Final-State Interaction in the Presence of Absorption*

ROLAND OMNES
Lawrence Radiation Laboratory, University of California, Berkeley, California
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There is by now a good agreement between the distorted-wave approximation and the sharpening of the peripheral peak by competing processes, although the theory itself has not been correctly derived at high energy. We show that the main result of the distorted-wave approximation can be derived by applying the dispersion theory of final-state interactions to the eigenstates of the S matrix. Some assumptions about the mean values and the variation of the eigen-phase-shifts that fit with scattering experiment have to be made. The theory does not apply to low angular momenta. It is valid in nuclear physics.

I. INTRODUCTION

IT was first indicated by Chew and Low that the exchange of a pion in a pion-nucleon inelastic collision should dominate the cross section for low values of the momentum transfer between the initial and final nucleons.[1] This effect was emphasized as the basis of phenomenological calculation of high-energy forward inelastic processes, the so-called peripheral processes, by Drell and Salzman.[2]

If the system of final particles, excluding the nucleon (or more generally the baryon in processes involving strange particles), has a zero total angular momentum in its own rest frame and if the exchange particle has zero spin and mass μ, the theory predicts a distribution in the square of the momentum transfer Δ^2 of the form $(\Delta^2+\mu^2)^{-2}$. This form can be significantly modified if higher values of the spin are involved, as, for instance, in the production of a ρ. In that case the peaking predicted by the theory is much less marked.[3]

Experimentally, the agreement between theory and experiment is rather good at very low values of Δ^2. Furthermore, Yang and Treiman have given a test which relates the angular correlations to the spin of the exchanged particle, and this test is generally in very good agreement with the dominant peripheral mechanism.[4] However, for values of Δ^2 larger than a few μ^2, the experimental peaks are often much narrower than the theory predicts.[5]

An obvious drawback of the theory is that it predicts in fact inelastic cross sections which, for high energies and low angular momenta, exceed the limit imposed by unitarity. However, even if this difficulty is removed by reducing to a reasonable value the first few partial cross sections in low angular momenta, the correction is much too small to explain the sharpness of the experimental peaks.

It has been suggested by Ferrari and Selleri that one should not consider the exchanged particle as giving rise to a pole in Δ^2 of the matrix element, but should compute the peripheral graph as in perturbation theory, allowing the matrix element at the pion vertex to depend on Δ^2.[5] This proposal is, however, defective in two respects: first, its theoretical basis is not clear, and second, it leads to a very rapid variation in Δ^2 which cannot be reconciled with our understanding of form factors.

It has been suggested independently by Durand and Chiu and by Gottfried and Jackson that the sharpening of the peak could be due to the existence of other inelastic processes, which would compete with the process considered.[6] The effect is, in essence, very simple: the lower the angular momentum, the larger is the cross section of nonperipheral processes that have to share with the peripheral process the common amount of $\pi\lambda\!\!\!^{-2}$, and the larger the reduction is. As low angular momenta mainly determine the distribution for large Δ^2, the net effect must be a sharpening of the peripheral peak.

In order to give a quantitative form to their argument these authors have used the distorted-wave approximation, which has already had good success in nuclear physics.[7] However, this leads to very severe difficulties: the first is the use of the Schrödinger equation with an optical potential which, while it can in principle reproduce any angular distribution, has no reason to give good information on the wave functions. Furthermore, the use of a potential at high energy for low angular momenta is well known to have no sense. Much worse is the fact that the essential formula which they use is derived in the limit where the interaction considered has a much smaller range than the competing processes: the cases of interest do not satisfy that limitation.

While these authors have recognized the difficulties, they have nevertheless applied the formulas of the distorted-wave approximation to the production of ρ mesons in pion-nucleon collisions, with amazing success.

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[1] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

[2] S. D. Drell and K. Hiida, Phys. Rev. Letters **7**, 199 (1961); F. Salzman and G. Salzman, *ibid.* **5**, 377 (1960) and Phys. Rev. **121**, 1541 (1961).

[3] This can be attributed, in a perturbative field-theoretic approach, to the necessity of derivative couplings.

