system, and it can be shown that in more realistic models unitarity yields Regge-type singularities in $T_{MM'}{}^{J}$. Nevertheless, as real-life situations are usually more complicated than simple models, we cannot help feeling that they are not going to be the only singularities in the J plane, because it seems very difficult to get in general only clean Regge singularities from singularities which depend upon so many subenergy parameters. If this is the case, then the Regge pole concept will not be very useful in analyzing the properties of three-body systems.

For the sake of completeness, let us note that the same analysis could be done if, in place of fixing the values of M and M', we were to fix the relative angular momentum of one of the "electrons" with the "nucleus." (That choice has been made by Newton[1] and Choudhury.[4]) Here again, and in a much more elementary way, one could get poles dependent upon the subenergies. The same conclusion, namely that no Fredholm equations can give this result, would stand.

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Poles in the Vertex Function, Zeros of the Propagator, and Bounds on Coupling Constants*

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Extending the arguments by Goebel and Sakita, it is shown in a general framework that a pole of the proper vertex function does not lead to a pole in the scattering amplitude. Connections between zeros of the propagator, poles of the proper vertex function, and upper bounds on the coupling constant are discussed in rather general terms as well as in terms of the Zachariasen model. By making use of the analytic continuation of the partial-wave scattering amplitude into the complex angular-momentum plane, a possible physical interpretation of the pole of the vertex function is given.

1. INTRODUCTION

IN quantum-field theory the masses of the interacting particles and the coupling constants are introduced from the outset as mutually independent parameters. Recently, however, Geshkenbein and Ioffe[1] have obtained an upper bound on the coupling constant for the Yukawa interaction of three particles a, b, and c, in terms of the masses of the particles alone. The starting point for their derivation was the consideration of the Lehmann representation for the propagator $\Delta(s)$ of one of the particles, for instance particle a, and the analyticity properties of the function $\Lambda(s)=g\Gamma(s)\Delta(s)$ where $\Gamma(s)$ is the proper vertex function with b and c on the mass shell. However, the assumption was also made that $\Gamma(s)$ has no pole in the complex s plane, on the grounds that a pole of $\Gamma(s)$ would also appear in the scattering amplitude through the term $(g^2/8\pi)\Gamma\Delta\Gamma$ and therefore would correspond to a bound state of the system (b,c) with the same quantum numbers of particle a. It has been pointed out by Goebel and Sakita[2] that it is possible for the propagator to have a zero, coupled to a pole of Γ. Such a pole would not appear in the physical amplitude and would have no direct physical significance. They have constructed specific models in which Δ develops a zero and the G–I bound is violated. They have also shown in nonrelativistic elastic models the dynamical origin of the pole of Γ and how the pole of $(g^2/8\pi)\Gamma\Delta\Gamma$ is canceled out in the b-c scattering amplitude.

In this paper we shall generalize the arguments of Goebel and Sakita and give an interpretation for the pole of the vertex function.

In Sec. 2 we have shown that within the framework of field theory a zero of the propagator is always associated with a pole of the vertex function. The basic assumption for our argument is the possibility of analytically continuing Δ, Λ and the scattering amplitude f as functions of the coupling constant.

In Sec. 3 we discuss the relation between the G–I bound and the condition $Z_a\geq 0$ where $Z_a{}^{1/2}$ is the wave

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[1] B. V. Geshkenbein and B. L. Ioffe, Phys. Rev. Letters **11**, 55 (1963); Zh. Eksperim. i Teor. Fiz. 44, 1211 (1963) [English transl.: Soviet Phys.—JETP **17**, 820 (1963)], hereafter will be referred to as G–I.
[2] C. J. Goebel and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).

function renormalization constant, and in Sec. 4 a discussion of the Zachariasen model illustrates our considerations. It is interesting to remark that, in this model, the condition $Z_a \geq 0$ implies a restriction on the coupling constant although one may have a pole of the vertex function as close to $s=m_a^2$ as we like, which would invalidate the argument of G–I.

In Sec. 5 we give an interpretation of the pole of the vertex function. We have shown, within the approximation of elastic unitarity, that if particle a is not a Regge pole and if a Regge trajectory with the quantum numbers of a and the proper signature crosses the values $s=s_1$ in the interval $(m_a^2,(m_b+m_c)^2)$, then Γ will have a pole at s_1 and the term $(g^2/8\pi)\Gamma\Delta\Gamma$ exactly cancels the Regge pole, so that it does not appear in the scattering amplitude. On the basis of the asymptotic behavior for large momentum transfer of Feynman diagrams for the scattering amplitude, it is argued that this interpretation is correct in the case of the Yukawa interaction of three scalar particles and in that of a spin-$\frac{1}{2}$ fermion interacting with a spin-zero boson. The latter case is studied in detail in Sec. 5. A summary of the conclusions arrived at is given in the last section.

2. ZEROS OF THE PROPAGATOR

Let us consider three scalar particles a, b, c, with a Yukawa interaction expressed by the virtual transition

$$a \rightleftharpoons b+c\,,$$

and let us assume that the least-massive state with the quantum numbers of particle a is a two-particle state of b and c.

