the third component of isospin I_3. Usually one also adds the hypercharge Y to this list of particle attributes. Because of the Gell-Mann–Nishijima relation, the hypercharge may be considered redundant, however. All the quantum numbers mentioned above are supposedly conserved in strong interaction processes, but, of course, not in the presence of electromagnetic and weak interactions. Therefore, if it is legitimate to neglect all but the strong interactions at least as a first approximation, then it is consistent to use these quantum numbers to characterize the hadrons. In doing so we introduce the additional assumption that the corresponding infinitesimal generators of the internal symmetry group can be included in an Abelian set of operators which is sufficient to label the basis vectors in the representation space of the fundamental group G (maximal set of commuting observables). In this way we are supplied with additional restrictions on the commutation relations. However, they are only approximate and good to the extent that electromagnetic and weak interactions do not alter them in a significant way. Keeping in mind that the three types of interactions considered here differ both in strength and with respect to the symmetry properties they exhibit, it is alluring to speculate whether one in general can separate out the contributions that each kind of interaction gives to the commutators of the theory. It still remains to be shown that it is a consistent procedure to neglect certain contributions in an approximation scheme.

Some final remarks regarding our assumptions for the group G are in order. At first sight the assumptions may seem to be very general in nature and highly plausible from the point of view of physics. It should be kept in mind, though, that these assumptions are quite restrictive and one may have to relax some of them if the program described above fails to work. The necessity for the fundamental group G to contain S as a subgroup may well be questioned. It may also be worthwhile to consider discrete internal symmetry groups rather than continuous Lie groups. After all, physically realizable transformations belonging to the internal symmetry group are discrete. It seems to be primarily for historical reasons that continuous internal symmetry groups have been preferred so far. Finally, we emphasize once more that much work remains to be done on the problem of identification of the generators with physical observables.

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Approximate Method for Determining the Elastic-Scattering Amplitude for Strong Interactions*

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A method for calculating the elastic-scattering amplitude in the S-matrix theory of strong interactions is proposed which does not require a partial-wave expansion of the amplitude. Crossing symmetry is satisfied by the amplitude, but unitarity is imposed only approximately. Equations are derived for the case of scattering of two spinless particles of unequal mass. The special case where the masses are equal is considered in detail for the input assumption that the scattering is predominantly S wave. Crossing symmetry introduces higher partial-wave contributions to the amplitude. The amplitude calculated in this way is in good agreement with the input assumption. The amount of violation of unitarity is least near threshold, but is only on the order of a percent at $s=8m^2$. In spite of this, there are serious problems with low-energy resonances and bound states. It is concluded that both unitarity and crossing symmetry are important in the production of resonances and bound states and that the modification of either may lead to difficulties. The total cross section derived from the approximate amplitude is compared with that obtained using the partial-wave expansion and keeping only the S wave. The results are in good agreement with each other.

I. INTRODUCTION

IN the S-matrix theory of strong interactions,[1] if the Mandelstam representation is taken to embody the assumption of analyticity for processes which go from a two-particle initial state to a two-particle final state, then the problem reduces to the determination of the

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[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

spectral functions in the representation from the unitarity condition.

This was the original program of calculation suggested by Mandelstam[2] for the π-π problem. It was found that the program could be simplified by using the Mandelstam representation to derive equations for the partial wave amplitudes.[3] If only the lowest partial waves are considered, these equations should give adequate answers in the low-energy region, and become appreciably inaccurate only after inelastic effects become important. Inelastic processes present sufficient difficulties and are best avoided if possible, so this limitation of the equations is not serious.

The implications of unitarity for the partial-wave amplitudes are not very simple. The discontinuity across the right-hand cut is simple enough but the left-hand singularities, which must be there because of crossing symmetry, are a source of great difficulty. These singularities introduce coupling between the partial wave amplitudes. The determination of the discontinuity on the left-hand cut which specifies this coupling is the basic problem faced in solving the low-energy pion scattering equations.