[4] Aachen-Birmingham-Bonn-Hamburg-London-München collaboration, Nuovo Cimento **31**, 729 (1964).

[5] E. Ferrari and F. Selleri, Nuovo Cimento Suppl. **24**, 453 (1962).

[6] L. Durand and Y. T. Chiu, Phys. Rev. Letters **12**, 399 (1964); K. Gottfried and J. D. Jackson, Nuovo Cimento (to be published).

[7] See, for instance, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

There is no doubt that the effect is present and, furthermore, that the formulas used are essentially correct.

It is the aim of this paper to derive these formulas with only very general assumptions, which have the advantage of fitting quite naturally with our present theoretical and experimental knowledge of high-energy phenomena.

The essential idea is to apply the now standard dispersion theory of final-state interactions not to the actually measured states but to the eigenstates of the S matrix. Furthermore, we have to make some hypothesis on the mean values and the energy variation of these eigen-phase-shifts, which are suggested by the fact that high-energy collisions give rise mainly to uncorrelated particles and resonances. Our theory is not restricted to the case in which the final system of particles excluding the nucleon can be treated as a particle.

In Sec. II we recall the final results of the distorted-wave approximation approach, which is essentially what has to be proved. In Sec. III we introduce the eigen-phase-shifts together with their essential properties. In Sec. IV, the theory of final-state interactions is applied to the peripheral matrix element between eigenstates of the S matrix. The main formula of the distorted-wave approximation is derived while some of its limitations are indicated. In Sec. V, we indicate the problems met with in the very low angular momenta. In Sec. VI we extend these considerations to nuclear physics, where it is stated that our main assumptions are satisfied, so that the present work could be considered as a general and simple derivation of the results of the standard approach, free from its customary limitations.

III. THE DISTORTED-WAVE APPROXIMATION

The distorted-wave approximation approach is best expressed in the case in which the process to be studied is a two-body reaction

$$a+A \rightarrow b+B. \tag{1}$$

One represents the elastic scattering of particles a and A as resulting from some optical potential U, and the elastic scattering of particles b and B as resulting from a potential U'. Then it is supposed that the process (1) takes place through the effect of a third interaction V, which can be treated in Born approximation. Accordingly the scattering matrix element for process (1) can be written as

$$M_{fi}=\langle \psi_f^{(-)} | V | \psi_i^{(+)} \rangle, \tag{2}$$

where $\psi_i^{(+)}$, for instance, is the incoming wave of particles a and A interacting through potential U. A word of caution is in order at this stage: Although it is always possible to represent the elastic-scattering amplitude of particles a and A by a Schrödinger equation with an optical potential which varies with energy, there is as far as we know no proof that the corresponding wave functions $\psi_i^{(+)}$ are reliable. Notwithstanding that difficulty, one writes for $\psi_i^{(+)}$ an optical approximation, valid when U varies slowly on a de Broglie wavelength of the incoming particle, i.e.,

$$\psi_i^{(+)}=e^{iqz}e^{-i\phi_i(\mathbf{b},z)}, \tag{3}$$

where $\mathbf{b}$ is the impact parameter vector normal to the z axis which is chosen along the direction of the incoming particle. The Schrödinger equation in semiclassical approximation ($|\nabla\phi| \ll q^{-1}$) gives, then,

$$\phi_i(\mathbf{b},z)=-\frac{1}{v}\int_{-\infty}^{z} U(\mathbf{b}+\hat{k}z')dz'. \tag{4}$$

Here $\hat{k}$ is the unit vector along the z direction.

