Internal and Space-Time Symmetries

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To account for the observed multiplet structure and the mass relations among the known elementary particles, a coupling of internal and space-time properties is considered. Earlier attempts in this direction have failed. The problem is re-examined from a more general point of view, and it is found that under these more relaxed conditions a coupling is possible and nontrivial mass relations may be obtained. The coupling can be described as a "minimal internal coupling," which essentially consists of a replacement of the original generators of the inhomogeneous Lorentz group. The characteristic features of the mass formula are exhibited in a simple model. A number of questions related to the interpretation of such quantities as mass and energy are raised.

1. INTRODUCTION

DURING this century the exploitation of symmetry properties has come to play an ever-increasing role in many branches of physics. It has been known for a long time that the invariance under a symmetry operation for a particular problem often implies rather severe restrictions on the solutions. However, it was not until the advent of quantum mechanics that the power of group-theoretical methods was fully realized, and the elegant analysis by Wigner and others of the atomic spectra, based on the three-dimensional rotation group, cleared the way for more extensive use of group theory in physics.

In elementary particle physics one lacks a satisfactory dynamical theory. Even in those cases where the basic equations are believed to be known, we still do not know how to solve these equations or how to obtain reliable approximations. For that reason it has become even more important in the field of elementary particle physics to exploit the symmetry properties which have been found empirically. The initial step along these lines was taken by Wigner in his detailed study of the irreducible representations of the inhomogeneous Lorentz group (IHLG).[1] The experiments known so far are all in agreement with relativistic invariance for the dynamical laws which govern physical processes. In this way, one is led to study the IHLG. In his paper, Wigner goes further, however, and identifies the infinitesimal generators of the group with physical quantities, viz., momenta and angular momenta. The irreducible representations are characterized by two numbers (m,s) which with the conventional identification of the generators correspond to the mass and the spin of the elementary system constituting the representation. Therefore, the elementary particles are in a natural way connected to elementary representations of the IHLG.

Already in 1936, Cassen and Condon[2] introduced the concept of isospin, and since then the isospin formalism has been used extensively to express charge independence for the strong nuclear forces. With the number of different elementary particles rapidly growing and with access to more experimental information, new selection rules were found, reflecting the possible existence of additional symmetries. Great efforts have been made to find a large symmetry group incorporating all these fragments of internal symmetries which have been found empirically. These attempts all have in common that the internal symmetry properties are taken into account by introducing internal degress of freedom which are independent of space and time. Group theoretically, this means that the internal symmetry group and the IHLG are completely disentangled and appear as a direct product. For the isospin group this is not at all unreasonable because all the members of an isomultiplet have the same spin and the same or very nearly the same mass so that they may be referred to the same irreducible representation of the IHLG. Isospin rotations will then mix states corresponding to the same mass and spin only. However, those symmetry groups which have been considered more recently violate very strongly this equality rule for the masses of the different particles within a multiplet. A transformation belonging to the internal symmetry group will then transform a state belonging to a given irreducible representation of the IHLG into a state belonging to a different irreducible representation. For that reason the internal symmetry group cannot be completely disentangled from the IHLG, that is, the internal symmetry group and the IHLG cannot be simply put together in a direct product.

Much attention has recently been paid to the octet model[3] which is based on the group SU_3. To explain the large mass splittings within the various SU_3 multiplets it is customary to assume a hierarchy of interactions with different symmetry properties. Only the strong interactions are SU_3-invariant. The semistrong interactions are assumed to break the SU_3 symmetry in such a way that the existing mass differences are obtained. By perturbation methods, mass formulas are derived which in many cases are quite successful. However, the underlying assumptions seem doubtful, and there is no

[1] E. P. Wigner, Ann. Math. 40, 149 (1939).
[2] B. Cassen and E. U. Condon, Phys. Rev. 50, 846 (1936).
[3] M. Gell-Mann, California Institute of Technology Synchrotron Report CTSL-20, 1961 (unpublished); Phys. Rev. 125, 1067 (1962). Y. Ne'eman, Nucl. Phys. 26, 222 (1961).

way of checking the reliability of such computations. At first sight, it may seem that the octet model does not relate the internal symmetry group to space and time, but the effect of the symmetry-breaking interactions may well be the same as introducing a coupling between internal degrees of freedom and space-time, so that the two approaches are in some sense equivalent. In this latter approach, one obtains exact mass formulas in contrast to the approximate ones obtained in symmetry-breaking models. To introduce the coupling between the internal symmetry group and the IHLG from the very beginning also has the distinct advantage of putting the theory on a sound mathematical basis.

With these last remarks in mind, we shall approach the problem of finding the fundamental symmetry group G by imposing some rather general conditions on the group. In particular, we shall require that all physical transformations, internal as well as external, shall be comprised in the group G. Further requirements are essentially based on physical arguments although some simplifying assumptions are also introduced to make more definite conclusions possible. It will be seen that within this general frame one can construct a nontrivial product of the IHLG and the internal symmetry group with a coupling between the two groups, which is consistent with nonvanishing mass splittings within the multiplets. Similar attempts[4] under somewhat more restrictive assumptions have failed in this respect. They all[5] result in a complete decoupling of internal and external degrees of freedom, which is incompatible with different masses for the members in one and the same multiplet.

In the following section, we state the assumptions which are made regarding the fundamental symmetry group G.

Section 3 is devoted to the mathematical aspects of the problem. The results are given in a number of theorems followed by some remarks spelling out the immediate consequences of the theorems. Possible extensions of the theorems are briefly discussed. To make the results independent of any specific model, the general notion of Lie groups is used.

Section 4 is devoted to a discussion of the physical implications which follow from the mathematical analysis. Two sets of Lorentz generators are introduced and interpreted. The question of selecting particle labels is discussed.

In Sec. 5, we give the mass and spin formulas which arise naturally in this group-theoretical approach. Some characteristic features are discussed in an explicit model.

In conclusion, some possible changes in the underlying principles are mentioned. Some of the possible extensions meet with severe mathematical difficulties, and little can be said about them at present. In any case, until it has been shown that the approach presented here is inconsistent with experimental findings, more ambitious programs are not called for.