We shall consider the propagator of particle a defined by

$$\Delta(s)=i\int \langle T(\phi_a(x/2)\phi_a(-x/2))\rangle e^{-ip\cdot x}d^4x\,, \qquad (1)$$

where $s=p^2$. The general axioms of local field theory imply the Lehmann representation for the propagator.[3] We shall assume no subtractions and write

$$\Delta(s)=\frac{1}{m_a^2-s}+\int_{s_0}^{\infty}\frac{\sigma(s')}{s'-s}ds'\,, \qquad (2)$$

where $s_0=(m_b+m_c)^2$,

$$\sigma(s)={\sum}'|\langle\phi_a(0)|n\rangle|^2\delta(p_n-p)(2\pi)^3\,, \qquad (3)$$

with the sum $\sum'$ taken over all states in the continuum.

The spectral function $\sigma(s)$ is positive definite, wherefrom it follows that $\Delta(s)$ is a Herglotz function. It has no complex zeros but it may have one real zero in the interval (m_a^2,s_0). For sufficiently weak coupling, $\Delta(s)$ has no zeros, as it reduces to the free propagator in the limit of zero coupling and at least for small g^2, an asymptotic expansion in power series of the coupling constant holds for small values of s. Then, since $\Delta(s)$ cannot have complex zeros, as one increases the coupling constant a zero would emerge on the physical sheet only if it moves from the second sheet, across the cut, around the branch point at $s=s_0$. The second sheet is reached from the first or physical sheet by crossing the cut in the interval between the threshold s_0 and the next branch point. The discontinuity of $\Delta(s)$ across this cut is given by the contribution to $\sigma(s)$ of the two-particle state $|p_bp_c\rangle$. One obtains

$$\Delta(s+i\epsilon)-\Delta(s-i\epsilon)=2i\frac{k}{\sqrt{s}}\frac{1}{8\pi}|\Lambda|^2=2i\sigma_{el}(s)\,, \qquad (4)$$

where

$$k=\frac{1}{2\sqrt{s}}\{[s-(m_b+m_c)^2][s-(m_b-m_c)^2]\}^{1/2} \qquad (5)$$

and

$$\Lambda(s)=(2\pi)^3(2p_{b0}2p_{c0})^{1/2}\langle\phi_a(0)|p_bp_c\rangle \qquad (6)$$

which is related to the form factor $F(s)$ by $F(s)=(m_a^2-s)\Lambda(s)$ and to the proper vertex function $\Gamma(s)$ by

$$g\Gamma(s)=\Lambda(s)/\Delta(s)\,, \qquad (7)$$

where g is the coupling constant.

Now, it can also be deduced (provided a certain condition is satisfied by the masses m_a, m_b, m_c of the particles), that, $\Lambda(s)$ is an analytic function of s in the plane cut along the real axis in the interval $(s_0,+\infty)$, with a pole at $s=m_a^2$. In the interval below the second branch point only the two-particle state $|p_b,p_c\rangle$ contributes to the discontinuity of $\Lambda(s)$ across the cut, which then will be given by

$$\Lambda(s+i\epsilon)-\Lambda(s-i\epsilon)=2i\frac{k}{\sqrt{s}}\Lambda f^*\,, \qquad (8)$$

where f is the s-wave amplitude for (b,c) elastic scattering, which is of the form

$$f=\frac{\sqrt{s}}{k}e^{i\delta}\sin\delta\,. \qquad (9)$$

In the elastic region, unitarity gives

$$f(s+i\epsilon)-f(s-i\epsilon)=2i\frac{k}{\sqrt{s}}ff^*\,. \qquad (10)$$

The relations (4), (8), and (10) allow us to express the continuation of f, Λ, and Δ onto the second sheet in terms of the functions in the first sheet. After straightforward computation, one obtains

$$f_{\rm II}=f_{\rm I}S^{-1}\,, \qquad (11)$$

$$\Lambda_{\rm II}=\Lambda_{\rm I}S^{-1}\,, \qquad (12)$$

$$\Delta_{\rm II}=\Delta_{\rm I}US^{-1}\,, \qquad (13)$$

[3] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

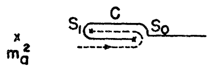

FIG. 1. Complex s plane, showing singularities of Δ^{-1} and the contour of integration C which encircles the pole s_1 as it moves onto the physical sheet along the trajectory indicated by the broken line.

where

$$S=1+2i\frac{k}{\sqrt{s}}f_{\mathrm{I}}\,, \tag{14}$$

$$U=1+2i\frac{k}{\sqrt{s}}h_{\mathrm{I}}\,, \tag{15}$$

and

$$h=f-(1/8\pi)\Lambda\Delta^{-1}\Lambda=f-(g^2/8\pi)\Gamma\Delta\Gamma. \tag{16}$$

