Evidence That the Deuteron Is Not an Elementary Particle*

STEVEN WEINBERG†
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 30 September 1964)

If the deuteron were an elementary particle then the triplet n-p effective range would be approximately $-ZR/(I-Z)$, where $R=4.31$F is the usual deuteron radius and Z is the probability of finding the deuteron in a bare elementary-particle state. This formula is model-independent, but has an error of the order of the range $m_\pi^{-1}=1.41$F of the n-p force, so it becomes exact only in the limit of small deuteron binding energy, i.e., $R\gg m_\pi^{-1}$. The experimental value of the effective range is not of order R and negative, but rather of order m_π^{-1} and positive, so Z is small or zero and the deuteron is mostly or wholly composite.

I. INTRODUCTION

MANY physicists believe that low-energy experiments can never decide whether a given particle is composite or elementary. I will try to show here that low-energy n-p scattering data already provide very strong model-independent evidence that the deuteron is in fact composite, or more precisely, that the probability Z of finding the deuteron in a bare elementary-particle state is very small.

This conclusion is based on a theorem proven in Secs. II and III, which give formulas[1] for the triplet n-p scattering length and effective range in the limit of small deuteron binding energy:

$$a_s=[2(1-Z)/(2-Z)]R+O(m_\pi^{-1}), \tag{1}$$

$$r_e=[-Z/(1-Z)]R+O(m_\pi^{-1}), \tag{2}$$

where Z is the famous deuteron "field renormalization" constant, and R is the usual deuteron radius

$$R\equiv(2\mu B)^{-1/2}=4.31\text{ F} \tag{3}$$

with B the deuteron binding energy and μ the n-p reduced mass. The first terms in (1) and (2) are model-independent and become very large for small B, while the second terms called $O(m_\pi^{-1})$ cannot be calculated without specific information on the n-p interaction but are expected to be of the order of magnitude of the range $m_\pi^{-1}=1.41$ F, and will in any case become negligible for $B\to 0$. In actuality R is three times larger than m_π^{-1}, so the separation between terms in (1) and (2) is reasonably clear cut.

If the deuteron is purely composite then $Z=0$,[2] and (1) and (2) give in this case

$$a_s\approx R;\quad r_e=O(m_\pi^{-1}). \tag{4}$$

This is in agreement with the conclusions of simple potential theory, and, as is well known, it also agrees with the experimental values:

$$a_s=+5.41\text{ F};\quad r_e=+1.75\text{ F}. \tag{5}$$

In contrast, if the deuteron had an appreciable probability Z of being found in an elementary bare-particle state then a_s would be less than R, and more striking, *r_e would be large and negative.* This is clearly contradicted by the experimental values (5), so we may conclude that Z is small (say <0.2), and therefore the deuteron is at least mostly composite.[3]

The large values for both a_s and r_e when Z is not zero may suggest to the reader that the effective-range approximation,

$$k\cot\delta\cong-1/a_s+r_ek^2/2, \tag{6}$$

may itself break down when the deuteron is elementary. In fact, we will see that this does not happen; it is only the first two terms in the expansion of $k\cot\delta$ in powers of k^2 that become of order R^{-1} for $Z\neq 0$ and $k\cong 1/R$, the third and higher terms being smaller by powers of $(Rm_\pi)^{-1}$. One well-known consequence of (6) is the relation between a_s, r_e, and R

$$1/R=1/a_s+r_e/2R^2 \tag{7}$$

which is satisfied by (1) and (2) for all Z. It should be stressed that (7) itself tells us nothing about the elementarity of the deuteron, since (7) follows directly from the requirement that (6) give $\cot\delta=+i$ (i.e., $e^{2i\delta}=\infty$) when k is extrapolated to the deuteron pole at $k=i/R$. The true token that the deuteron is com-

* Research supported in part by the U. S. Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63 and in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.
[1] After deriving these formulas I became aware that they could also be obtained in the nonrelativistic limit of the Zachariasen model, as treated by J. S. Dowker, Nuovo Cimento **25**, 224 (1962), by using his Eq. (9) in his Eq. (13), and then passing to the limit $\mu R\gg 1$. However, Dowker's derivation does not show that for small binding energy this result is actually model-independent and hence applicable to the deuteron, and he does not make this application. (There seems to be a factor of 4 lost from Dowker's equation for the effective range, but his equation for $k\cot\delta$ is correct.)
[2] The use of $Z=0$ to distinguish composite from elementary particles has been discussed by many authors, including J. C. Howard and B. Jouvet, Nuovo Cimento **18**, 466 (1960); M. T. Vaughan, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961); R. Acharya, Nuovo Cimento **24**, 870 (1962); S. Weinberg, *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 683; A. Salam, Nuovo Cimento **25**, 224 (1962); J. S. Dowker, *ibid.* **25**, 1135 (1962); S. Weinberg, Phys. Rev. **130**, 776 (1963).
[3] The point that the experimental values (5) of r_e and a_s are consistent with $Z=0$ has been made by H. Ezawa, T. Muta, and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **29**, 877 (1963). However, these authors do not compute r_e and a_s for $Z\neq 0$, and hence miss the point that an elementary deuteron would entail a large negative n-p effective range.

posite is that r_e is small and positive rather than large and negative.