Several methods have been suggested for determining the discontinuity[3,4]; each has its weak points.[5] It is sufficient to note that they all have a common feature; they sacrifice crossing symmetry in order to maintain unitarity in the physical region of the amplitude.

Two recent investigations of exactly soluble models indicate that the violation of crossing symmetry can have important consequences for the low-energy region.[6,7]

It is the purpose of this paper to investigate the possibility of an approximation method for the low-energy region which maintains crossing symmetry. In order to do this we must avoid the partial-wave expansion and treat the complete amplitude. The starting point for the method will be the fixed momentum transfer dispersion relation, and an approximation will be made in the unitarity condition which turns the dispersion relation into a soluble integral equation.

In the next section the general formulas will be derived for a neutral scalar model. The solutions in the lowest approximation will then be discussed for the case of equal masses in Sec. 3.

2. THE APPROXIMATE EQUATIONS

In this section we will consider the scattering of a spinless particle of mass m from a spinless particle of mass M. The fixed momentum transfer dispersion relation for the elastic scattering amplitude is[8]

$$T(s,t)=\frac{g^2}{s-M^2}+\frac{g^2}{s+t-M^2-2m^2}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}ds'\,\mathrm{Im}T(s',t)\times\left[\frac{1}{s'-s}+\frac{1}{s'+s+t-2M^2-2m^2}\right]. \quad (1)$$

Unitarity requires that

$$\mathrm{Im}T(s,t)=\frac{-\{[s-(M+m)^2][s-(M-m)^2]\}^{1/2}}{64\pi^2 s}\times\int d\Omega\, T^*(s,t')T(s,t'')+\text{inelastic terms}. \quad (2)$$

The integral is over the angles ϑ, and φ where

$$t=-\frac{[s-(M+m)^2][s-(M-m)^2]}{2s}(1-\cos\theta), \quad (3a)$$

$$t'=-\frac{[s-(M+m)^2][s-(M-m)^2]}{2s}(1-\cos\vartheta), \quad (3b)$$

$$t''=-\frac{[s-(M+m)^2][s-(M-m)^2]}{2s}(1-\cos\vartheta'), \quad (3c)$$

and

$$\cos\vartheta'=\cos\theta\cos\vartheta+\sin\theta\sin\vartheta\cos\varphi. \quad (3d)$$

This equation for $\mathrm{Im}T$ is not very useful in its present form; what is needed is an approximation which makes the unitarity condition tractable and is accurate in the low-energy region. Equation (2) can be rewritten

$$\mathrm{Im}T(s,t)=\frac{-\{[s-(M+m)^2][s-(M-m)^2]\}^{1/2}}{16\pi s}\times K(s,t)|T(s,t)|^2, \quad (4)$$

where

$$K(s,t)=\frac{1}{4\pi}\int d\Omega\frac{T^*(s,t')T(s,t'')}{T^*(s,t)T(s,t)}+\text{inelastic terms}. \quad (5)$$

Next, it will be shown that $K(s,t)$ can be approximated in a simple way for the low-energy region, and that, given $K(s,t)$, Eq. (1) can be solved for $T(s,t)$.

In the low-energy region we assume that the inelastic

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[8] The incident particles have masses M and m and four-momenta p_1 and p_2 and the scattered particles p_3, M and p_4, m. The variables are the usual $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $u=(p_1-p_4)^2$, which are related by $s+t+u=2M^2+2m^2$. The pole terms are included so that this model corresponds to the neutral scalar model for pion-nucleon scattering; $\hbar=c=1$.

contribution to K can be neglected and that only the lowest partial waves contribute to T. The partial-wave expansion is

$$T(s,t)=-16\pi\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta). \quad (6)$$

Let $X_l(s)=a_l(s)/a_0(s)$ for $l\geqslant 1$. If we assume that the X_l are small, substitution of the partial-wave expansion into the definition of K gives

$$\begin{aligned}K(s,t)\approx 1&-3(X_1^*+X_1)P_1(\cos\theta)\\&+3|X_1|^2P_1(\cos\theta)[1+3P_1(\cos\theta)]\\&+9(X_1^{*2}+X_1^2)[P_1(\cos\theta)]^2\\&-5(X_2^*+X_2)P_2(\cos\theta)+\cdots. \quad (7)\end{aligned}$$

This indicates that if the coupling between the partial waves due to crossing symmetry does not require that the higher partial waves be too strong then K may be approximated quite simply in the low-energy region by 1.