2. PHYSICAL RESTRICTIONS ON THE GROUP G

Contrary to the conventional approach, we shall require at the very outset that the fundamental symmetry group G shall not only explain the observed multiplet structure among the elementary particles and resonances but also give rise to nontrivial mass relations between the members of each multiplet. Therefore, our first requirement is:

(i) The group G shall explain the multiplet structure among the elementary particles and give rise to nontrivial mass formulas.

Since we take for granted that any physical theory must satisfy relativistic invariance, the group G must include all the transformations of the IHLG. Thus our second requirement is:

(ii) The group G shall contain the IHLG as a subgroup.

Furthermore, in this paper we also make the following simplifying assumption[6]:

(iii) Every transformation of the group G can be written as a product of two elements, of which the first belongs to the IHLG and the second belongs to another subgroup S which will be taken to be the internal symmetry group.

As a matter of convenience, we shall assume that the internal symmetry group S is compact. It will be recalled that for every compact Lie group S the corresponding Lie algebra[7] S decomposes into a direct sum of a finite number of noncommutative simple algebras $S_1, S_2, \cdots, S_n$ and its center S_0. Therefore, the case of a semisimple internal symmetry group is contained within this more general frame. By retaining the commutative center of the group we leave room for various gauge groups, however. We shall return to this question in Sec. 4. Thus, our next assumption is:

(iv) The internal symmetry group S shall be compact.

Besides these rather general assumptions, there are others which are more closely related to the physical identification of the generators of the group G with physical quantities. At this point, we shall refrain from

[4] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964); M. E. Mayer, H. J. Schnitzer, E. C. G. Sudarshan, R. Acharya, and M. Y. Han, Phys. Rev. **136**, B888 (1964); O. W. Greenberg, Phys. Rev. **135**, B1447 (1964); A. Beskow and U. Ottoson, Nuovo Cimento **34**, 248 (1964).

[5] See, however, H. Bacry and J. Nuyts, Phys. Letters **12**, 156 (1964).

[6] An attempt to relax this condition on the group G has recently been made by H. Bacry and J. Nuyts (Ref. 5).

[7] To simplify our notations we use the same symbol for a Lie group and its corresponding Lie algebra.

making any additional assumptions which would distinguish the effects of the three types of interactions that are normally considered, namely the strong, the electromagnetic, and the weak interactions. Therefore, in a sense, we are really looking for the fundamental symmetry group G in the theory of elementary particles.

If all types of interactions are taken into account, there are only three internal constants of motion—the electric charge Q, the baryon number N, and the lepton number L.[8] From our discussion so far, it is clear that we must identify the corresponding operators with generators of the internal symmetry group S. We postpone till Sec. 4 a discussion of whether or not they should be referred to the center of the group. To avoid difficulties in interpretation, we must require that these operators commute with all the generators of the IHLG, because otherwise a measurement of the charge carried by a particle would yield different results depending on which reference system the observer is in, etc. Therefore, as our final assumption, we introduce:

(v) The operators corresponding to charge, baryon number, and lepton number shall be included among the generators of the group S and they shall commute with all the generators of the IHLG.

The existence of at least one quantum number which is strictly conserved in the presence of all types of interaction is very essential for the mathematical treatment of the problem posed here. By assuming the existence of three such quantum numbers, we conform with physical reality rather than keeping the weakest possible conditions for the mathematical treatment.

Before entering the mathematical section, we introduce the notation which will be used throughout the paper (unless otherwise specified). The elements of the Lie algebra of the IHLG are denoted L_A, L_B, $\cdots$, or when it is necessary to be more specific, p_μ and $M_{\mu\nu}$ $(\mu,\nu=0,1,2,3)$. They satisfy the commutation relations[9]

$$[L_A,L_B]=C_{AB}{}^{C}L_C \tag{2.1}$$

or[10]

$$\begin{aligned}&[p_\mu,p_\nu]=0,\\ &[M_{\mu\nu},p_\lambda]=i(g_{\nu\lambda}p_\mu-g_{\mu\lambda}p_\nu),\\ &[M_{\mu\nu},M_{\kappa\lambda}]=i(g_{\nu\kappa}M_{\mu\lambda}+g_{\mu\lambda}M_{\nu\kappa}-g_{\mu\kappa}M_{\nu\lambda}-g_{\nu\lambda}M_{\mu\kappa}).\end{aligned} \tag{2.1$'$}$$

The generators of the internal symmetry group S are written S_ρ, S_σ, $\cdots$. When the group S is assumed semisimple, we sometimes express the generators also in the Cartan basis,[11] which is commonly written H_i, H_j, $\cdots$; E_α, E_β, $\cdots$. The commutation relations are

$$[S_\rho,S_\sigma]=C_{\rho\sigma}{}^{\tau}S_\tau \tag{2.2}$$

or, alternatively, in the case that S is semisimple,

$$\begin{aligned}&[H_i,H_j]=0,\\ &[E_\alpha,E_{-\alpha}]=\alpha^iH_i,\\ &[H_i,E_\alpha]=\alpha_iE_\alpha,\\ &[E_\alpha,E_\beta]=N_{\alpha\beta}E_{\alpha+\beta},\quad(\alpha\neq-\beta).\end{aligned} \tag{2.2$'$}$$

3. MATHEMATICAL RESTRICTIONS ON THE GROUP G

In accordance with the results in the preceding paragraph, we shall now analyze the case when the Lie group G is a product of a Lie group L and a compact semisimple Lie group S. We shall further assume that there are a sufficient number of operators H_m in S that commute with all the generators of L so that there is at least one component of these operators in every simple part of S. In particular, we shall study the case with the weakest assumption, i.e., when there is one H_m which has components in every simple part of S. We shall exploit the full content of the Jacobi identities for such a product to find out under what circumstances a nontrivial product can be formed. To clarify the discussion, we will not write down the Jacobi identities but rather state from which operators we form the identities and which relations we can derive from them. In the discussion, we use α, β, γ as indices for the E's that do not commute with H_m, and ϵ and δ for those that do.