From the definitions (7) and (16) one can also deduce expressions for the continuation of Γ and h onto the second sheet. One readily obtains

$$\Gamma_{\mathrm{II}}=\Gamma_{\mathrm{I}}/U \tag{17}$$

and

$$h_{\mathrm{II}}=h_{\mathrm{I}}/U. \tag{18}$$

The last relation shows that in the elastic region h_{I} satisfies the unitarity condition (10) and therefore one can write

$$h=\frac{\sqrt{s}}{k}e^{i\delta_0}\sin\delta_0. \tag{19}$$

From (17) it follows that in the elastic region Γ has the phase of h just as Λ has the phase of f. Now (11), (12), and (13) tell us that the poles of Δ_{II} coincide with the poles of Λ_{II} and f_{II} and are given by the zeros of S. On the other hand, the zeros of Δ_{II} coincide with the poles of Γ_{II} and h_{II} and are given by the zeros of U. Let us suppose that the coupling constant is sufficiently small so that Δ_{I} has no zeros; we recall that in the limit of weak coupling Δ_{I} approaches the free-particle propagator. Let us then assume that U has a zero on the real axis below the threshold $s=s_0$. As one increases the coupling constant this zero may move toward the threshold and eventually turn around it onto the first sheet. From (16) it is clear that the zeros of U and S do not coincide; since $\Delta_{\mathrm{I}}<0$ for $m_a^2<s<s_0$, it follows that in this interval $h_{\mathrm{I}}>f_{\mathrm{I}}$ and the first zero of U is located to the right of the first zero of S. In this way, h and Γ develop a pole at the same position where Δ has a zero and before a pole of f reaches the physical sheet. Thus the poles of Γ and h *are not* poles of f and Λ. As will be shown later, in the expression (16) there is complete cancellation of the poles of h and $-(g^2/8\pi)\times\Gamma\Delta\Gamma$ by virtue of the relation

$$\mathrm{Res}h=-(g^2/8\pi)(\mathrm{Res}\Gamma)^2/\mathrm{Res}\Delta^{-1}. \tag{20}$$

We also remark that $\Lambda_{\mathrm{I}}(s)$ may have zeros, but according to our analysis they would correspond to zeros in $\Gamma_{\mathrm{I}}(s)$ and not in the propagator $\Delta_{\mathrm{I}}(s)$. These results contradict the assertion of G–I to the effect that Γ has no pole where Δ has a zero because such a pole would appear in the scattering amplitude through the term $(g^2/8\pi)\Gamma\Delta\Gamma$.

3. THE GESHKENBEIN-IOFFE BOUND

If the propagator has no zeros, the Lehmann representation (2) is equivalent to the following representation for the inverse of the propagator:

$$\Delta^{-1}(s)=(m_a^2-s)\times\left[1-(m_a^2-s)\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\lambda(s')ds'}{(s'-m_a^2)^2(s'-s)}\right] \tag{21}$$

with positive definite spectral function $\lambda(s')$. In the elastic region

$$\lambda_{\mathrm{el}}(s')=\frac{g^2}{8\pi}|\Gamma|^2\frac{k}{\sqrt{s}}. \tag{22}$$

In the previous section we have seen that as one varies the coupling constant a pole of Γ may move from the second sheet, around the branch point at $s=s_0$, onto the physical sheet. When this happens one has to deform the contour of integration in the expression (21) so as to go around the pole in the way shown in Fig. 1. As a result one picks up an additional term

$$\frac{1}{\pi}\oint\frac{g^2}{8\pi}\frac{\Gamma_{\mathrm{I}}^2}{1+2i(k'/\sqrt{s'})h_{\mathrm{I}}}\frac{k'}{\sqrt{s'}}\frac{1}{(s'-m_a^2)^2}\frac{ds'}{s'-s}$$
$$=-\frac{g^2}{8\pi}\frac{(\mathrm{Res}\Gamma)^2}{\mathrm{Res}h}\frac{1}{(s_1-m_a^2)^2}\frac{1}{s_1-s}, \tag{23}$$

which gives a pole of Δ^{-1} whose residue satisfies relation (20).

The Lehmann representation for the propagator implies the restriction $0\leq Z_a\leq 1$ for the wave function renormalization constant $Z_a^{1/2}$. Hence, the condition

$$Z_a=1-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\lambda(s')ds'}{(s'-m_a^2)^2}\geq 0. \tag{24}$$

This expression for the renormalization constant holds in general, provided the path of integration is taken along the contour C, when Δ^{-1} has a pole at s_1 in the physical sheet.

Let us suppose that Z_a is a function of the coupling constant g^2, such that for $g^2<g_0^2$, $Z_a>0$, and $Z_a\to 0$ as $g^2\to g_0^2$. We shall assume that $\Delta(s)$ has a certain domain of analyticity as a function of the coupling constant which includes the real open interval $(0,g_0^2)$, and admits the Lehmann representation for g^2 in this interval. As g^2 varies from the left to the right of g_0^2, one may have the following behavior of $\Delta(s)$:

(i) A complex pole of $\Delta(s)$ in its second sheet moves to infinity and onto the first sheet from $-\infty$ along the real axis. The residue of this pole when it emerges in the physical sheet will be positive corresponding to a state of negative norm, a so-called "ghost."

(ii) A zero of Δ moves from the right to the left of the pole at $s=m_a^2$ and at the same time either:

(a) There is a physical pole in the interval (m_a^2,s_0) whose residue tends to $-\infty$ and changes sign. Thus the norm of the state associated with this pole becomes negative; or

(b) A complex pole of Δ in some sheet adjacent to the physical sheet approaches the cut and crosses over onto the physical sheet.

In the next section we shall discuss a model which exhibits these features. We shall not consider here other possibilities for the behavior of the propagator as a function of the coupling constant, as well as the complications that arise when dealing with particles with spin.