II. RELATIONS BETWEEN Z AND THE p-n-d COUPLING CONSTANT

We[4] split the total Hamiltonian $\mathcal{H}$ into a free-particle part $\mathcal{H}_0$ and an unspecified interaction V:

$$\mathcal{H}=\mathcal{H}_0+V. \tag{8}$$

We will restrict ourselves throughout to the pieces of $\mathcal{H}$, $\mathcal{H}_0$, and V referring to the subspace with the quantum numbers of a deuteron at rest, i.e., $J^P=1^+$, $N=2$, $T=0$, $S=0$, and $\mathbf{P}=0$. The continuum eigenstates of $\mathcal{H}_0$ with these quantum numbers will be called $|\alpha\rangle$:

$$\mathcal{H}_0|\alpha\rangle=E(\alpha)|\alpha\rangle, \tag{9}$$

$$\langle\beta|\alpha\rangle=\delta(\beta-\alpha), \tag{10}$$

the label α referring to the types of particles present and to their momenta and spins. In addition there may or may not be discrete "bare elementary-particle" states $|n\rangle$, with

$$\mathcal{H}_0|n\rangle=E_n|n\rangle, \tag{11}$$

$$\langle\alpha|n\rangle=0, \tag{12}$$

$$\langle m|n\rangle=\delta_{mn}. \tag{13}$$

The completeness relation for the eigenstates of $\mathcal{H}_0$ reads

$$1=\sum_n |n\rangle\langle n|+\int d\alpha\ |\alpha\rangle\langle\alpha|. \tag{14}$$

It is the presence of discrete terms in this sum that distinguishes an elementary from a composite particle, both from the point of view of Levinson's theorem,[5] and in our low-energy approach.

The physical one-deuteron state is a normalized eigenstate of $\mathcal{H}$:

$$(\mathcal{H}_0+V)|d\rangle=-B|d\rangle, \tag{15}$$

$$\langle d|d\rangle=1. \tag{16}$$

Using the completeness relation (14) lets us write the normalization condition (16) as

$$1=Z+\int d\alpha|\langle\alpha|d\rangle|^2, \tag{17}$$

where

$$Z\equiv\sum_n |\langle n|d\rangle|^2. \tag{18}$$

We use Schrödinger's equation (15) to put Eq. (17) into its final exact form[6]

$$1-Z=\int d\alpha\frac{|\langle\alpha|V|d\rangle|^2}{(E(\alpha)+B)^2}. \tag{19}$$

We now suppose that B is very small, or more precisely, that $|\langle\alpha|V|d\rangle|$ is essentially equal to a constant g (the p-n-d coupling constant) for $E(\alpha)$ over an energy range from zero to some value E_0 large compared with B:

$$|\langle\alpha|V|d\rangle|\cong g \quad\text{for}\quad 0\leqq E(\alpha)\leqq E_0 \tag{20}$$

$$B\ll E_0. \tag{21}$$

We expect $E_0\approx m_\pi^2/2\mu$ so (21) is the same as the statement $R^2\gg m_\pi^{-2}$, which is reasonably well satisfied in reality. For small B the integral (19) nearly diverges, so it can be approximately evaluated by restricting $|\alpha\rangle$ to low-energy n-p states, replacing $|\langle\alpha|V|d\rangle|$ by g, and replacing the α integral with an integral over the energy E of the two-particle state, with

$$d\alpha=\rho(E)^{1/2}dE, \tag{22}$$

where ρ is the constant

$$\rho=4\pi k^2dk/(E)^{1/2}dE=4\pi/(2\mu^3)^{1/2} \quad (E\equiv k^2/2\mu). \tag{23}$$

Hence for small B we have

$$1-Z\cong g^2\rho\int_0^\infty\frac{(E)^{1/2}dE}{(E+B)^2} \tag{24}$$

or

$$g^2=2(B)^{1/2}(1-Z)/\pi\rho. \tag{25}$$

We see that g^2 takes its maximum value[7] when the deuteron is composite and has $Z=0$, while an elementary deuteron would have $0<Z<1$ and a coupling constant smaller by a factor $(1-Z)^{1/2}$.

Since Z determines the residue of the one deuteron pole, it can be measured by studying the effect of this pole where it shows up most clearly, that is, in n-p scattering at low energy.