The Jacobi identity for the operators L_A, E_α, H_m gives

$$C_{A\alpha}{}^{k}=0, \tag{3.1}$$

$$C_{A\alpha}{}^{\beta}=0\quad\text{if}\quad\beta_m\neq\alpha_m, \tag{3.2}$$

$$C_{A\alpha}{}^{B}=0, \tag{3.3}$$

$$L_AE_\epsilon H_m:\quad C_{A\epsilon}{}^{\alpha}=0, \tag{3.4}$$

$$L_AH_iH_m:\quad C_{Ai}{}^{\alpha}=0. \tag{3.5}$$

Lemma: The roots α with $\alpha_m\neq0$ span the root space if H_m has a component $\neq0$ in every simple part.[12]

$L_AE_\alpha E_{-\alpha}$: $C_{Ai}{}^{B}\alpha^i=0$. By the preceding lemma then

$$C_{Ai}{}^{B}=0, \tag{3.6}$$

$$L_AE_\alpha H_i\text{ and the lemma:}\quad C_{Ai}{}^{j}=0, \tag{3.7}$$

$$(\alpha_i-\beta_i)C_{A\alpha}{}^{\beta}=N_{\beta-\alpha,\alpha}C_{Ai}{}^{\beta-\alpha}, \tag{3.8}$$

$$C_{A-\alpha}{}^{-\alpha}=-C_{A\alpha}{}^{\alpha}. \tag{3.9}$$

$$L_AE_\epsilon H_i:\qquad C_{A\epsilon}{}^{B}=0, \tag{3.10}$$

$$-\epsilon^jC_{Ai}{}^{-\epsilon}=\epsilon_iC_{A\epsilon}{}^{j}, \tag{3.11}$$

$$(\epsilon_i-\delta_i)C_{A\epsilon}{}^{\delta}=N_{\delta-\epsilon,\epsilon}C_{Ai}{}^{\delta-\epsilon}. \tag{3.12}$$

$$L_AE_\epsilon E_{-\epsilon}:\qquad C_{A-\epsilon}{}^{-\epsilon}=-C_{A\epsilon}{}^{\epsilon}. \tag{3.13}$$

[8] For the present discussion we disregard the possible existence of two different lepton numbers, the muonic lepton number L_μ and the electronic lepton number L_e, suggested after the discovery that there are two kinds of neutrinos.

[9] We use the Einstein convention with summation over repeated indices unless otherwise stated.

[10] We do not distinguish between the generators of the group and the operators representing them.

[11] G. Racah, Cern Report 61-8 (unpublished).

[12] The complete proof is given in U. Ottoson, Chalmers University Report (unpublished).

$L_AE_\alpha E_\beta$: $$C_{A\alpha+\beta}{}^{\alpha+\beta}=C_{A\alpha}{}^{\alpha}+C_{A\beta}{}^{\beta}. \quad (3.14)$$

$L_AE_\alpha E_\epsilon$: $$C_{A\alpha+\epsilon}{}^{\alpha+\epsilon}=C_{A\alpha}{}^{\alpha}+C_{A\epsilon}{}^{\epsilon}. \quad (3.15)$$

$L_AE_\epsilon E_\delta$: $$C_{A\epsilon+\delta}{}^{\epsilon+\delta}=C_{A\epsilon}{}^{\epsilon}+C_{A\delta}{}^{\delta}. \quad (3.16)$$

$L_AL_BE_\alpha$: $$C_{A\beta}{}^{\gamma}C_{B\alpha}{}^{\beta}-C_{B\beta}{}^{\gamma}C_{A\alpha}{}^{\beta}=C_{D\alpha}{}^{\gamma}C_{AB}{}^{D}. \quad (3.17)$$

The remaining operator triplets $L_AH_{m_1}H_{m_2}$, $L_AH_iH_j$, $L_AL_BE_\epsilon$, $L_AL_BH_m$, and $L_AL_BH_i$ do not introduce any further conditions. So, if we define

$$J_{A\alpha}=C_{A\alpha}{}^{\alpha}, \quad (3.18)$$

$$K_{A\epsilon}=C_{A\epsilon}{}^{i}/\epsilon^i \quad \text{(no summation)}, \quad (3.19)$$

the structure constants that cannot be proved to be zero are of the form

$$C_{A\alpha}{}^{\alpha}=J_{A\alpha},$$

$$C_{A\epsilon}{}^{\epsilon}=J_{A\epsilon},$$

$$C_{A\alpha}{}^{\beta}=N_{\alpha,\beta-\alpha}K_{A\alpha-\beta}, \quad \alpha_m=\beta_m, \quad (3.20)$$

$$C_{A\epsilon}{}^{\delta}=N_{\epsilon,\delta-\epsilon}K_{A\epsilon-\delta}, \quad (3.21)$$

$$C_{A\epsilon}{}^{i}=\epsilon^iK_{A\epsilon}, \quad (3.22)$$

$$C_{Ai}{}^{\epsilon}=\epsilon_iK_{A-\epsilon}. \quad (3.23)$$

From the relations (3.9) and (3.13)–(3.16) we find that

$$J_{A-\alpha}=-J_{A\alpha},\quad J_{A-\epsilon}=-J_{A\epsilon},\quad J_{A\alpha+\beta}=J_{A\alpha}+J_{A\beta},$$
$$J_{A\alpha+\epsilon}=J_{A\alpha}+J_{A\epsilon}, \quad\text{and}\quad J_{A\epsilon+\delta}=J_{A\epsilon}+J_{A\delta}.$$

By the additivity in $J_{A\alpha}$ we can define $J_D{}^i$ such that $J_{D\alpha}=\alpha_iJ_D{}^i$ and $J_{D\epsilon}=\epsilon_iJ_D{}^i$. According to (3.8) or (3.20), $C_{A\alpha}{}^{\beta}$ can be different from zero only if $\alpha_m=\beta_m$ and $\beta-\alpha$ is a root. From (3.17) we get that $C_{A\alpha}{}^{\beta}$ form a finite dimensional matrix representation of the generators L_A. When L has no Abelian factor group, which means that all operators L_A can be expressed as commutators, then the representation is traceless, so that $C_{A\alpha}{}^{\alpha}$ has to be zero for all α such that there are no other roots β with $\alpha_m=\beta_m$. We can write (3.17) as

$$K_{A\epsilon}J_{B\epsilon}-J_{A\epsilon}K_{B\epsilon}+\textstyle\sum_\delta N_{-\delta\epsilon}K_{A\delta}K_{B\epsilon-\delta} =\textstyle\sum_D K_{D\epsilon}C_{AB}{}^{D}, \quad (3.24)$$

$$\textstyle\sum_\delta \delta^iK_{A-\delta}K_{B\delta}=\textstyle\sum_D J_D{}^iC_{AB}{}^{D}. \quad (3.25)$$

These observations are summarized in the following two theorems[7]:

Theorem 1

Let the Lie group G be a product of the Lie group L with infinitesimal generators L_A and the compact semisimple Lie group S with infinitesimal generators H_i and E_α, such that one H_m with components in every simple part of S commutes with all L_A.