To solve the integral equation, obtained by the substitution of Eq. (4) in Eq. (1), it is convenient to proceed by defining a function $D(s,t)$ related to T by

$$T(s,t)=\left[\frac{g^2}{s-M^2}+\frac{g^2}{s+t-M^2-2m^2}+\lambda(t)\right]\Big/D(s,t). \quad (8)$$

It follows that

$$\begin{aligned}D(s,t)=\Lambda(t)&+\frac{(2s+t-2M^2-2m^2)}{16\pi^2}\\&\times\int_{(M+m)^2}^{\infty}ds'\frac{\{[s'-(M+m)^2][s'-(M-m)^2]\}^{1/2}}{s'(2s'-2M^2-2m^2+t)}\\&\times K(s',t)\left\{\frac{g^2}{s'-M^2}+\frac{g^2}{s'+t-M^2-2m^2}+\lambda(t)\right\}\\&\times\left\{\frac{1}{s'-s}-\frac{1}{s'+s+t-2M^2-2m^2}\right\} \quad (9)\end{aligned}$$

is a solution of the integral equation. Since the residues of the poles are g^2, $\Lambda(t)$ is given by the condition that

$$D(M^2,t)=1. \quad (10)$$

Once an approximate $K(s,t)$ is given, $T(s,t)$ is specified by Eqs. (8), (9), and (10). In the following sections the consequences of making the approximation that $K(s,t)=1$ will be investigated.

3. THE EQUAL-MASS CASE

To illustrate some of the properties of the approximate solution, we consider the case where $m=M$, and there is no single-particle pole, so $g^2=0$. Without loss of generality $\lambda(t)$ can be set equal to one in Eqs. (8), and (9).

In the case when $K=1$ it is convenient to introduce the function

$$\phi(x)=\int_1^{\infty}\frac{dx'}{[x'(x'-1)]^{1/2}}\frac{1}{(x'-x)}. \quad (11)$$

Some properties of this function are given in the appendix. In terms of this function and with the above assumptions Eq. (9) can be written

$$\begin{aligned}D(s,t)=\Lambda(t)+\frac{1}{64m^2\pi^2}\bigg\{&(s-4m^2)\phi\left(\frac{s}{4m^2}\right)\\&-(s+t)\phi\left(\frac{4m^2-s-t}{4m^2}\right)+(t+4m^2)\phi\left(\frac{4m^2-t}{4m^2}\right)\bigg\}. \quad (12)\end{aligned}$$

The coupling constant is defined as the value of T at the symmetry point

$$\lambda=T(4m^2/3,4m^2/3)=1/\Lambda(4m^2/3).$$

The functional form of $\Lambda(t)$ is determined from crossing symmetry. Symmetry under the interchange of s and u has already been imposed but, in the equal mass case, symmetry under the interchange of s and t is also expected. With this additional condition we arrive at the result

$$\begin{aligned}D(s,t)=\frac{1}{\lambda}-\frac{1}{8\pi^2}\phi(\tfrac{1}{3})+\frac{1}{64m^2\pi^2}\{&(s-4m^2)\phi(s/4m^2)\\&+(t-4m^2)\phi(t/4m^2)+(u-4m^2)\phi(u/4m^2)\}. \quad (13)\end{aligned}$$

The first question to investigate is whether the approximation is consistent. In particular, by how much is unitarity violated, and by how much does $K(s,t)$ calculated from the new amplitude differ from 1.