In both cases (i) and (ii), g_0^2 is an upper bound of g^2 resulting exclusively from the requirement of the validity of the Lehmann representation. However, at least in case (ii), the method of Geshkenbein and Ioffe would certainly not apply because as we approach the bound there will be a zero of the propagator moving towards m_a^2.

In principle, $\Delta(s)$ may have a certain number of zeros and poles in the interval (m_a^2,s_0) in alternated positions. However the information on the existence of poles is readily available, and there is no instance of two stable particles with exactly the same quantum numbers. One can then impose the condition that there is no pole of Δ other than that at $s=m_a^2$. In this case Δ may have only one zero at some value s_1 of s in the interval (m_a^2,s_0). Therefore, (2) together with (4) gives

$$\frac{1}{m_a^2-s_1}+\frac{1}{8\pi^2}\int_{s_0}^{\infty}\frac{|\Lambda|^2}{s'-s_1}\frac{k'}{s'}ds'\leq 0. \tag{25}$$

For $s_1=s_0$, this inequality reduces to that used by G–I. If no further information on Λ is given in addition to its analyticity properties it is clear that this inequality would yield only a bound on the product $g^2(s_1-m_a^2)/(s_0-m_a^2)$ and not on g^2 itself.

4. A MODEL

In this section we shall discuss the Zachariasen model[4] for combined three- and four-particle interactions. This model has sufficient structure to illustrate the foregoing considerations. In this model the scattering amplitude is pure s wave, has no left-hand cuts and satisfies elastic unitarity. It is obtained by iteration of the diagrams shown in Fig. 2.

Each higher order term has of course divergent closed loops, but they can be renormalized. The result in terms of renormalized quantities may be determined from

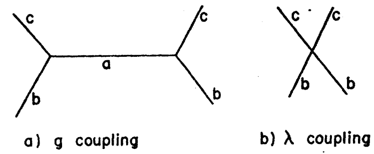

FIG. 2. Lowest order graphs for bc scattering in the combined version of the Zachariasen model.

the relation

$$f=\frac{1}{8\pi}[g^2+\lambda(m_a^2-s)]\frac{\Lambda}{g}. \tag{26}$$

Using dispersion relations and elastic unitarity, one obtains

$$\Lambda=\frac{g}{m_a^2-s}\left[1-\frac{m_a^2-s}{8\pi^2}\int(g^2+\lambda(m_a^2-s'))\times\frac{k'}{\sqrt{s'}}\frac{ds'}{(s'-m_a^2)^2(s'-s)}\right]^{-1}, \tag{27}$$

which is the solution of the dispersion equations without Castillejo-Dalitz-Dyson (CDD) poles. Now Δ and Γ can be obtained from (2) and (7), respectively, with $\sigma(s)$ given by (4). However, it is simpler to use (16) and remark that since f has no left-hand cut, neither has h, and therefore h is simply proportional to Γ.[5] Setting $h=\alpha\Gamma$, one obtains

$$\Gamma=f\left(\frac{g}{8\pi}\Lambda+\alpha\right)^{-1},$$

and from (7)

$$\Delta=\frac{8\pi}{\lambda}\left(\frac{g}{8\pi}\Delta+\alpha\right)\bigg/(s_1-s),$$

where

$$s_1=m_a^2+g^2/\lambda. \tag{28}$$

Since f and therefore Δ have no poles at $s=s_1$, it follows that $\alpha=-(g/8\pi)\Lambda(s_1)$. Hence

$$\Delta=\frac{g}{\lambda}[\Lambda(s)-\Lambda(s_1)]/(s_1-s) \tag{29}$$

and

$$\Gamma=\frac{8\pi}{g}f(\Lambda(s)-\Lambda(s_1))^{-1}. \tag{30}$$

One can now actually verify that

$$\mathrm{Im}\Delta=(1/8\pi)(k/\sqrt{s})|\Lambda|^2.$$

The wave function renormalization constant for particle a will be given by

$$Z=\left(-\frac{g}{\lambda}\Lambda(s_1)\right)^{-1}=1-\frac{g^2}{8\pi^2}\int\frac{k'}{\sqrt{s'}}\frac{ds'}{(s'-m_a^2)^2} \tag{31}$$

and is independent of λ. The condition $Z\geq 0$ will then impose a restriction on the coupling constant g^2.

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[4] F. Zachariasen, Phys. Rev. **121**, 1851 (1961); S. D. Drell and F. Zachariasen, Phys. Rev. **119**, 463 (1960).

[5] M. L. Whippman and I. S. Gerstein, Phys. Rev. **134**, B1123 (1964).

If $\lambda<0$, (28) gives $s_1<m_a^2$ and by (31), $g\Lambda(s_1)>0$; one then finds that Λ always has a pole below s_1. The value of λ should be fixed in such a way that the pole occurs at a positive $s=m_{a'}^2$ corresponding to a physical particle a'. One can rewrite the expression (27) so as to explicitly exhibit the lowest pole. One obtains an expression formally identical to (27) but in terms of new parameters λ', g', which will be taken as coupling constants. One can easily show that the renormalized and unrenormalized quantities are related by

$$(g_0^2/\lambda_0)+m_0^2=(g^2/\lambda)+m^2 \qquad (32)$$

$$g_0^2/\lambda_0^2=(g^2/\lambda^2)Z. \qquad (33)$$

These relations would, of course, hold with the replacement of the unprimed by the primed set of parameters. The connection between the two sets is thereby established.