III. CALCULATION OF LOW-ENERGY n-p SCATTERING

Let us first recall the derivation of the Low equation. The S matrix between general continuum states is

---

[4] The material of this section is largely contained in Sec. V of S. Weinberg, Phys. Rev. **130**, 776 (1963), and is repeated here for the reader's convenience.

[5] A proof of Levinson's theorem using this completeness relation was given by J. M. Jauch, Helv. Phys. Acta **30**, 143 (1957).

[6] A relativistic form of this sum-rule was given in articles by R. Acharya, J. S. Dowker, and S. Weinberg (1962); see Ref. 2. Relativistic calculations generally give Z^{-1} as a divergent integral, but it seems reasonable to hope that this is a failure of perturbation theory, and not an indication that Z is really zero for all particles. I am grateful to F. Low and S. Mandelstam for discussions of this point.

[7] This upper limit on g is of the same sort as discovered by M. A. Ruderman and S. Gasiorowicz, Nuovo Cimento **8**, 861 (1958); see also M. A. Ruderman, Phys. Rev. **127**, 312 (1962). The negative effective range for $Z\neq0$ can be understood as due to the decrease of the scattering length from its maximum value R, by reference to the relation (7).

given by

$$S_{\beta\alpha}=\delta(\beta-\alpha)-2i\pi\delta(E(\beta)-E(\alpha))T_{\beta\alpha}, \qquad (26)$$

where

$$T_{\beta\alpha}=\langle\beta|T(E_\alpha+i\epsilon)|\alpha\rangle$$

the operator $T(W)$ being defined by the Lippmann-Schwinger equation

$$T(W)=V+V[W-\mathcal{H}_0]^{-1}T(W). \qquad (27)$$

The formal solution of Eq. (27) is

$$T(W)=V+V[W-\mathcal{H}]^{-1}V. \qquad (28)$$

The second term in (28) can be evaluated by summing over the "in" continuum eigenstates of $\mathcal{H}$, plus the exact one-deuteron state $|d\rangle$:

$$T_{\beta\alpha}(W)=V_{\beta\alpha}+\frac{\langle\beta|V|d\rangle\langle\alpha|V|d\rangle^*}{W+B}+\int d\gamma\frac{\langle\beta|V|\gamma,\text{in}\rangle\langle\alpha|V|\gamma,\text{in}\rangle^*}{W-E(\gamma)}.$$

Setting $W=E_\alpha+i\epsilon$ and recalling that

$$T_{\beta\alpha}=\langle\beta|V|\alpha,\text{in}\rangle,$$

we obtain the Low equation in its exact form:

$$T_{\beta\alpha}=V_{\beta\alpha}+\frac{\langle\beta|V|d\rangle\langle\alpha|V|d\rangle^*}{E(\alpha)+B}+\int d\gamma\frac{T_{\beta\gamma}T_{\alpha\gamma}^*}{E(\alpha)-E(\gamma)+i\epsilon}. \qquad (29)$$

Let us now specialize, and suppose that α and β are n-p states with relative momenta $\mathbf{k}$ and $\mathbf{k}'$ respectively, the energies $\mathbf{k}^2/2\mu$ and $\mathbf{k}'^2/2\mu$ being taken of order B or less. The second term in (29) then becomes $|g|^2/(E(a)+B)$ which is of order $1/(B)^{1/2}$ and hence much larger than the first term $V_{\beta\alpha}$. The deuteron pole may give a large value to the unitarity integral so it would not be consistent to neglect the third term[8] in (29), but we can assume it to be dominated by terms for which γ is itself a low-energy $(E(\gamma)\lesssim B)$ n-p state. Therefore for small B the Low equation becomes

$$T_{\mathbf{k}'\mathbf{k}}=\frac{|g|^2}{E(\mathbf{k})+B}+\int d^3\mathbf{k}''\frac{T_{\mathbf{k}'\mathbf{k}''}T_{\mathbf{k}\mathbf{k}''}^*}{E(\mathbf{k})-E(\mathbf{k}'')+i\epsilon} \qquad (30)$$

with $E(\mathbf{k})=\mathbf{k}^2/2\mu$. The solution can evidently be found in the form

$$T_{\mathbf{k}'\mathbf{k}}=t(E(\mathbf{k})) \qquad (31)$$

with $t(E)$ satisfying the one-dimensional integral equation

$$t(E)=\frac{|g|^2}{E+B}+\rho\int_0^\infty(E'')^{1/2}\frac{|t(E'')|^2}{E-E''+i\epsilon}dE'' \qquad (32)$$

and ρ given again by

$$\rho=4\pi k^2dk/(E)^{1/2}dE=4\pi/(2\mu^3)^{1/2}. \qquad (23)$$

Equation (25) gives the S-wave phase shift $\delta(E)$ in terms of the solution of (32) as

$$e^{2i\delta(E)}=1-2i\pi\rho(E)^{1/2}t(E). \qquad (33)$$

The procedure for solving equations like (32) is well known, but for completeness we review it in an Appendix. The answer is

$$t(E)=\left[\frac{(E+B)}{g^2}+\frac{\pi\rho(B-E)}{2(B)^{1/2}}+i\pi\rho(E)^{1/2}\right]^{-1}. \qquad (34)$$