The unitarity of the approximation can be checked by use of the optical theorem. The theorem is obtained by taking $t=0$ in Eq. (2). For s less than the lowest inelastic threshold we have the exact result that

$$\mathrm{Im}T(s,0)=-\frac{1}{32\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}\int d\cos\theta|T(s,t)|^2. \quad (14)$$

On the other hand, the approximate amplitude, Eq. (13), satisfies

$$\mathrm{Im}T(s,0)=-\frac{1}{16\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}|T(s,0)|^2. \quad (15)$$

The right-hand side of Eq. (14) can be numerically integrated when T is the approximate amplitude and the result can be compared with the right-hand side of Eq. (15). The result of such a comparison is shown in Table I for various values of λ and s; the values of λ are chosen to maximize the difference.

Some information about the relative importance of the various partial waves can be obtained in the follow-

TABLE I. Tests of the consistency of the approximation $K=1$. The third column gives the fractional difference between $\frac{1}{2}\int d\cos\theta|T(s,t)|^2$ and $|T(s,0)|^2$. The fourth column gives the coefficient of $P_2(\cos\theta)$ in the expansion
$K(s,t)\approx 1-5(X_2^*+X_2)P_2(\cos\theta)$.

$s/4m^2$	λ/λ_c	Fractional difference	First correction to $K(s,t)$
1.2	1.76	0.002	−0.002
	1.00	0.003	−0.003
	0.97	0.002	−0.002
1.5	1.76	0.006	−0.006
	1.00	0.008	−0.009
	0.97	0.007	−0.009
2.0	1.76	0.014	−0.014
	1.00	0.021	−0.020
	0.97	0.019	−0.019
4.0	1.76	0.046	−0.043
	1.00	0.068	−0.052
	0.97	0.069	−0.050

ing way. Near threshold we can make an effective-range expansion of the partial-wave amplitude. If the phase shift for the lth partial wave is defined by

$$a_l(s)=\left(\frac{s}{s-4m^2}\right)^{1/2}\sin\delta_l(s)e^{i\delta_l(s)} \qquad (16)$$

then

$$(s-4m^2)^{l+\frac{1}{2}}\cot\delta_l=\alpha_l^{-1}+r_l(s-4m^2)+\cdots. \qquad (17)$$

If only the first term in this effective-range expansion is kept, then the amplitude is

$$T(s,t)=-16s^{1/2}\times\sum_{l=0}^{\infty}\frac{(2l+1)(s-4m^2)^l\alpha_lP_l(1+2t/(s-4m^2))}{[1-i(s-4m^2)^{l+\frac{1}{2}}\alpha_l]}. \qquad (18)$$

Very near threshold this series converges for $-4m^2<t<4m^2$; if, in addition, $t\ll s-4m^2$ then the lth Legendre polynomial can be approximated by

$$\frac{1\times3\times5\times\cdots\times(2l-1)}{l!}[1+2t/(s-4m^2)]^l.$$

After making this approximation and taking $s=4m^2$, we obtain

$$T(4m^2,t)=-32\pi m\sum_{l=0}^{\infty}\frac{(2l+1)!}{(l!)^2}\alpha_lt^l. \qquad (19)$$

A similar power series expansion of the approximate amplitude can be made; comparison of the two expansions will determine α_l.

By using the expansion (A5) for $\phi(x)$, we obtain the expansion of the approximate solution,

$$T(4m^2,t)=\frac{4\pi^2\lambda}{C}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{4^nC^n}\times\left[\sum_{p=1}^{\infty}\frac{16^p[(2p-1)!]^2t^{2p}}{(4p-1)!(4p+1)(4m^2)^{2p}}\right]^n, \qquad (20)$$

where $C=4\pi^2-\lambda(1+\frac{1}{2}\phi(\frac{1}{3}))$.