Since $m_{a'}^2<s_1$ it is clear that λ' is positive. From now on we shall drop the primes with the understanding that λ is positive.

Let us investigate the behavior of the propagator as a function of g^2, keeping s_1 constant. For small g^2, Δ has two complex poles and two zeros in the second sheet, their trajectories starting at $s=0$, for $g^2=0$. One can easily verify that as g^2 increases one of the zeros moves onto the first sheet. When g^2 approaches the critical value g_c^2 for which $Z=0$, this zero tends to m_a^2, exhibiting the behavior referred to in (ii) of the preceding section. Likewise, the poles of Δ behave in the way described in either case (a) or (b), for $s_1<s_0$ or $s_1>s_0$, respectively.

If one demands that there is no bound state other than that corresponding to particle a, the inequality

$$m_a^2+\frac{g_c^2}{\lambda}>s_0 \qquad (34)$$

gives an upper bound on λ, independent of g^2.

A special case of interest of the Zachariasen model is the case $\lambda=0$. We have then

$$\Delta=\Lambda/g. \qquad (35)$$

The renormalization constant will still be given by (31). However, in this case Δ never has a zero and the Geshkenbein-Ioffe method is valid. As g^2 approaches the critical value, a pole in the second sheet moves to infinity and onto the physical sheet along the negative real axis. This example belongs to case (i) considered in Sec. 2.

5. INTERPRETATION OF THE POLES OF THE VERTEX FUNCTION

In this section we shall investigate the physical meaning of the zeros of the propagator or the poles of the vertex function.

Let us take the scattering of spinless particles b and c as in Sec. 2 and suppose that one can write a dispersion relation for the scattering amplitude at fixed energy, with N subtractions. Then for $l\geq N$ the partial-wave amplitudes will be given by

$$h_l^{\pm}(s)=\frac{1}{\pi}\int_{t_0}^{\infty}A_3(s,t')Q_l(z')\frac{dt'}{2k^2} \pm\frac{1}{\pi}\int_{u_0}^{\infty}A_2(s,u')Q_l(z'')\frac{du'}{2k^2}, \qquad (36)$$

where the indices $\pm$ refer to the signature, and

$$z'=1+\frac{t'}{2k^2},\; z''=\frac{1}{2k^2}\left(\tfrac{1}{2}s-m_b^2-m_c^2-\frac{1}{2s}(m_b^2-m_c^2)^2+u'\right).$$

The functions $h_l^{\pm}(s)$ defined by (36) are analytic in the half-plane $\mathrm{Re}\,l\geq N$. The analytic continuation of $h_l^{\pm}(s)$ (with the proper signature) may or may not coincide with the physical amplitudes for real integer values of $l<N$. If $h_l^{\pm}$ does not coincide with $f_l(s)$, their difference must be an analytic function which has the physical cut, but no left-hand cut.[6] Let us write (for $l=0$)

$$f_0(s)-h_0^+=(g/8\pi)\Lambda\Gamma, \qquad (37)$$

where Λ and Γ are as yet not identified but are analytic in the cut plane except for poles on the real axis. In the elastic region both f_0 and h_0^+ satisfy elastic unitarity so that their difference must be of the form

$$(g/8\pi)\Lambda\Gamma=\frac{\sqrt{s}}{k}e^{i(\delta_0+\delta_0')}\sin(\delta_0-\delta_0'), \qquad (38)$$

where δ_0 and δ_0' are the phases of f_0 and h_0^+, respectively. Now one can choose the phases of Γ and Λ to be δ_0' and δ_0 in the elastic region, so that one can write

$$\Gamma(s)=\gamma(s)\exp\frac{1}{\pi}\int_{s_0}^{\infty}\frac{s-m_a^2}{s'-m_a^2}\frac{\delta_0'(s')ds'}{(s'-s)} \qquad (39)$$

and

$$\Lambda(s)=\lambda(s)\exp\frac{1}{\pi}\int_{s_0}^{\infty}\frac{s-m_a^2}{s'-m_a^2}\frac{\delta_0(s')ds'}{(s'-s)}, \qquad (40)$$

where for the analytic functions $\gamma(s)$ and $\lambda(s)$ the cut starts at the first inelastic threshold. Let us assume that the pole associated with particle a is not a Regge pole. Then $(g/8\pi)\Lambda\Gamma$ has a pole at $s=m_a^2$ with residue $-(g^2/8\pi)$. One can choose $\gamma(s)$ to be regular at $s=m_a^2$, with $\gamma(m_a^2)=1$, so that $\lambda(s)$ has a pole at that point with residue $-g$. If h_0^+ develops a pole the contour of integration in (39) is to be deformed in the manner shown in Fig. 1. Within the approximation of elastic unitarity $\Gamma(s)$ and $\Lambda(s)$ are determined by (39) and

[6] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **42**, 1260 (1962) [English transl.: Soviet Phys.—JETP **15**, 873 (1962)].