(i) Then S is an invariant subgroup of G.

(ii) The only mixed structure constants that can be different from zero are those of the form $C_{A\alpha}{}^{\alpha}$; $C_{A\alpha}{}^{\beta}$, if $\alpha_m=\beta_m$ and $\alpha-\beta$ is a root; $C_{A\epsilon}{}^{i}$, if $\epsilon_m=0$; or $C_{Ai}{}^{\epsilon}$, if $\epsilon_m=0$; and they fulfull the relations

$$C_{A-\alpha}{}^{-\alpha}=-C_{A\alpha}{}^{\alpha};\, C_{A\alpha+\beta}{}^{\alpha+\beta}=C_{A\alpha}{}^{\alpha}+C_{A\beta}{}^{\beta};$$

$$\frac{C_{A\alpha}{}^{\alpha-\epsilon}}{N_{\alpha,-\epsilon}}=\frac{C_{A\epsilon}{}^{i}}{\epsilon^i}=\frac{C_{Aj}{}^{-\epsilon}}{-\epsilon_j} \quad\text{if}\quad \epsilon_m=0.$$

(iii) The $C_{A\alpha}{}^{\beta}$ with $\alpha_m=\beta_m\neq0$ are matrix elements in a representation of the generators L_A.

Theorem 2

Consider a Lie group L, a finite-dimensional representation of L, and a compact semisimple Lie group S with a selected generator H_m. Let the generator H_m be such that there are roots α, β, $\cdots$ of S with equal nonzero m components, and the differences $\alpha-\beta$, $\cdots$ are different roots. Suppose that the number of roots is greater than or equal to the dimension of the representation of L. Then there is a product G of L and S such that the $C_{A\alpha}{}^{\beta}$'s are equal to the matrix elements of the representation of L_A, and H_m commutes with all L_A.

Remarks

(i) The extension to a noncompact semisimple Lie group S is straightforward. The prescription for the H_m will be lengthier in this case.

(ii) The generalization to the case with more than one H_m is trivial. The dimension of the space of the ϵ's will be smaller in this case.

(iii) As every compact Lie group is a direct product[13] of the center S_0 of S and a semisimple group, Theorems 1 and 2 are valid for a general compact Lie group, if we add the requirement that the generators of S_0 shall commute with all L_A. [From the requirements of the theorems it is easily shown that $C_{A\alpha}{}^{u}$, $C_{A\epsilon}{}^{u}$, and $C_{Ai}{}^{u}$ (u stands for an index of a generator in S_0) must be zero. Then the remark follows immediately.]

(iv) When we take the IHLG as L, we have to calculate all the finite-dimensional representations of the IHLG. An interesting fact is that in the one-, two-, three-, and four-dimensional representations the translations are represented trivially.

(v) With the IHLG as L, the generators p_μ and H_i can not simultaneously be represented by Hermitian operators in a representation of G, because then the commutator $[p_\mu,H_i]$ should be anti-Hermitian. According to (3.23), this requires that the finite-dimensional representation of IHLG be unitary, but there is no such representation.[14] We return to this question below and show that Remark (v) does not exclude physically interesting cases.

[13] L. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, New Jersey, 1939), Theorem 86.

[14] This has been shown independently by Dr. E. C. G. Sudarshan (private communication).

Theorem 3

Let the Lie group G be a product of the Lie group L with infinitesimal generators L_A and a compact semisimple Lie group S with infinitesimal generators H_i and E_α, such that at least one H_m with components in every simple part of S commutes with all L_A. Then G is a direct product of S and a group $\tilde{L}$ locally isomorphic with L.

Proof. Define

$$\tilde{L}_A = L_A - \sum_i J_A{}^i H_i + \sum_\epsilon K_{A-\epsilon} E_\epsilon,$$

where $\epsilon_m = 0$ as before.

Then

$$[\tilde{L}_A, H_j] = [L_A, H_j] - \sum_\epsilon K_{A-\epsilon}\epsilon_j E_\epsilon = 0,$$

$$[\tilde{L}_A, E_\alpha] = [L_A, E_\alpha] - J_{A\alpha}E_\alpha - \sum_\epsilon K_{A-\epsilon} N_{\alpha\epsilon} E_{\alpha+\epsilon} = 0,$$

$$[\tilde{L}_A, E_\delta] = [L_A, E_\delta] - J_{A\delta}E_\delta - \sum_\epsilon K_{A-\epsilon} N_{\delta\epsilon} E_{\delta+\epsilon} - K_{A\delta}\sum_i \delta^i H_i = 0,$$

and

$$\begin{aligned}[\tilde{L}_A, \tilde{L}_B] &= [L_A, \tilde{L}_B] = [L_A, L_B] - \sum_i J_B{}^i [L_A, H_i] \\ &\quad + \sum_\epsilon K_{B-\epsilon}[L_A, E_\epsilon] = [L_A, L_B] - \sum_i J_B{}^i \sum_\epsilon K_{A-\epsilon}\epsilon_i E_\epsilon \\ &\quad + \sum_\epsilon K_{B-\epsilon}(J_{A\epsilon}E_\epsilon + \sum_\delta K_{A-\delta} N_{\delta+\epsilon} + K_{A\epsilon}\sum_i \epsilon^i H_i) \\ &= [L_A, L_B] - \sum_i (\sum_\epsilon K_{A-\epsilon} K_{B\epsilon}\epsilon^i) H_i + \sum_\epsilon (K_{A-\epsilon} J_{B-\epsilon} \\ &\quad - J_{A-\epsilon} K_{B-\epsilon} + \sum_\delta N_{-\delta-\epsilon} K_{A\delta} K_{B-\epsilon-\delta}) E_\epsilon \\ &= \sum_D C_{AB}{}^D (L_D - \sum_i J_D{}^i H_i + \sum_\epsilon K_{D-\epsilon} E_\epsilon) \\ &= \sum_D C_{AB}{}^D \tilde{L}_D.\end{aligned}$$

We have shown that the $\tilde{L}_A$'s commute with the generators of S and have the same structure constants as the L_A's, and that is all we have to show.