From comparison of the two series, it is seen that the first three nonzero α_l are

$$\alpha_0=-\pi\lambda/8mC;\quad \alpha_2=\pi\lambda^2/[8\times15^2m(4m^2)^2C^2];$$

$$\alpha_4=\frac{\pi\lambda^2}{9^2\times5^2\times7m(4m^2)^4C^2}\left(\frac{1}{7}-\frac{\lambda}{20C}\right). \qquad (21)$$

Notice that there is a critical value of λ for which all of the α_l become infinite. This value of λ is

$$\lambda_c=4\pi^2/[1+\tfrac{1}{2}\phi(\tfrac{1}{3})]\quad(=13.81).$$

The magnitude of the ratio of α_2 to α_0 is small if λ is kept away from λ_c. Since there are other factors which place limits on the values of λ which may be used, further discussion of this point is left until later.

It is also possible to check the consistency of the approximation by calculating the first correction to $K(s,t)$ in Eq. (7) by doing a partial-wave analysis of the amplitude. The results of such a calculation are also shown in Table I.

If it is assumed that the amplitude given by Eq. (13) is reasonably consistent with the approximation made in deriving it, then it is interesting to see how the amplitude changes as the coupling constant is varied. Are there values of λ for which there are bound states? The answer to this question is a qualified "yes."

The bound states will manifest themselves as zeros of $D(s,t)$ for $0<s<4m^2$. It is obvious from the form of $D(s,t)$ in Eq. (13) that, for a given value of t, λ can be chosen to make D zero at any selected value of s in that interval. If λ is fixed at a value which gives a zero of D and then t is varied, the location of the zero changes. This behavior is not consistent with the interpretation of the zero as a bound state since the location of the zero is the square of the mass of the bound state and should be independent of t.

A more detailed investigation shows that for small binding energy the change in position is not very great. Figure 1 illustrates this by means of a contour chart of $16\pi^2[D(s,t)-1/\lambda+\phi(\frac{1}{3})/8\pi^2]$; an acceptable bound state would appear as a vertical straight line in the figure.

It is possible for D to have zeros for other values of s.

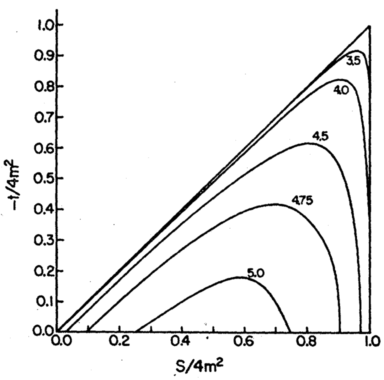

FIG. 1. Contour chart of the function $-16\pi^2\{D(s,t)-\lambda^{-1}+[\phi(\frac{1}{3})/8\pi^2]\}$.

TABLE II. Location of zeros of $D(s,t)$ as a function of λ; $-4m^2 \leqslant t \leqslant 0$.

Value of λ/λ_c	Location of zeros
$\lambda/\lambda_c < 0.89$	One in complex plane with $\text{Re}s = \frac{1}{2}(4m^2 - t)$ and one zero in ReD with $s > 4m^2$.
$0.89 < \lambda/\lambda_c < 1.20$	One zero in the ReD with $s > 4m^2$ and one on real axis with $\frac{1}{2}(t - 4m^2) < s < 4m^2$ at $t = 0$ which moves into the complex plane as t is decreased.
$\lambda/\lambda_c > 1.20$	No zeros.

With each zero below $s = 4m^2$ there is a zero of the real part of D above $4m^2$. This corresponds to a resonance and suffers from the same disease as the bound state.

Associated with the movement of the bound state zeros is the possibility of zeros at complex values of s. These occur on the line $\text{Re}s = \frac{1}{2}(4m^2 - t)$. As t is decreased the zero moves down the real axis to the symmetry

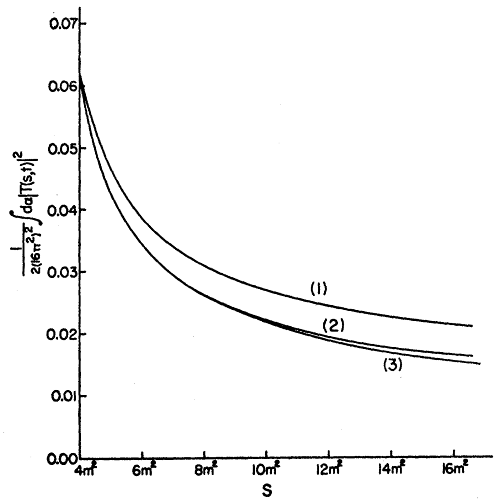

FIG. 2. Comparison of $[2(16\pi^2)^2]^{-1}\int d\Omega |T(s,t)|^2$ for (1) a calculation omitting crossing symmetry, (2) a solution of the partial-wave equations keeping only the S wave, and (3) the approximation discussed here with $\lambda = 1.76\lambda_c$.