Using (34) in (33) gives

$$\cot\delta=i-1/\pi\rho(E)^{1/2}t(E)$$

$$=-\frac{1}{\pi\rho(E)^{1/2}}\left[\frac{(E+B)}{g^2}+\frac{\pi\rho(B-E)}{2(B)^{1/2}}\right]; \qquad (35)$$

so in terms of $k\equiv(2\mu E)^{1/2}$ and $R^{-1}\equiv(2\mu B)^{1/2}$, we find that $\cot\delta$ is "exactly" given by the effective-range formula:

$$k\cot\delta=-1/a_s+r_ek^2/2 \qquad (36)$$

with

$$a_s=2R\left[1+\frac{2(B)^{1/2}}{\pi\rho g^2}\right]^{-1}, \qquad (37)$$

$$r_e=R\left[1-\frac{2(B)^{1/2}}{\pi\rho g^2}\right]. \qquad (38)$$

It now only remains to use Eq. (25), which gives $\pi\rho g^2/2(B)^{1/2}=1-Z$, and we finally have the promised formulas:

$$a_s=[2(1-Z)/(2-Z)]R \qquad (39)$$

$$r_e=-[Z/(1-Z)]R. \qquad (40)$$

It need not be reiterated that our derivation is only valid in the limit of small binding energy.

IV. REMARKS

We have found the deuteron to be composite, but this hardly comes as a surprise. Is there any particle of high-energy physics whose elementary or composite character can be unmasked by the methods of this article?

The requirements for our analysis to be applicable to a given particle are:

[8] Using the pole term alone would give a nonunitary S-matrix except at zero energy, and would give a scattering length $a_s=2R(1-Z)$, which is wrong by a factor $2-Z$. I would like to thank F. Low for warning me that ignoring the unitarity integral gives a scattering length too large by a factor 2 in the usual case of $Z=0$.

(i) The particle must be stable; else Z is undefined. (However, it may be an adequate approximation to ignore the decay modes of a very narrow resonance.)

(ii) The particle must couple to a two-particle channel with threshold not too much above the particle mass.

(iii) It is crucial that this two-body channel have zero orbital angular momentum l, since for $l\neq 0$ the factor $(E)^{1/2}$ in the integrands of (24) and (32) would be $E^{l+(1/2)}$, and the integrals could not be approximated by their low-energy parts.

In every case where (i) and (ii) are satisfied we find that (iii) is not, usually because the negative parity of most mesons forces the two-particle channel to have $l=1$. (For instance, the nucleon couples to the p-wave pion-nucleon channel.) One begins to suspect that Nature is doing her best to keep us from learning whether the "elementary" particles deserve that title.

APPENDIX: SOLUTION OF THE INTEGRAL EQUATION

We wish to solve the nonlinear integral equation

$$t(E)=\frac{g^2}{E+B}+\rho\int_0^\infty \frac{(E')^{1/2}|t(E')|^2}{E-E'+i\epsilon}dE'. \tag{A1}$$

First define a function $\tau(W)$ of a complex energy variable W, by

$$\tau(W)=\frac{g^2}{W+B}+\rho\int_0^\infty \frac{(E)^{1/2}|t(E)|^2}{W-E}dE \tag{A2}$$

so that

$$t(E)=\tau(E+i\epsilon). \tag{A3}$$

The function $\tau(W)$ is analytic with negative-definite imaginary part for $\mathrm{Im}W>0$, so $\tau^{-1}(W)$ is analytic with positive definite imaginary part for $\mathrm{Im}W>0$; it therefore has the representation[9]

$$\tau^{-1}(W)=\tau^{-1}(W_0)+(W-W_0)\tau^{-1\prime}(W_0)+(W-W_0)^2\int_{-\infty}^\infty \frac{\sigma(E)}{(E-W_0)^2(E-W)}dE, \tag{A4}$$

where $\sigma(E)$ is real and positive, and W_0 is an arbitrary subtraction point. We note from (A2) that

$$\tau^{-1}(-B)=0 \quad\text{and}\quad \tau^{-1\prime}(-B)=1/g^2; \tag{A5}$$

so taking $W_0=-B$ lets us write (A4) as

$$\tau^{-1}(W)=\frac{(W+B)}{g^2}+(W+B)^2\int_{-\infty}^\infty \frac{\sigma(E)}{(E+B)^2(E-W)}dE. \tag{A6}$$