Remarks

(i) The necessary and sufficient condition for the conclusion of Theorem 3 to hold is that

$$C_{Aj}{}^D = 0, \quad C_{Aj}{}^i = 0, \quad C_{Aj}{}^\alpha = \alpha_j K_{A-\alpha},$$

$$C_{A\alpha}{}^D = 0, \quad C_{A\alpha}{}^i = \alpha^i K_{A\alpha},$$

$$C_{A\alpha}{}^\alpha = \sum_i J_A{}^i \alpha_i = J_{A\alpha}, \quad C_{A\beta}{}^{\alpha+\beta} = N_{\beta\alpha} K_{A-\alpha},$$

where $K_{A\alpha}$ and $J_A{}^i$ fulfill the relation

$$K_{A\alpha}J_{B\alpha} - J_{A\alpha}K_{B\alpha} + \sum_\beta N_{-\beta\alpha} K_{A\beta} K_{B\alpha-\beta} = \sum_D K_{D\alpha} C_{AB}{}^D$$

and

$$\sum_\alpha \alpha^i K_{A-\alpha} K_{B\alpha} = \sum_D J_D{}^i C_{AB}{}^D.$$

These conditions are not independent but related by the Jacobi identities. A smaller set of conditions is therefore sufficient. We have given the redundant set above, because a minimal set can be chosen in different ways. Then the operators $\tilde{L}_A = L_A - \sum_i J_A{}^i H_i + \sum_\alpha K_{A-\alpha} E_\alpha$ commute with the generators of S, and the $\tilde{L}_A$'s have the same structure constants as the L_A's. These requirements on the structure constants are somewhat more general than in Theorem 3, where all $K_{A\alpha} = 0$ for α outside a certain plane.

(ii) Under the conditions of Theorem 3 with the IHLG as L, G has an invariant Abelian subgroup, which is isomorphic with the four-dimensional translation group.

4. PHYSICAL INTERPRETATION OF THE THEOREMS

The theorems of Sec. 3 strongly limit the type of coupling of the internal group S to the IHLG. We recall that the sufficient conditions for these theorems to be valid are that S is compact, and that there exists a generator H_m of S which has a nonvanishing component in every simple part of S and which, together with the center S_0, commutes with the IHLG. In the discussion which follows, we shall call H_m a c-type generator, and the operators H_i, which do not commute with the IHLG, are termed nc-type generators. Theorem 1 tells us which of the mixed structure constants can be different from zero. The existence of nc-type generators is seen to be essential for a nontrivial coupling.

The strictly conserved quantum numbers, namely the electric charge Q, the baryon number N, and the lepton number L, shall be identified with c-type generators or with generators in the center S_0. For historical reasons, one is tempted to place N in S_0 and Q in the semisimple part.[15] We need not be specific on this point but only require that at least one of these strictly conserved quantities be identified with H_m so that the conditions for Theorem 1 are fulfilled.

As long as we assumed the group G to be a direct product of the IHLG and S, all the operators H_i of S were of c type and, as a consequence, they corresponded to constants of the motion. If one conforms with the convention of treating the baryon number and the lepton number separately by means of gauge groups, then available data do not seem to be consistent with more than two internal constants of the motion. For that reason, we limited our attention to semisimple Lie groups of rank two. With a coupling between the IHLG and the group S the situation is different. Clearly an nc-type H which has a large commutator with p_0 is so strongly nonconserved that it may not be possible to use it as a particle label. Therefore, we are now free to consider semisimple groups of higher rank as long as we do not require all the operators H_i to commute with the IHLG. Those nc-type generators which have small commutators can still be used as fairly good quantum numbers, while those which have large commutators cannot. They could possibly be used to distinguish different decay modes of resonances.

Theorem 3 completely determines the structure of the group G. G is simply a direct product of S and a group $\tilde{L}$ which is locally isomorphic to the IHLG. The group $\tilde{L}$

[15] Compare the Gell-Mann–Nishijima relation $Q = I_3 + \frac{1}{2}Y$; $I_3 + Y \in S$ but not S_0. If we put particles with different charge into the same multiplet, Q obviously has to be in the semisimple part of S.

is generated by the operators

$$\tilde{L}_A = L_A + b_A{}^\rho S_\rho ,$$

where

$$b_A{}^\epsilon = K_{A-\epsilon}, \quad b_A{}^\alpha = 0, \quad b_A{}^i = -J_A{}^i .$$

Theorem 3 does not imply that we have arrived at a trivial result, however, since the IHLG generated by L_A is not a factor in the product but rather has components in both factors. The coupling between S and the IHLG can simply be described as the replacement

$$L_A \to \tilde{L}_A = L_A + b_A{}^\rho S_\rho \tag{4.1}$$

of the generators of the IHLG. Therefore, in analogy with the electromagnetic coupling, we might say that we are uniquely led to a "minimal internal coupling" which consists of the substitution (4.1). The "free generators" $\tilde{L}_A$, which commute with S, are the generators of a free, i.e., noninteracting system, while the generators L_A give rise to the physical operators such as momentum, energy, and spin in the presence of interaction. In accordance with this interpretation, we also suggest that the generators $\tilde{L}_A$ should be represented by Hermitian operators, while the physical generators should not. Actually, it is not possible to have a representation of G in which both the generators of the IHLG and S are Hermitian. This follows from Theorem 3. The expectation value of the physical operators will then acquire an imaginary part. This is an interesting feature and in conformity with general principles of quantum mechanics. It has long been customary to regard the mass operator of an unstable particle as a complex quantity. For other observables such as spin, it is only recently that such a generalization to non-Hermitian operators has been suggested.[16]

The assumption of Hermitian generators $\tilde{L}_A$ immediately simplifies the problem of finding the relevant representations of G. One only has to take the direct product of a unitary representation of $\tilde{L}$ and a unitary representation of S, and these are well known.