When the range of V is much smaller than the range of U and U' it is possible to give a simple expression to Eq. (2) by using Eq. (4). To do so one notices that the parameter b can be replaced by the angular momentum $l=qb$ and one introduces the Born approximation $B(l)$ for M_{fi}, i.e.,

$$B(l)=\langle \phi_{fl} | V | \phi_{il} \rangle, \tag{5}$$

where ϕ_{il}, for instance, is the free-particle wave of a and A in the angular momentum l. Then one gets

$$T(l)=S_{ff}^{1/2}(l)B(l)S_{ii}^{*1/2}(l), \tag{6}$$

where

$$S_{ii}(l)=e^{-i\phi(\mathbf{b},\infty)} \tag{7}$$

is the matrix element of the S matrix for elastic scattering.[8] Equation (6) is the key formula for all applications, and its success is sometimes very remarkable.[7] However, it should be stressed that the present arguments do not provide any justification for it in the case of a peripheral interaction at high energy, since

(a) the Schrödinger equation formulation for the wave functions is not justified, and

(b) the range of a peripheral interaction is very large. It is therefore important to know whether or not, and under which conditions, Eq. (6) is correct.

In the present discussion as well as in the following considerations, we have neglected the spin of the particles, but it should be obvious that no point of principle is left out by such a simplification.

III. EIGEN-PHASE-SHIFTS AND THEIR PROPERTIES

We shall denote by $|a\rangle$, $|b\rangle \cdots$ a complete set of states which are diagonal in the energy, the total angular momentum, and the number of particles, i.e., the states which are commonly used in the interpretation of an experiment.

The unitarity of S allows us to introduce another

[8] N. J. Sopkovitch, Nuovo Cimento **26**, 186 (1962).

complete set of states $|\alpha\rangle$, $|\beta\rangle\cdots$, which are eigenstates of S, i.e.,

$$S|\alpha\rangle=e^{2i\delta_\alpha}|\alpha\rangle, \tag{8}$$

where δ_α is a real number, an "eigen-phase-shift."

Actually, as soon as three-particle states can be produced, some of the eigenstates of S are not normalizable. We shall ignore this difficulty in the following. In practice it would amount to writing Stieljes integrals where we shall write summations.

The states $|a\rangle$ and $|\alpha\rangle$ are related by a unitary transformation

$$|a\rangle=\sum_\alpha U_{a\alpha}|\alpha\rangle. \tag{9}$$

Moreover, if one uses time-reversal invariance and chooses for $|a\rangle$ time-reversal-invariant states, the matrix U will be real, i.e., orthogonal. This property implies in particular

$$\sum_\alpha U_{a\alpha}^2=1. \tag{10}$$

While the existence of eigen-phase-shifts is acknowledged, their properties have not been investigated.[9] In the case in which only a finite number of states can be produced—i.e., where the only open channels are two-body channels—the eigenvalues of S, $e^{2i\delta_\alpha}$, and the matrix elements $U_{a\alpha}$ are piecewise analytic, but not analytic. In the following, we shall assume that this property remains true in general.

One can get useful experimental information about these eigen-phase-shifts by relating them to the elastic scattering amplitude. According to Eqs. (8) and (9), one has

$$\langle a|S|a\rangle=\sum_\alpha U_{a\alpha}^2 e^{2i\delta_\alpha}. \tag{11}$$

For energies of a few GeV, when $|a\rangle$ is a two-particle state (a pion-nucleon state, for instance), there is good experimental evidence that the scattering amplitude is pure imaginary, i.e., that the matrix elements (11) are real. Moreover, $\langle la|S|la\rangle$, as a function of the angular momentum l, is a function that is small (of the order of 0.3) for small l and tends to 1 for large l.

The comparison of Eqs. (10) and (11) shows that $\langle a|S|a\rangle$ is generated by adding complex numbers $U_{a\alpha}^2 e^{2i\delta_\alpha}$ whose moduli add up to 1. The simplest hypothesis that fits the experimental data is that most of the eigen-phase-shifts which communicate with the elastic channel have a value around 0 or $\frac{1}{2}\pi$. This can be understood in the following way: the matrix U decouples the particles produced in a strong interaction into states in which there is not much correlation, i.e., $\delta_\alpha\approx 0$. However, many states contain resonances which have to be kept as such by U. A state made up of one resonance and uncorrelated particles has essentially for phase shift the phase shift of the resonance, i.e., $\delta\approx\frac{1}{2}\pi$. Obviously, these values 0 and $\frac{1}{2}\pi$ have to be taken as they stand for large values of l for which generally only one resonance is produced, and have to be taken modulo π for small l, for which several resonances can be produced.