For $E>0$ we can calculate $\sigma(E)$ as

$$\sigma(E)=\mathrm{Im}\tau^{-1}(E+i\epsilon)/\pi=-\mathrm{Im}\tau(E+i\epsilon)/\pi|t(E)|^2;$$

so (A2) gives

$$\sigma(E)=\rho(E)^{1/2}. \tag{A7}$$

For $E\leqq 0$, $\tau^{-1}(E)$ is real, so $\sigma(E)$ vanishes except for possible δ functions where $\tau(E)=0$. We will first look for a solution with no zeros, and will come back to the general case later. Thus $\sigma(E)$ is given by (A7) for all E, and

$$\tau^{-1}(W)=\frac{(W+B)}{g^2}+\rho(W+B)^2\int_0^\infty \frac{(E)^{1/2}}{(E+B)^2(E-W)}dE =\frac{(W+B)}{g^2}+\frac{\pi\rho[(-W)^{1/2}-(B)^{1/2}]^2}{2(B)^{1/2}}. \tag{A8}$$

Setting $W=E+i\epsilon$ gives finally

$$t(E)=[(E+B)/g^2+\pi\rho(B-E)/2(B)^{1/2}+i\pi\rho(E)^{1/2}]^{-1}. \tag{A9}$$

It is easy to check directly that (A9) does in fact satisfy (A1).

There may also be solutions for which $\tau(W)$ has Castillejo-Dalitz-Dyson zeros on the negative real axis. It is easy to see that there can be at most one zero, at a value $W=-C$ with

$$0<C<B,$$

because $\tau(W)$ is negative for $W<-B$, and decreases monotonically for $-B\leqq W\leqq 0$. If $\tau(W)$ does have such a zero then (A8) is replaced by

$$\tau^{-1}(W)=-\frac{(W+B)^2}{f^2(W+C)}+\frac{(W+B)}{g^2}+\frac{\pi\rho[(-W)^{1/2}-(B)^{1/2}]^2}{2(B)^{1/2}}. \tag{A10}$$

In this case the effective-range expansion of $k\cot\delta$ in powers of E will have radius of convergence equal to C, and will therefore be useless at energies $E\cong B>C$. We reject this class of solutions because the effective-range approximation (6) gives good agreement with experiment up to energies larger than B.

Note added in proof. The conclusion stated in the preprint of this article has been challenged by R. Blankenbecler, M. L. Goldberger, K. Johnson, and S. B. Treiman. The very cogent criticisms of this formidable team (called BGJT below) force me to state more precisely what I claim to prove, and how this applies to the real deuteron.

Our theorem is that the triplet n-p scattering length and effective range are given exactly by formulas (39) and (40), in the limit as the deuteron binding energy B

[9] A. Herglotz, Ber. Verhandl. Sachs. Ges. Wiss. Leipzig. Math-Phys. **63**, 501 (1911); J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943), Chap. II.

vanishes, if Z is held fixed at any value $0<Z<1$. However, B and Z depend on all the parameters of the Hamiltonian, so it is necessary to clarify what we mean by $B\to 0$ with Z fixed. Suppose there to be one bare elementary deuteron state $|d_0\rangle$ with unrenormalized energy $-B_0$:

$$H_0|d_0\rangle=-B_0|d_0\rangle. \tag{N1}$$

We may write its coupling to the continuum free-particle states $|\alpha\rangle$ in terms of a form factor U_α and an unrenormalized coupling constant g_0:

$$\langle\alpha|V|d_0\rangle=g_0U_\alpha. \tag{N2}$$

Then B and Z may be regarded as functions of B_0 and g_0, or vice versa. We claim that (39) and (40) become exact if B_0 and g_0 are varied in such a way that $B\to 0$ with fixed Z, the form factor U_α and interaction $V_{\beta\alpha}$ being held fixed throughout.

This theorem could not be completely proved by using the Low equation (30), since (as pointed out by BGJT) its solutions can have an unlimited number of CDD zeros in the unitarity cut, as well as between $E=-B$ and $E=0$, and Eq. (30) tells us nothing about the dependence on B and Z of the location and strength of these zeros. Instead, we shall prove our theorem by explicitly isolating all effects of the virtual bare deuteron. To this end, we define an operator $T^{(1)}(W)$ as what $T(W)$ would be if we could ignore $|d_0\rangle$ in sums over intermediate states. That is,

$$T^{(1)}(W)=V+V\Lambda[W-H_0]^{-1}T^{(1)}(W), \tag{N3}$$

where $\Lambda\equiv 1-|d_0\rangle\langle d_0|$ is the projection operator on the continuum. It is well known[10] that the T matrix defined by Eq. (27) can be expressed in terms of $T^{(1)}(W)$, as