point and then out into the complex plane. Table II summarizes the locations of the zeros of D and the values of λ for which they occur.

4. DISCUSSION OF RESULTS

We conclude this investigation of the approximation with some comments on the possible significance of the properties of the solution mentioned in the preceding section, and with a comparison with other approximations which can be used for the problem.

First, the approximation does not lead to any violent departures from unitarity. In terms of the scattering lengths α_l, the $l = 2$ contribution is negligible (one the order of a percent or less) provided λ is not too close to λ_c. A sufficient requirement in this regard is that there be no zeros in D.

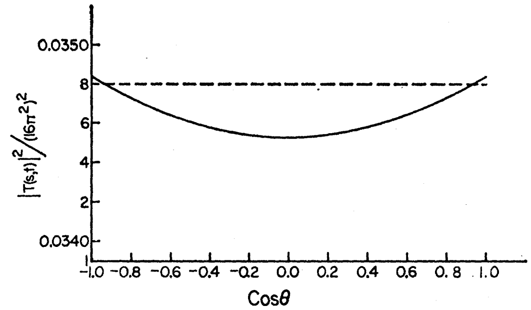

FIG. 3. Angular distribution for $\lambda = 1.76\lambda_c$, at $s = 6m^2$. The dashed line is the solution of the partial-wave equations keeping only the S wave.

An examination of the solutions for values of λ chosen to emphasize the violation of unitarity shows that the violation is only on the order of a percent even at the high energy of $s = 8m^2$. As seen in Table I, this is true whether the deviation from unitarity is measured by the violation of the optical theorem or by the correction term to K.

The behavior of the solutions which contain the pseudobound states and resonances presents a more serious problem. It is not altogether clear what mechanism prevents the mass of the bound state from depending on t in the exact amplitude. It would seem that some subtle interplay between unitarity and crossing symmetry is needed, and the results presented here raise questions not only about the approximation used in this case but also about those approximations where crossing symmetry is sacrificed.

The difficulty seems particularly serious when the coupling constant is essentially a subtraction parameter. In this case a contour chart like Fig. 1 is a convenient way to picture the situation and it is difficult to imagine

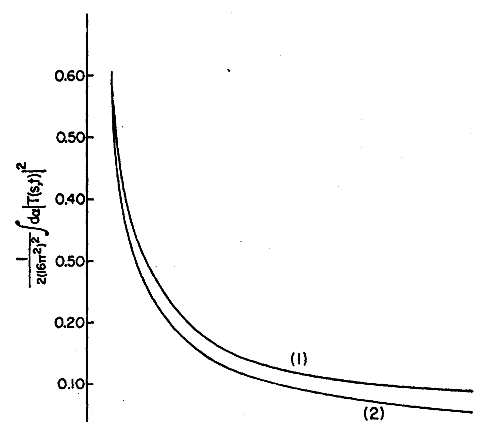

FIG. 4. Comparison of $[2(16\pi^2)^2]^{-1}\int d\Omega |T(s,t)|^2$ for (1) a calculation omitting crossing symmetry, and (2) the approximation discussed here with $\lambda = 0.97\lambda_c$. The solution of the partial-wave equations is indistinguishable from (2).