The introduction of a free system or particle in addition to the real physical system is here forced upon us. For the free system, the particle labels can be chosen in the conventional manner. One merely selects a maximal Abelian set of Hermitian operators from $\tilde{L}$ and from S. From $\tilde{L}$ one can choose the free momentum $\tilde{p}_\mu$ and the free spin. From S one may choose the H_i and a set of polynomials.[17] For the free noninteracting particle, one can thus retain the conventional classification scheme and associate one or more particles with each point in the weight diagrams of the various representations of S. The free particle does not decay when it is transformed by a "free time translation" $\exp(i\tilde{p}_0\tilde{t})$, but it can be transformed into other particles under a physical time translation $\exp(ip_0t)$, since p_0 does not commute with the internal quantum numbers. Therefore, the term "free system" represents a property which has relevance only when one specifies at the same time under which group it should transform. By retaining the original IHLG as the group of physical transformations, the interaction is introduced. This is in complete analogy with nonrelativistic systems where one studies, on the one hand, a free Hamiltonian H_0 and, on the other hand, a physical Hamiltonian H. The eigenfunctions of H_0 are no longer eigenfunctions of the physical Hamiltonian H. H therefore causes transitions among the free states. As long as H is Hermitian and has an attractive potential, it is possible to find stationary physical states. Also, in our case, the free particle ceases to be free when we make the physical Lorentz transformations. However, the possibility of finding stationary states, i.e., eigenfunctions of p_0, is not necessarily possible here; in any case, some eigenfunctions will certainly be only quasistationary, i.e., decaying.

It is really alluring to speculate whether one might have the whole S matrix given here, so that no further assumptions about interactions need be given.

5. MASS AND SPIN FORMULAS

The possibility of obtaining mass formulas has recently been one of the strongest reasons for considering internal symmetry groups. When the group G was assumed to be a direct product of S and the IHLG, a symmetry-breaking mechanism had to be introduced *ad hoc*. In our approach, this is not desirable and may not be necessary. In fact, the spectrum of the mass operator $m^2 = p_\mu p^\mu$, if it exists at all, is given as soon as one has an irreducible representation of G. An irreducible unitary representation of G is characterized by $2+n$ invariants, since the group $\tilde{L}$ has two invariants

$$\tilde{m}^2 = \tilde{p}^\mu \tilde{p}_\mu , \tag{5.1}$$

$$\tilde{w}^2 = \epsilon^{\mu\nu\lambda\kappa}\epsilon^{\mu\nu'\lambda'\kappa'}\tilde{p}_\nu\tilde{p}_{\nu'}\tilde{M}_{\lambda\kappa}\tilde{M}_{\lambda'\kappa'} , \tag{5.2}$$

and S has n invariants, where n is the sum of the rank of the semisimple part and the number of parameters in the center S_0. In analogy with earlier terminology, we call $\tilde{m}^2$ the square of the "free mass" and $\tilde{w}^2/\tilde{m}^2$ the square of the "free spin." These numbers are real since the representation is unitary. Since $\tilde{p}_\mu = p_\mu + b_\mu{}^\rho S_\rho$, we get for the square of the physical mass

$$m^2 = p_\mu p^\mu = \tilde{m}^2 - 2g^{\mu\nu} b_\mu{}^\rho \tilde{p}_\nu S_\rho + g^{\mu\nu} b_\mu{}^\rho b_\nu{}^\sigma S_\rho S_\sigma . \tag{5.3}$$

This equation is the mass formula in our purely group-theoretic approach.

Until recently, attention was concentrated on the problem of obtaining relations between the mass and the internal quantum members of a particle. Some even went so far as to postulate that other external quantum numbers should be the same within a multiplet. We are forced to put all external properties on an equal footing

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[16] See, e.g., S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963).

[17] G. E. Baird and L. C. Biedenharn, Proceedings of the Coral Gables Conference, January 1964, p. 58 (unpublished).

and postulate spin formulas as well. Of course, these are obtained from the fourth-order invariant

$$W^2=\epsilon^{\mu\nu\lambda\kappa}\epsilon_\mu{}^{\nu'\lambda'\kappa'}(\tilde{p}_\nu-b_\nu{}^\rho S_\rho)(\tilde{p}_{\nu'}-b_{\nu'}{}^\rho S_\rho)\times(\tilde{M}_{\lambda\kappa}-b_{\lambda\kappa}{}^\rho S_\rho)(\tilde{M}_{\lambda'\kappa'}-b_{\lambda'\kappa'}{}^\rho S_\rho)\,. \quad (5.4)$$

It is interesting that one may also obtain in this way a relation between masses and spins via the internal state. Such connections between the mass and spin of a particle have been suggested before on quite different grounds. We have in mind especially the theory of Regge poles based on analyticity of the scattering amplitudes[16] and the theory of Corben based on generalized relativistic equations.[18] The spin operator W^2 will, in general, not be Hermitian and not even diagonalizable, so that the particles can not be eigenstates of this operator. The spin values of particles then emerge as expectation values of the spin operator rather than as eigenvalues.

Returning to the mass formula (5.3), we see that a necessary condition for this to be nontrivial is that the constants $b_\mu{}^\rho$ be different from zero. Now, according to Theorem 1, this means that at least some of the numbers $K_{\mu\epsilon}$, $J_\mu{}^i$ have to be different from zero. Then the finite-dimensional representation of the IHLG introduced in Theorem 2 must not be a trivial representation of the translations. According to Remark (iv), the dimension of the representation has to be at least five. Again, according to Theorem 1, the root space of the semisimple part of S must then contain more than four roots α, β, $\cdots$, with equal nonzero m components. This immediately rules out a number of possible choices for S, among them SU_3. In SU_3, the root space is two-dimensional and, irrespective of how we choose the c-type operator H_m, we cannot find more than two roots with equal nonzero m components.

Postponing the choice of a realistic group S, we shall look at a simple model. In order to make this sufficiently simple and at the same time retain the essential features, we shall choose as the group L of Theorem 1 the (1+1) IHLG instead of the physical (1+3) IHLG. In this group there is only one space dimension. The generators of time translation, space translation, and acceleration

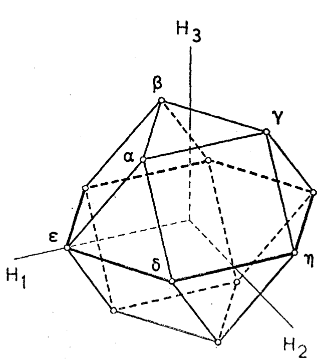

FIG. 1. Root diagram for SU_4.

