_{11} from the uncoupled value $\tilde{K}_{11}$ we find L_2 to be approximately a constant[17] and in good agreement with its theoretical estimate as calculated from (3.18).