While we acknowledge the empirical character of these results, we believe that they provide the simplest fit of the data together with a promising statistical simplicity. It should be mentioned in that respect that the absence of correlation of the particles in the eigenstates is essentially what is necessary in order to get an exponential diffraction peak, as has been shown by Van Hove under slightly different assumptions.[10]

It should be mentioned that, since the eigen-phase-shifts vary around fixed values, it can be assumed that they vary slowly with energy.

IV. FINAL-STATE INTERACTIONS

Let us now consider an inelastic process $a\to b$. For more clarity we shall consider the case in which a is a pion-nucleon state and b a nucleon-ρ state, neglecting the spins. We shall split the collision matrix for a given total angular momentum into two parts: T_1^0, which is the pion pole contribution to $a\to b$, and the rest of the T matrix:

$$T=T_1^0+T_2.$$

When the total angular momentum is large, the cross section due to T_1^0 is smaller than the one due to T_2. Consequently, we shall make the approximation that T_2 satisfies the unitarity condition, i.e., that $S_2=I+2iT_2$ is a unitary operator. This approximation has some relation with the range hypothesis in the distorted-wave approach. It is not that the range of T_1^0 is smaller than the range of T_2 but that, for large ranges, T_1^0 is smaller than T_2. In the following we shall accordingly write S in place of S_2 and use the results of the preceding section.

The quantity $\langle|b|T_1^0|a\rangle$ is real. The existence of initial and final-state interaction will tend to modify it and, in this present case, ultimately to reduce it. However, the usual treatment of final-state interactions cannot be directly applied to the case in which there is absorption in $|a\rangle$ and $|b\rangle$.[11] In order to apply it, we shall first pass to the $|\alpha\rangle$ basis:

$$\langle\beta|T_1^0|\alpha\rangle=U_{a\alpha}U_{b\beta}^{-1}\langle b|T_1^0|a\rangle. \tag{12}$$

Now, whereas $\langle\beta|T_1^0|\alpha\rangle$ is only piecewise analytic as a function of the total energy, we shall nevertheless apply the final-state dispersion correction to it. To do so, let us make explicit its dependence on the total energy E by writing it as $T_{\beta\alpha}^0(E)$, and introduce $\delta_{\beta\alpha}(E)=\delta_\beta(E)+\delta_\alpha(E)$. The interaction in the initial and final states

[9] Y. Yamaguchi, Suppl. Progr. Theoret. Phys. (Kyoto) 7, 1 (1959).

[10] L. Van Hove, Nuovo Cimento 25, 392 (1962).

[11] A model of multichannel N/D formalism leading to formula (6) has been recently proposed by E. J. Squires, University of Edinburgh (unpublished). Unfortunately there is no obvious justification for the statistical assumptions which have to be made. This formula has also been used for large l by M. H. Ross and G. L. Shaw, Phys. Rev. Letters 12, 627 (1964), and by M. Baker and R. Blankenbecler, Phys. Rev. 128, 415 (1962)

will modify $T_{\beta\alpha}{}^0(E)$ into

$$T_{\beta\alpha}(E)=\int O_{\beta\alpha}(E,E')T_{\beta\alpha}{}^0(E')dE', \tag{13}$$

where the operator $O_{\beta\alpha}(E,E')$ is defined as

$$O_{\beta\alpha}(E,E')=\delta(E-E')+\frac{1}{\pi}\frac{f(E')h(E')}{f(E)(E'-E-i\epsilon)} \tag{14}$$

with

$$h(E)=e^{i\delta_{\beta\alpha}(E)}\sin\delta_{\beta\alpha}(E) \tag{15}$$

and

$$f(E)=\exp\left(\frac{1}{\pi}\int\frac{\delta_{\beta\alpha}(E')dE'}{E'-E}\right) \tag{16}$$

up to necessary subtractions.[12]

When $\delta_\beta(E)$ and $\delta_\alpha(E)$ vary slowly in the neighborhood of 0 and $\frac{1}{2}\pi$, and when $T_1{}^0(E)$ does not vary too rapidly with energy, Eq. (13) can be simplified into

$$T_{\beta\alpha}(E)=e^{i[\delta_\beta(E)+\delta_\alpha(E)]}T_{\beta\alpha}{}^0(E). \tag{17}$$

It is to be expected that this simple result would hold in more general conditions than the conditions which insure the validity of Eq. (13).