$$T_{\beta\alpha}(W)=T_{\beta\alpha}^{(1)}(W)+|g_0|^2\Gamma_\beta(W)\Gamma_\alpha^*(W^*)\Delta(W), \tag{N4}$$

with vertex functions defined by

$$g_0\Gamma_\beta(W)=\langle\beta|T^{(1)}(W)|d_0\rangle \tag{N5}$$

and with propagator

$$\Delta(W)=[W+B_0-|g_0|^2F(W)]^{-1}, \tag{N6}$$

$$|g_0|^2F(W)\equiv\langle d_0|T^{(1)}(W)|d_0\rangle. \tag{N7}$$

The deuteron binding energy B and renormalization constant Z are determined by the requirement[11] that $\Delta(W)$ have a pole at $W=-B$ with residue Z. This gives B_0 and $|g_0|$ explicitly, as

$$|g_0|^{-2}=-(Z/(1-Z))F'(-B), \tag{N8}$$

$$B_0=B+|g_0|^2F(-B). \tag{N9}$$

[10] See, e.g., G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955), or Ref. 11.

[11] S. Weinberg, Phys. Rev. **130**, 776 (1963), Sec. IV.

Hence, we can eliminate B_0 and g_0 from (N4), obtaining

$$T_{\beta\alpha}(W)=T_{\beta\alpha}^{(1)}(W)+\Gamma_\beta(W)\Gamma_\alpha^*(W^*)\times\left[-\left(\frac{Z}{1-Z}\right)F'(-B)(W+B)+F(-B)-F(W)\right]^{-1}. \tag{N10}$$

It is crucial that $\Gamma_\alpha(W)$, $T_{\alpha\beta}^{(1)}(W)$, and $F(W)$ are entirely independent[12] of B_0 and g_0, being determined by U_α and $V_{\beta\alpha}$ through the equations

$$T_{\beta\alpha}^{(1)}(W)=V_{\beta\alpha}+\int d\gamma V_{\beta\gamma}[W-E(\gamma)]^{-1}T_{\gamma\alpha}^{(1)}(W),$$

$$\Gamma_\beta(W)=U_\beta+\int d\alpha T_{\beta\alpha}^{(1)}(W)[W-E(\alpha)]^{-1}U_\alpha,$$

$$F(W)=\int d\alpha U_\alpha^*[W-E(\alpha)]^{-1}\Gamma_\alpha(W).$$

Therefore, these functions are to be regarded as independent of B and Z (in the sense discussed above), so that Eq. (N10) displays explicitly the complete dependence of $T_{\beta\alpha}(W)$ on B and Z.

So far, everything has been general and exact. Now let us specialize the transition $\alpha\to\beta$ to be an elastic triplet n-p scattering $\mathbf{k}\to\mathbf{k}'$, so that $W=E+i\epsilon$, with $E=E(\mathbf{k})=E(\mathbf{k}')$. Let us also take $B\to 0$ (with fixed Z), keeping E of order B or less. In this limit, the vertex function $\Gamma_{\mathbf{k}}(E\pm i\epsilon)$ approaches some constant Γ_0. In order to compute the behavior of the denominator in (N10), we note that (N3) gives the exact relation

$$T^{(1)}(W)=T^{(1)}(0)+WT^{(1)}(0)[W-H_0]^{-1}H_0^{-1}T^{(1)}(W), \tag{N11}$$

so that

$$F(W)=F(0)+W\int d\alpha\frac{\Gamma_\alpha^*(0)\Gamma_\alpha(W)}{[W-E(\alpha)]E(\alpha)}. \tag{N12}$$

In the limit of small W, the integrand in (N12) will diverge at $E(\alpha)=0$, and may therefore be evaluated by integrating only over the very low-energy n-p states, for which we may replace $\Gamma_\alpha(0)$ and $\Gamma_\alpha(W)$ by Γ_0, and $d\alpha$ by $\rho E^{1/2}dE$. In this limit, then,

$$F(W)-F(0)\to W\rho|\Gamma_0|^2\int_0^\infty\frac{E^{1/2}dE}{(W-E)E}=-i\pi\rho|\Gamma_0|^2W^{1/2},$$

[12] This is not quite true if we allow the bare elementary deuteron to show up in states like $d_0+\pi^++\pi^-$, which would have to be included in a relativistic theory. However these states do not contribute to $T^{(1)}(W)$ for $B\to 0$, since (N8) and (N12) show that g_0 vanishes like $B^{1/4}$ in this limit. The reason that the virtual deuteron state can contribute to low-energy n-p scattering is that the vanishing of $|g_0|^2$ is more than compensated by a vanishing denominator; this happens only for states near zero energy.