(40) with $\gamma(s)\equiv 1$ and

$$\lambda(s)\equiv\frac{g}{m_a^2-s}\left(1-\frac{\delta_0(s_0)}{\pi}\frac{s-m_a^2}{s_0-m_a^2}\right).$$

In this approximation $\Lambda(s)$ coincides with the function defined by (6). We also have

$$\Delta=\frac{\Lambda}{g\Gamma}=\frac{s}{k}e^{i(\delta_0-\delta_0')}\sin(\delta_0-\delta_0')\left(\frac{g^2}{8\pi}|\Gamma|^2\right)^{-1}. \quad (41)$$

Hence,

$$\mathrm{Im}\Delta^{-1}=-\frac{k}{\sqrt{s}}\frac{g^2}{8\pi}|\Gamma|^2 \quad (42)$$

and

$$\mathrm{Im}\Delta=-|\Delta|^2\,\mathrm{Im}\Delta^{-1}=\frac{1}{8\pi}\frac{k}{\sqrt{s}}|\Lambda|^2, \quad (43)$$

which shows that $\Delta(s)$ coincides with the propagator and $\Gamma(s)$ with the proper vertex function.

Moreover, by making use of the representation (21), one can construct $\Delta(s)$ and then determine Λ by means of (41). Therefore the scattering amplitude $f_0(s)$ will become completely determined in terms of the Froissart amplitude $h_0^+(s)$. This is in agreement with Martin's[7] result that the double spectral functions completely determine the physical amplitude.

We conjecture that these results might be valid not only within the approximation of elastic unitarity but more generally when full unitarity is taken into account. Let us investigate this possibility within the framework of Feynman diagrams.

We shall consider the case in which the only interaction of particles b and c is a vertex interaction with particle a. In the perturbation expansion for the scattering amplitude one can distinguish two classes of diagrams. The first class consists of all diagrams which contain a single line corresponding to the one-particle intermediate state in channel s. These diagrams are functions of the energy only and contribute to the s-wave amplitude alone. The sum of these diagrams is just $(g^2/8\pi)\Gamma\Delta\Gamma=(g/8\pi)\Lambda\Gamma$ where Γ is the vertex function and Δ the propagator for particle a. The remaining diagrams are functions of the momentum transfer variables and contribute to all partial waves. It is easy to verify that, at fixed energy, each one of these diagrams tends to zero for infinite momentum transfer so that one can write a dispersion relation without subtractions. Therefore the contribution of each diagram to *every* partial wave will be given by an expression of the form (36). Let us assume that for sufficiently small values of the coupling constant the sum of the partial-wave contributions of all these diagrams may also be written in the form (36) and but for the s wave, correctly gives the physical amplitude. Now, to obtain the s-wave amplitude one simply has to add the diagrams of the first class. Hence, one can write

$$f_l(s)=(g/8\pi)\Lambda\Gamma\delta_{l0}+h_l^{\pm}(s). \quad (44)$$

For the actual value of the coupling constant, a representation (36) for $h_l^{\pm}(s)$ may break down for $l<N$ but if each term in (44) can be analytically continued as a function of the coupling constant, then $h_l^{\pm}(s)$ will coincide with the analytic continuation in the l variable of the function defined by (36) for $\mathrm{Re}\,l\geq N$. Thus, in this case, perturbation expansion provides a plausibility argument for the validity of the relation (44). According to this analysis $\Gamma(s)$ has a pole at the point where the Regge trajectory crosses the value $l=0$. By the mechanism discussed in Sec. 2 this pole exactly cancels the Regge pole, so that it does not appear in the scattering amplitude. It should be pointed out, however, that this interpretation depends essentially on the type of coupling and on the spins of the interacting particles. In the next section we discuss in detail the interaction of a spin $\frac{1}{2}$ with a spin-zero particle.

6. THE PION-NUCLEON INTERACTION AND THE NUCLEON PROPAGATOR

Let us now consider the interaction of a spin-$\frac{1}{2}$ fermion n with a spin-zero boson π. The Lehmann representation for the fermion propagator is

$$\Delta(P)=\frac{m+\boldsymbol{P}}{m^2-s}+\int_{s_0}^{\infty}[(\boldsymbol{P}+\sqrt{s'})\rho_1(s')-\rho_2(s')]\frac{ds'}{s'-s}$$
$$=\boldsymbol{P}\Delta_1+\Delta_2, \quad (45)$$

where $s=P^2$ and $s_0=(m+\mu)^2$. The spectral functions satisfy the inequalities[3]

$$2(\sqrt{s})\rho_1\geq\rho_2\geq 0. \quad (46)$$

The two-particle contribution to $\mathrm{Im}\Delta$ is given by

$$\mathrm{Im}\Delta=\frac{1}{8\pi}\Lambda\left(\frac{E}{\sqrt{s}}\boldsymbol{P}+m\right)\bar{\Lambda}\frac{k}{\sqrt{s}}, \quad (47)$$

where

$$E=(1/2\sqrt{s})(s+m^2-\mu^2). \quad (48)$$

For pseudoscalar meson one can write

$$\Lambda=(\Lambda_1\boldsymbol{P}+\Lambda_2)\gamma_5, \quad (49)$$

where $P=p+q$. At $s=m^2$, Λ_1 and Λ_2 have a pole with residue -1 and $-m$, respectively. In the interval $s_0<s<(m+2\mu)^2$, unitarity gives

$$\mathrm{Im}\Delta_1=\frac{1}{8\pi}\frac{k}{\sqrt{s}}\left\{\frac{E-m}{2\sqrt{s}}|(\sqrt{s})\Lambda_1+\Lambda_2|^2\right.$$
$$\left.+\frac{E+m}{2\sqrt{s}}|(\sqrt{s})\Lambda_1-\Lambda_2|^2\right\}=\rho_1, \quad (50)$$