Using the fact that, by its very definition, $T_1{}^0$ has matrix elements only between $|a\rangle$ and $|b\rangle$, we get from Eqs. (12) and (17):

$$\langle b|T_1|a\rangle=(\textstyle\sum_\beta U_{b\beta}{}^2e^{i\delta_\beta})(\sum_\alpha U_{a\alpha}{}^2e^{i\delta_\alpha})\langle b|T_1{}^0|a\rangle. \tag{18}$$

All the effects of initial- and final-state interactions are concentrated into the factors in parenthesis. It is easy to evaluate them for large values of the total angular momentum, where all phase-shifts are near 0 or $\frac{1}{2}\pi$ and

$$\langle a|S|a\rangle=1-\epsilon_a, \tag{19}$$

$$\langle b|S|b\rangle=1-\epsilon_b, \tag{20}$$

where ϵ_a and presumably ϵ_b are small and real. Equation (11) can then be written

$$\langle a|S|a\rangle=(1-\tfrac{1}{2}\epsilon_a)+e^{2i\pi/2}(\tfrac{1}{2}\epsilon_a), \tag{21}$$

so that

$$\textstyle\sum_\alpha U_{a\alpha}{}^2e^{i\delta_\alpha}=(1-\frac{1}{2}\epsilon_a)+e^{i\pi/2}(\frac{1}{2}\epsilon_a) \tag{22}$$

and

$$|\textstyle\sum_\alpha U_{a\alpha}{}^2e^{i\delta_\alpha}|=|1-\frac{1}{2}\epsilon_a|=[\langle a|S|a\rangle^{1/2}].$$

Therefore, Eq. (18) can be approximated by

$$\langle b|T_1|a\rangle=[\langle b|S|b\rangle]^{1/2}\langle b|T_1{}^0|a\rangle[\langle a|S|a\rangle^*]^{1/2}, \tag{23}$$

which is precisely the result of the distorted-wave approximation as shown in Eq. (6).

V. THE CASE OF LOW ANGULAR MOMENTA

The foregoing analysis cannot be applied when the absorption is large (ϵ_a and ϵ_b significantly different from zero) and when $\langle\beta|T_1{}^0|a\rangle$ is larger or of the order of $\langle a|T_2|a\rangle$. This is precisely the case for low angular momenta, where it can even happen that the peripheral model predicts matrix elements larger than the unitarity limit.

The distorted-wave approximation cannot be applied to this case, either in the potential form (small angular momenta at high energy cannot be represented by a potential), nor in the form of the preceding section. In principle, one could use a many-channel N/D calculation, but our understanding of high-energy interactions is yet too primitive to carry out such a program in a practical way.[11]

We shall therefore agree that because of the competing channels and the unitarity limit, the peripheral matrix element must be significantly decreased although there is no reliable theory for this reduction.

VI. CONCLUSIONS

We have shown that under very wide conditions, the reduction of matrix elements predicted by the distorted-wave approximation can be justified without appealing to considerations of potentials. The hypotheses needed bear a direct relation with already known experimental results.

It should be stressed at that point that the properties of the eigen-phase-shifts that we have assumed are presumably satisfied in nuclear physics at intermediate energies. In fact, the outcome of most nuclear collisions are systems of nucleons and excited nuclei which are not very much correlated, so that the eigen-phase-shifts can be expected to run around 0 and $\frac{1}{2}\pi$ modulo π. Therefore, the present approach constitutes another justification of the distorted-wave approximation in nuclear physics.

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[12] M. Jacob, G. Mahoux, and R. Omnes, Nuovo Cimento 23, 838 (1962); J. D. Jackson and G. L. Kane, *ibid.* 23, 444 (1962).