the phase of W being taken between 0 and 2π. The second term in (N10) will now be of order $B^{-1/2}$, while the first merely approaches a constant, and hence may be ignored. Putting all this together, we find that the unknown constant $|\Gamma_0|^2$ cancels, and

$$\pi\rho T_{\mathbf{k}'\mathbf{k}}\to\left[\left(\frac{Z}{1-Z}\right)\left(\frac{E+B}{2B^{1/2}}\right)+iE^{1/2}+B^{1/2}\right]^{-1},$$

so finally

$$-E^{1/2}\cot\delta(E)\to\left(\frac{Z}{1-Z}\right)\left(\frac{E+B}{2B^{1/2}}\right)+B^{1/2},\quad\text{(N13)}$$

this result being valid over a range $0<E\lesssim B$, in the limit $B\to 0$. The effective range and scattering length deduced from (N13) agree with our previous formulas (39) and (40).

It is of course impossible to say with absolute certainty that the actual value $B=2.2$ MeV is small enough to allow this theorem to be applied to the real deuteron. If B were 2.2 eV instead of 2.2 MeV we would have little doubt; a Z value as small as 0.01 would then give an effective range of -43 F, and this could hardly be mistaken for the one or two Fermis expected for $Z=0$! However, there is one objective indication that even $B=2.2$ MeV is small enough: The success of the effective-range approximation (36) over a range of energies larger than B.

To see the relevance of the empirical success of the effective-range approximation, we may try to go one step beyond the limiting formula (N13). The most convenient formalism for this purpose is that of the Heitler reaction-matrix theory. As long as E is below all inelastic thresholds (and low enough to ignore the D-wave admixture) the $J=1^+$ T operator will be given by

$$T^{(1)}(E+i\epsilon)=K^{(1)}(E)-i\pi\rho E^{1/2}\phi(E)K^{(1)}(E)|E\rangle\langle E|K^{(1)}(E),\quad\text{(N14)}$$

where $|E\rangle$ is the S-wave state with energy E, normalized to

$$\rho E^{1/2}\langle E'|E\rangle=\delta(E-E')$$

and $K^{(1)}(E)$ is the reaction operator corresponding to $T^{(1)}(E+i\epsilon)$, defined by

$$K^{(1)}(E)=V+V\Lambda P[E-H_0]^{-1}K^{(1)}(E),\quad\text{(N15)}$$

with P denoting "principal value." The function $\phi(E)$ is

$$\phi(E)\equiv[1+i\pi\rho E^{1/2}\kappa(E)]^{-1},\quad\text{(N16)}$$

$$\kappa(E)\equiv\langle E|K^{(1)}(E)|E\rangle.\quad\text{(N17)}$$

Equation (N14) gives the $T^{(1)}$-matrix, vertex functions, and self-energy function as

$$\langle E|T^{(1)}(E+i\epsilon)|E\rangle=\kappa(E)\phi(E),\quad\text{(N18)}$$

$$\Gamma_E(E+i\epsilon)=\gamma(E)\phi(E),\quad\text{(N19)}$$

$$\Gamma_E^*(E-i\epsilon)=\gamma^*(E)\phi(E),\quad\text{(N20)}$$

$$F(E+i\epsilon)=\lambda(E)-i\pi\rho E^{1/2}|\gamma(E)|^2\phi(E),\quad\text{(N21)}$$

where

$$g_0\gamma(E)\equiv\langle E|K^{(1)}(E)|d_0\rangle,\quad\text{(N22)}$$

$$|g_0|^2\lambda(E)\equiv\langle d_0|K^{(1)}(E)|d_0\rangle.\quad\text{(N23)}$$

Since $K^{(1)}(E)$ is Hermitian, $\kappa(E)$ and $\lambda(E)$ are real, and (N4) now gives the (manifestly unitary) result

$$-\pi\rho E^{1/2}\cot\delta(E)=\left[\kappa(E)+\frac{|\gamma(E)|^2|g_0|^2}{E+B_0-|g_0|^2\lambda(E)}\right]^{-1}.\quad\text{(N24)}$$

We would normally expect the operator $K^{(1)}(E)$ to be constant for energies below a few MeV, so that we can take κ, γ, and λ as constants in this energy range. [Recall that $\kappa(E)$, $\gamma(E)$, and $\lambda(E)$ are independent of B_0 and g_0, and hence of B.] With this assumption, $F(W)$ is given by

$$F(W)=\lambda-i\pi\rho W^{1/2}|\gamma|^2/(1+i\pi\rho\kappa W^{1/2}).\quad\text{(N25)}$$

We can then find $|g_0|$ and B_0 from (N8) and (N9), and using these values in (N24) yields

$$-(E)^{1/2}\cot\delta(E)=B^{1/2}-\frac{(E+B)(E_0+B)}{2B^{1/2}(E-E_0)}\left(\frac{Z}{1-Z}\right),\quad\text{(N26)}$$

with

$$E_0=-B\left\{1+2\left(\frac{1-Z}{Z}\right)\left(\frac{1}{\pi\rho\kappa B^{1/2}}-1\right)\right\}.\quad\text{(N27)}$$