[7] A. Martin, Phys. Rev. Letters 9, 410 (1962).

$$\mathrm{Im}\Delta_2=\frac{1}{8\pi}\frac{k}{\sqrt{s}}\left\{\frac{E-m}{2}|(\sqrt{s})\Lambda_1+\Lambda_2|^2\right.$$
$$\left.-\frac{E+m}{2}|(\sqrt{s})\Lambda_1-\Lambda_2|^2\right\}=\rho_1(\sqrt{s})-\rho_2. \quad (51)$$

Taking linear combinations of (50) and (51), one obtains

$$\mathrm{Im}[(\sqrt{s})\Delta_1\pm\Delta_2]=\frac{1}{8\pi}\frac{k}{\sqrt{s}}(E\mp m)|(\sqrt{s})\Lambda_1\pm\Lambda_2|^2. \quad (52)$$

The discussion of Sec. 2, concerning the behavior of the propagator as $Z\to\infty$ would apply here, but a zero of Δ_1 will not correspond to a pole of the vertex function. One defines the vertex function by

$$g\Gamma=g(\Gamma_1\boldsymbol{P}+\Gamma_2)\gamma_5=\Delta^{-1}\Lambda. \quad (53)$$

Then

$$(\sqrt{s})\Lambda_1\pm\Lambda_2=\pm g[(\sqrt{s})\Gamma_1\pm\Gamma_2][(\sqrt{s})\Delta_1\pm\Delta_2]. \quad (54)$$

It can readily be established that the amplitudes $(\Lambda_1\sqrt{s}\pm\Lambda_2)$ correspond to the even (+) and odd (−) states of $n\pi$ in their center-of-mass system or to the $P_{1/2}$ and S states, respectively. If we denote by $f_{1/2+}$ and $f_{1/2-}$ the $P_{1/2}$ and S partial-wave amplitudes for $n\pi$ elastic scattering, unitarity in the elastic region gives

$$\mathrm{Im}(\Lambda_1\sqrt{s}\pm\Lambda_2)=k(\Lambda_1\sqrt{s}\pm\Lambda_2)f_{1/2\pm}{}^*. \quad (55)$$

It is now convenient to use the variable $\kappa=\sqrt{s}$ and regard $\Delta_\pm(\kappa)=\pm(\Delta_1\kappa\pm\Delta_2)$ as analytic continuations of one another in the κ plane. The function $\Delta_+(\kappa)$ is a Herglotz function analytic in the complex κ plane cut along the segments $(\pm\geq s_0,\pm\infty)$ and with a pole at $\kappa=m$. For sufficiently small values of the coupling constant, $(\kappa-m)\Delta_+$ is negative in the interval $(-\sqrt{s_0},+\sqrt{s_0})$. Any zero of Δ_+ has to come from adjacent sheets around the branch points at $\kappa=\pm\sqrt{s_0}$. Following the arguments developed in Sec. 2, one concludes from (54) and the unitarity relations (52) and (55) that the zeros of $\Delta_\pm$ are associated with poles of $(\Gamma_1\sqrt{s}\pm\Gamma_2)$. Since $\Delta_+(\kappa)-\Delta_+(-\kappa)=2\kappa\Delta_1$, as long as Δ_+ does not develop a zero, neither would Δ_1 have any zeros. As one increases the coupling constant $\Delta_+(\kappa)$ may develop a zero through either branch point $\pm\sqrt{s_0}$. If $\Delta_+(\kappa)$ develops a zero through both branch points and κ_+ and $-\kappa_-$ are the positions of these zeros, then $\Delta_1(s)$ will have a zero in the interval (κ_+^2,κ_-^2). Given the fact that $\Delta_1(s)$ has no poles other than that at $s=m^2$, one can derive an inequality similar to (25).

One may also obtain an upper bound on the renormalization constant Z_2 from the inequality

$$Z_2^{-1}\geq 1+\frac{1}{8\pi^2}\left(\int_{s_0}^{\infty}-\int_{-\infty}^{-s_0}\right)$$
$$\times(E'-m)|\Lambda_1\kappa'+\Lambda_2|^2\frac{k'}{\kappa'}d\kappa'=1+\Phi. \quad (56)$$

Since $(\Lambda_1\kappa+\Lambda_2)$ is analytic in the κ plane, Φ has a minimum which can be obtained by mapping the two cut plane onto the unit circle and making use of certain properties of orthogonal polynomials.[8] For pion-nucleon interaction (taking into account the isospin factor) one finds $\Phi\geq(1/85)g^2/4\pi$.