[18] H. C. Corben, Phys. Rev. **131**, 2219 (1963).

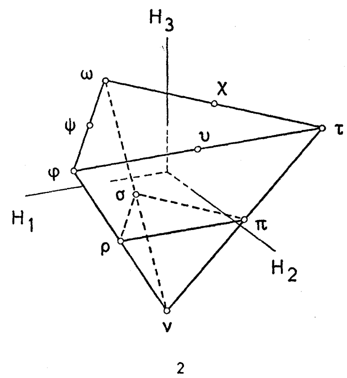

FIG. 2. Weight diagram for the ten-dimensional representation of SU_4.

are denoted p_0, p_1, and N, respectively. The (1+1) IHLG admits a three-dimensional representation, in which p_0, p_1, and N are represented by the matrices

$$p_0=\begin{bmatrix}-i\sin\theta\cos\theta & 0 & i\cos^2\theta\\ 0 & 0 & 0\\ -i\sin^2\theta & 0 & i\sin\theta\cos\theta\end{bmatrix},$$

$$p_1=\begin{bmatrix}0 & 0 & 0\\ -i\sin\theta & 0 & i\cos\theta\\ 0 & 0 & 0\end{bmatrix},$$

$$N=\begin{bmatrix}0 & i\cos\theta & 0\\ i\cos\theta & 0 & i\sin\theta\\ 0 & i\sin\theta & 0\end{bmatrix}.$$

Since we have found a three-dimensional representation, in which the translations are represented nontrivially, we can choose a semisimple group of rather low rank. The group SU_4 fulfills the requirements of Theorem 2 with L equal to the (1+1) IHLG, and its root diagram is shown in Fig. 1. We have chosen the third axis H_3 as H_m. Then there are three roots α, β, and γ with equal nonzero m components such that the differences $\alpha-\beta$, $\cdots$ are different roots. We can then construct a product group G of the (1+1) IHLG and SU_4 such that the $C_{A\alpha}{}^{\beta}$'s are equal to the matrix elements of the representation of the (1+1) IHLG. By means of the relations (3.18) and (3.20), we can calculate $J_{A\alpha}$ and $K_{A\epsilon}$, and furthermore $J_D{}^i$, and put these quantities into the mass formula:

$$\begin{aligned}m^2=\tilde{m}^2&-2\textstyle\sum_\epsilon g^{\mu\nu}K_{\mu-\epsilon}\tilde{p}_\nu E_\epsilon+2\sum_i g^{\mu\nu}J_\mu{}^i\tilde{p}_\nu H_i\\&+\textstyle\sum_\epsilon\sum_\delta g^{\mu\nu}K_{\mu-\epsilon}K_{\nu-\delta}E_\epsilon E_\delta\\&-\textstyle\sum_\epsilon\sum_i g^{\mu\nu}K_{\mu-\epsilon}J_\nu{}^i[E_\epsilon H_i+H_iE_\epsilon]\\&+\textstyle\sum_i\sum_j g^{\mu\nu}J_\mu{}^iJ_\nu{}^jH_iH_j\,. \quad (5.5)\end{aligned}$$

As a suitable representation of SU_4 we choose the lowest-dimensional representation that gives a nontrivial mass formula, and this is the ten-dimensional representation. Its weight diagram is shown in Fig. 2.

In the ten-dimensional representation all weights are simple. We see that the states are divided into three classes corresponding to different eigenvalues of H_m. In the first class there is only one state, here called ν; in another, three states called π, ρ, σ; and in the third class, six states named $\tau, v, \varphi, \chi, \psi, \omega$. From Eq. (5.5), we find the following expectation values of the square of the mass operator:

$$\begin{aligned}
\langle\nu|m^2|\nu\rangle &= \tilde{m}^2,\\
\langle\pi|m^2|\pi\rangle &= \tilde{m}^2 - i\sin2\theta\tilde{p}_0,\\
\langle\rho|m^2|\rho\rangle &= \tilde{m}^2 + i\sin2\theta\tilde{p}_0,\\
\langle\sigma|m^2|\sigma\rangle &= \tilde{m}^2,\\
\langle\tau|m^2|\tau\rangle &= \tilde{m}^2 + 2i\sin2\theta\tilde{p}_0 - \tfrac{1}{2}\sin^2 2\theta,\\
\langle v|m^2|v\rangle &= \tilde{m}^2 + \sin^2 2\theta,\\
\langle\varphi|m^2|\varphi\rangle &= \tilde{m}^2 - 2i\sin2\theta\tilde{p}_0 - \tfrac{1}{2}\sin^2 2\theta,\\
\langle\chi|m^2|\chi\rangle &= \tilde{m}^2 + i\sin2\theta\tilde{p}_0,\\
\langle\psi|m^2|\psi\rangle &= \tilde{m}^2 - i\sin2\theta\tilde{p}_0,\\
\langle\omega|m^2|\omega\rangle &= \tilde{m}^2.
\end{aligned}$$

According to Sec. 4, the states $|\nu\rangle, \cdots, |\omega\rangle$ correspond to free noninteracting particles. If we assume that the real part of the expectation value corresponds to the observed mass, then we see that we have obtained a mass splitting. Seven particles have the mass $\tilde{m}^2$, two have the mass $\tilde{m}^2-\frac{1}{2}\sin^2 2\theta$ and the tenth has the mass $\tilde{m}^2+\sin^2 2\theta$. The imaginary parts are all proportional to $\tilde{p}_0$, and these terms are by no means small. Assuming that the imaginary parts are essentially the inverse lifetimes, we see that six particles are highly unstable. Of course, since we use the free-particle basis, there are also off-diagonal matrix elements in the mass matrix. In connection with the large imaginary part in the expectation values, this raises the question whether we have used the correct definition of mass. One might think that one should first diagonalize the mass matrix and then obtain masses for certain particle mixtures. In that case, we would have a situation which is analogous to the $(K_0,\bar{K}_0)$ particle mixture. The particular states $(K_0\pm\bar{K}_0)/\sqrt{2}$ have definite and different masses and also lifetimes, while K_0 or $\bar{K}_0$ are not given a definite mass. Now, in general, it is not possible to diagonalize the mass operator, at least not as long as the operators $\tilde{L}_A$ and H_i are Hermitian, but in a specific model one may still have this possibility. Whether or not this is desirable, we feel that we have to give a clearer definition of the mass of a particle, if such a notion is to be meaningful at all for unstable particles.