Formula (N26) would give the scattering amplitude a CDD zero at E_0. This possibility was already encountered in the Appendix, and we may repeat the arguments made there. If $|E_0|$ were of order B or smaller then the effective-range expansion would fail for energies comparable to B, in contradiction with experiment. [For example, formula (7) would no longer hold.] On the other hand, if $|E_0|\gg B$ then for energies of order B or less (N26) becomes identical with (N13), and we regain our formulas (39) and (40) for a_s and r_e. Of course, (N27) tells us that $|E_0|\gg B$ if and only if

$$B\ll\left[\frac{1-Z}{\pi\rho\kappa Z}\right]^2\quad\text{(N28)}$$

and the success of the effective-range approximation tells us that (N28) *is* satisfied for $B=2.2$ MeV.

However, as pointed out by BGJT, there might be other kinds of CDD zero which can alter some of these conclusions. For instance, suppose that the n-p system had a very low-energy resonance or a very shallow bound state at energy E_1 in the absence of virtual one-deuteron states. Then $K^{(1)}(E)$ would have a pole at $E=E_1$, and might be approximated over the low-energy region by

$$K^{(1)}(E)=|\psi\rangle\langle\psi|/(E-E_1). \qquad \text{(N29)}$$

[Note that this makes $dK^{(1)}(E)/dE$ negative-definite, as is appropriate if the pole is to be regarded as a bound state of the inelastic channels.] We then find from (N17), (N22), and (N23) that

$$\kappa(E)\cong|\alpha|^2/(E-E_1)\,; \quad \alpha=\langle E_1|\psi\rangle, \qquad \text{(N30)}$$

$$\gamma(E)\cong\alpha\beta^*/(E-E_1)\,; \quad \beta=\langle d_0|\psi\rangle, \qquad \text{(N31)}$$

$$\lambda(E)\cong|\beta|^2/(E-E_1). \qquad \text{(N32)}$$

In this case, the CDD zero is at $E=-B_0$, since (N24) gives here

$$-\pi\rho E^{1/2}\cot\delta(E)=|\alpha|^{-2} \times\{E-E_1-|g_0|^2|\beta|^2/(E+B_0)\}. \qquad \text{(N33)}$$

Furthermore, (N21) gives

$$F(W)=|\beta|^2[W-E_1+i\pi\rho|\alpha|^2W^{1/2}]^{-1}, \qquad \text{(N34)}$$

so

$$|g_0|^2|\beta|^2=\frac{(B+E_1+2\eta B)^2}{(1+\eta)}\left(\frac{1-Z}{Z}\right), \qquad \text{(N35)}$$

and

$$B_0=B-\frac{|g_0|^2|\beta|^2}{B+E_1+2\eta B} =B-\frac{(B+E_1+2\eta B)}{1+\eta}\left(\frac{1-Z}{Z}\right), \qquad \text{(N36)}$$

with $\eta\equiv\pi\rho|\alpha|^2/2B^{1/2}$. In order to account for the validity of the effective-range approximation we must take $|B_0|\gg B$, and this is evidently only possible if Z is very small, which of course is the conclusion we are trying to draw. Incidentally, if $|B_0|\gg B$ then (N36) gives

$$-|g_0|^2|\beta|^2/B_0\cong B+E_1+2\eta B, \qquad \text{(N37)}$$

so (N33) becomes

$$-(E)^{1/2}\cot\delta(E)=B^{1/2}-(E+B)/2\eta. \qquad \text{(N38)}$$

In this case, the effective range can take any negative value it likes if Z is small, while a sizable Z is ruled out by the success of the effective-range approximation. Of course, these conclusions all depend upon the assumption that E_1 is comparable with B; if we were to let $B\to 0$ we should find ourselves back in the case already treated.

We could be even more general, and include several poles and a nonresonant background term in (N29). In this case the CDD zero is not at B_0, because we no longer have $|\gamma(E)|^2=\kappa(E)\lambda(E)$, and matters are much more complicated. We will leave it to the reader to decide for himself whether it is possible that the n-p forces could generate such a mess within a few MeV of zero energy and in just such a way as to preserve the effective-range formula. In any event, even though we can never be absolutely certain how shallow a bound state must be to allow the use of Eqs. (39) and (40), we are at least now sure that these formulas become exact in the limit of zero binding energy.

It would perhaps be worth mentioning that a preprint by G. Segré has quoted the present article as "Evidence that the Deuteron is an Elementary Particle." Lest the reader feel that I cannot make up my mind, let me stress that I have never wavered in my belief that the deuteron is composite.