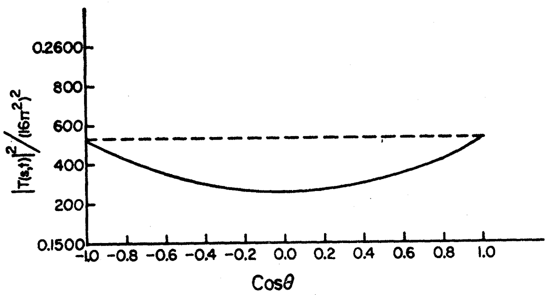

FIG. 5. Angular distribution for $\lambda=0.97\lambda_c$, at $s=6m^2$. The dashed line is the solution of the partial-wave equations keeping only the S wave.

how a bound state could occur except for a single combination of subtraction constant and bound-state mass.

There are several other approximations with which the results of the previous section can be compared. One possibility is to neglect crossing symmetry entirely and assume that the only interaction between the particles takes place in an S wave. This is a special case of the model proposed by Zachariasen.[9] A second is that of Efremov, Tzu, and Shirkov,[10] in which a partial-wave expansion is made and only the S-wave equation is solved. Crossing symmetry is only approximately satisfied.

The comparisons are shown graphically in Figs. 2 through 5, for two values of the coupling constant. Both the total cross section and the angular distributions are compared. The various approximations are normalized to give the same cross section at threshold.

These comparisons show that all three methods give similar results. The very good agreement between the method proposed here and the solution of the partial-wave equations give some evidence that the total cross section is insensitive to the details of the left-hand singularities as long as they are roughly what is required by crossing symmetry.

Finally, we mention one way in which the violation of unitarity enters when $K(s,t)$ is approximated by unity. Assume that the amplitude satisfies the Mandelstam representation, and calculate the double spectral functions. Unitarity requires that the double spectral functions be zero outside some region in the s-t plane. The double spectral functions of the approximate amplitude are not zero where unitarity requires it. A fairly complicated $K(s,t)$ will be required to make the double spectral function behave as it should.

[9] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

[10] A. V. Efremov, H.-Y. Tzu, and D. V. Shirkov, Zh. Eksperim. i Teor. Fiz. **41**, 603 (1961) [English transl.: Soviet Phys.—JETP **14**, 432 (1962)].

In spite of the bad features of the amplitude in this crude approximation, a more extensive investigation does seem worthwhile. However, any substantial improvement in $K(s,t)$ will require considerable use of computers in order to obtain the scattering amplitude.

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APPENDIX

In Sec. 3 the function $\phi(x)$ was introduced. Some of its properties are given below.

$$\phi(x)=\int_1^\infty \frac{dx'}{[x'(x'-1)]^{1/2}(x'-x)} = \frac{-2}{[x(x-1)]^{1/2}} Q_0([(x-1)/x]^{1/2}). \quad \text{(A1)}$$

This function is analytic in the complex x plane with a cut from $x=1$ to $+\infty$, and has $\text{Im}\phi(x)>0$ when $\text{Im}x>0$. These properties make $\phi(x)$ a Herglotz function.

In calculating the numerical value of the function, the following are useful:

$$\phi(x)=\frac{-2}{[x(x-1)]^{1/2}}\left[\tanh^{-1}([(x-1)/x]^{1/2})-\frac{i\pi}{2}\right]; \quad x>1 \quad \text{(A2a)}$$

$$=\frac{2}{[x(1-x)]^{1/2}}\left[\frac{\pi}{2}-\tan^{-1}([(1-x)/x]^{1/2})\right]; \quad 0\leqslant x\leqslant 1 \quad \text{(A2b)}$$

$$=\frac{2}{[-x(1-x)]^{1/2}}\tanh^{-1}([(-x)/1-x]^{1/2}); \quad x<0 \quad \text{(A2c)}$$

$$\phi(x)+\phi(1-x)=\frac{i\pi}{[x(x-1)]^{1/2}}. \quad \text{(A3)}$$

For large x,

$$\phi(x)\approx -\frac{1}{x}\ln|4x|. \quad \text{(A4)}$$

For small x,

$$\phi(x)=\sum_{n=0}^{\infty}\frac{2^{2n+1}[n!]^2x^n}{(2n+1)!}. \quad \text{(A5)}$$