The mass formula (5.5), if it can be successfully interpreted, applies equally well to bosons and fermions. Hitherto one has generally assumed that the formulas should be linear in the mass for fermions and quadratic for bosons, but there seems to exist neither strong theoretical nor experimental evidence for this difference.

6. DISCUSSION

In our previous discussion of the problem of combining internal and space-time symmetries we have refrained from making any specific choice of internal symmetry group S. Our starting point was rather some general requirements on the fundamental symmetry group G. From those we derived criteria which any possible candidate for S must fulfill. These still leave room for considerable choice, however, and the ultimate decision must be founded on more detailed calculations and a thorough analysis of the identification of the generators with physical quantities.

From the discussion leading to Theorem 1 it is seen that the root diagram of S must necessarily contain a sufficient number of roots $\alpha, \beta, \cdots$ with equal nonzero m-component, that is, $0\neq\alpha_m=\beta_m=\cdots$, so that the corresponding mixed structure constants $C_{A\alpha}{}^{\beta}$ can be made to form a finite-dimensional representation of the generators of the IHLG. It is further required that difference vectors $\alpha-\beta$, etc., all are roots. All these difference vectors obviously lie in a hyperplane perpendicular to the H_m axis. Finally, to obtain a nontrivial mass formula it is necessary that the translation operators be not represented in a trivial manner in the finite-dimensional representation of the IHLG. As noted already, this last requirement forces us to consider five-dimensional or higher dimensional representations of the IHLG. Therefore, the root space of S must contain a plane perpendicular to the H_m axis with at least five root vectors in it. This immediately rules out $SU_3\,(=A_2)$ or any other semisimple Lie group of rank 2. Among the groups of rank 3, it is found that $SU_4\ (=A_3=D_3)$ cannot yield any nontrivial mass formula. The groups $O_7\ (=B_3)$ and $Sp_6\ (=C_3)$ have root diagrams which meet most of our requirements. However, there is a large number of restrictive relations among the roots and for that reason it seems unlikely that sufficient freedom remains for the mixed structure constants $C_{A\alpha}{}^{\beta}$ so that they can be made to form a representation of the IHLG with the translation operators represented in a nontrivial way. A preliminary study rather seems to favor $SU_6\,(=A_5)$ as a realistic choice for S. It is interesting to note that on quite different grounds this group has lately received considerable attention.[19] To what extent SU_6 can account for all known facts in elementary particle physics is obviously still an open question, but work is in progress to examine its capacity in this respect.

The initial restrictions on the group G were chosen in such a way that there would be room for all the known types of interaction in the theory. Under special circumstances it may be meaningful to neglect, say, electromagnetic and weak interactions. This is, of course, the basic assumption underlying most of the work done in the field of strong interaction physics. Strongly interacting particles, the so-called hadrons, are usually labeled by baryon number B, charge Q, isospin I, and

[19] F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964); T. K. Kuo and Tsu Yao, Phys. Rev. Letters **13**, 415 (1964).

the third component of isospin I_3. Usually one also adds the hypercharge Y to this list of particle attributes. Because of the Gell-Mann–Nishijima relation, the hypercharge may be considered redundant, however. All the quantum numbers mentioned above are supposedly conserved in strong interaction processes, but, of course, not in the presence of electromagnetic and weak interactions. Therefore, if it is legitimate to neglect all but the strong interactions at least as a first approximation, then it is consistent to use these quantum numbers to characterize the hadrons. In doing so we introduce the additional assumption that the corresponding infinitesimal generators of the internal symmetry group can be included in an Abelian set of operators which is sufficient to label the basis vectors in the representation space of the fundamental group G (maximal set of commuting observables). In this way we are supplied with additional restrictions on the commutation relations. However, they are only approximate and good to the extent that electromagnetic and weak interactions do not alter them in a significant way. Keeping in mind that the three types of interactions considered here differ both in strength and with respect to the symmetry properties they exhibit, it is alluring to speculate whether one in general can separate out the contributions that each kind of interaction gives to the commutators of the theory. It still remains to be shown that it is a consistent procedure to neglect certain contributions in an approximation scheme.

Some final remarks regarding our assumptions for the group G are in order. At first sight the assumptions may seem to be very general in nature and highly plausible from the point of view of physics. It should be kept in mind, though, that these assumptions are quite restrictive and one may have to relax some of them if the program described above fails to work. The necessity for the fundamental group G to contain S as a subgroup may well be questioned. It may also be worthwhile to consider discrete internal symmetry groups rather than continuous Lie groups. After all, physically realizable transformations belonging to the internal symmetry group are discrete. It seems to be primarily for historical reasons that continuous internal symmetry groups have been preferred so far. Finally, we emphasize once more that much work remains to be done on the problem of identification of the generators with physical observables.

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Approximate Method for Determining the Elastic-Scattering Amplitude for Strong Interactions*

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A method for calculating the elastic-scattering amplitude in the S-matrix theory of strong interactions is proposed which does not require a partial-wave expansion of the amplitude. Crossing symmetry is satisfied by the amplitude, but unitarity is imposed only approximately. Equations are derived for the case of scattering of two spinless particles of unequal mass. The special case where the masses are equal is considered in detail for the input assumption that the scattering is predominantly S wave. Crossing symmetry introduces higher partial-wave contributions to the amplitude. The amplitude calculated in this way is in good agreement with the input assumption. The amount of violation of unitarity is least near threshold, but is only on the order of a percent at $s=8m^2$. In spite of this, there are serious problems with low-energy resonances and bound states. It is concluded that both unitarity and crossing symmetry are important in the production of resonances and bound states and that the modification of either may lead to difficulties. The total cross section derived from the approximate amplitude is compared with that obtained using the partial-wave expansion and keeping only the S wave. The results are in good agreement with each other.

I. INTRODUCTION

IN the S-matrix theory of strong interactions,[1] if the Mandelstam representation is taken to embody the assumption of analyticity for processes which go from a two-particle initial state to a two-particle final state, then the problem reduces to the determination of the

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[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).