We shall now see that the poles of Γ_1 and Γ_2 can be related to Regge poles as in the case of scalar particles. The sum of Feynman diagrams of the first class, in the terminology of Sec. 5, will be

$$g^2\bar{\Gamma}(s-i\epsilon)\Delta(s+i\epsilon)\Gamma(s+i\epsilon). \quad (57)$$

They contribute to the amplitudes $f_{1/2\pm}$ a term

$$\frac{g^2}{4\pi}\frac{E\mp m}{2\sqrt{s}}(\Gamma_1\sqrt{s}\pm\Gamma_2)^2(\Delta_1\sqrt{s}\pm\Delta_2). \quad (58)$$

Now one must write the representations for the covariant amplitudes A and B, with the minimum number of subtractions required by perturbation theory. According to Mandelstam[9] the absorptive amplitudes in the s channel A_1 and B_1 have a representation of the form

$$A_1(s,t)=\sigma_1(s)+\frac{1}{\pi}\int\frac{\rho_{12}(s,u')}{u'-u}du'+\frac{1}{\pi}\int\frac{\rho_{13}(s,t')}{t'-t}dt', \quad (59)$$

where the term $\sigma_1(s)$ comes from (57). It follows that the absorptive amplitudes for partial waves with $j>\frac{1}{2}$ can be expressed in terms of A_{1l} (and B_{1l}) given by

$$A_{1l}{}^\pm(s)=\frac{1}{\pi}\int_{t_0}^{\infty}\rho_{13}(s,t')Q_l(z')\frac{dt'}{2k^2}$$
$$\pm\frac{1}{\pi}\int_{u_0}^{\infty}\rho_{12}(s,u')Q_l(z'')\frac{du'}{2k^2}. \quad (60)$$

From now on we drop the superscripts indicating the signature. Likewise, the discontinuity across the left-hand cut of the partial-wave amplitude, as defined in the complex l plane by an expression like (36), is always analytic in the half-plane $\mathrm{Re}l>-1$.[6] Since the partial waves are determined by their discontinuities, then for small values of the coupling constant they will be given by

$$f_{j\pm}=\frac{g^2}{4\pi}\frac{E\mp m}{2\sqrt{s}}\Gamma_\pm(s)^2\Delta_\pm(s)\delta_{j1/2}+h_{j\pm}(s), \quad (61)$$

where $h_{j\pm}(s)$ is an analytic function of j and corresponds to the contribution of the diagrams of the second class. Since the Feynman amplitudes require only one subtraction, the partial waves with $j\geq\frac{3}{2}$ coincide with the analytic continuation of $h_{j\pm}$ as in the case of scalar particles. On the other hand, in the dispersion relations

[8] It is much simpler to obtain $\Phi_{\min}$ with this mapping than with that used in Ref. 1.

[9] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

for the helicity amplitudes with $j=\frac{1}{2}$ and free from kinematical singularities one subtraction is required. This would contribute an additional term to (61) of the form $b_\pm(s)=(c\sqrt{s})[1\pm(d/\sqrt{s})]$ where c and d are real constants. However, the unitarity relations in the elastic region for $\Lambda_\pm$, $\Delta_\pm^{-1}$ and $h_{1/2\pm}$ would imply that either $c=0$, or

$$h_{1/2\pm}=-\frac{b_\pm}{2}-\frac{1}{2ik}(1-(1-k^2b_\pm{}^2)^{1/2})$$

which has branch points at $k^2b_\pm{}^2=1$, in contradiction with the Mandelstam representation. This is analogous to the result obtained by Martin.[7]

Let us assume that both sides of (61) can be analytically continued as a function of the coupling constant, even though the representation (60) no longer holds for Re$l<N$. Then, for $l<N$, the partial-wave amplitudes will be given by (61) where the last term is obtained by analytic continuation in j, from the region Re$j\geq N\mp\frac{1}{2}$.

If $h_{j\pm}$ has a pole which for $j=\frac{1}{2}$ is on the real axis (below threshold s_0), $\Gamma_\pm(s)$ would also have a pole at the same point, but this would not be a pole of the scattering amplitude. The nucleon should therefore be a fixed pole. Since $\Delta_\pm(s)$ has at most one zero, there cannot be more than one pole of $h_{1/2\pm}(s)$ (with the proper signature) in the interval (m^2,s_0). In addition, the pole of $h_{1/2+}(s)$, if any, must lie to the right of $s=m^2$, because no zero of $\Delta_+(s)$ can reach that point.

7. CONCLUDING REMARKS

We have explored the possibility of defining the physical amplitudes by analytic continuation in the coupling constant and used the properties of the perturbation expansion to derive the following results:

(a) Whenever the propagator has a zero, the vertex function has a pole, but not the scattering amplitude. This invalidates G–I's argument leading to a bound on the scattering amplitude. However, the condition $Z\geq 0$ may actually imply a bound on the coupling constant as exemplified by the Zachariasen model.

(b) If the particle under consideration is not a Regge pole, the vertex function will have a pole whenever there is a Regge pole with the same quantum numbers and the proper signature. The Regge pole is cancelled in the physical-scattering amplitude by the pole of $(g^2/8\pi)\Gamma\Delta\Gamma$. However, it would influence the high-energy behavior of the scattering amplitude just as any Regge pole and could show up as a resonance in higher partial waves. Hence, in principle, a pole of Γ could be experimentally observed. The nucleon would fit into this picture if one assumes interactions involving only the coupling between spin-$\frac{1}{2}$ baryons and spin-zero mesons. On the other hand, if the particle corresponds to a Regge pole, there seems to be no simple physical interpretation for the pole of Γ.

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