Coupling of Internal and Quantized Space-Time Symmetries

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Commutation relations in the group LU_4, recently introduced by Kurşunoğlu for studying the coupling of internal and space-time symmetries, are examined in some detail. It is shown that the operators $\Gamma_{\mu 4}$ ($\mu=1,2,3,4$), defined by the author, may be formally identified with the noncommuting coordinate operators x_μ of a quantized space-time proposed by Snyder. This identification removes certain difficulties arising from associating the $\Gamma_{\mu 4}$, instead, with the momentum operators p_μ. It is also shown that the commutation relations $[p_\mu,p_\nu]=0$ follow naturally from the introduction of quantization into LU_4, and that the notion of quantization provides an intuitive basis for understanding the coupling of space-time and internal symmetry groups.

I. INTRODUCTION

It has been proposed recently[1] that a connection between the Lorentz group and the strong interaction symmetry group SU_3 may be established by considering a new group LU_4, which contains the former groups as subgroups. From the Lie algebra of LU_4 it may be shown that there exist four operators which are formally identifiable with the momentum operators p_μ[2] which, however, fail to commute. One method for removing this difficulty is to define, as does the author, a broader group ILU_4, whose generators include four commuting momentum operators in addition to the generators of LU_4. The author considers, however, an alternative to broadening the group LU_4, namely, viewing the commutation relations in question as reflecting the fact that translations in the realm of microphysics are discrete operations. In the following paper this suggestion is explored in some detail. It is shown that the operators Γ_{j4} and $\frac{1}{2}\Gamma_{44}$ defined by Kurşunoğlu and previously identified with the operators p_j and p_4, respectively, may be identified instead with the noncommuting coordinate operators of a quantized space-time first proposed by Snyder.[3] It is further demonstrated that, not only is a quantized space-time consistent with both relativity and quantum mechanics, but that in the present context (of the coupling of space-time and internal symmetry groups) it provides an intuitive basis for understanding the coupling of these symmetries.

In the following section Kurşunoğlu's derivation of the group LU_4 is summarized, followed in turn by a discussion of the quantization of space-time in Sec. III. In Sec. IV quantized space-time is introduced into LU_4 and the consequences of this are explored in detail.

II. THE GROUP LU_4

Let $M_{\mu\nu}$ denote the generators of the homogeneous Lorentz group, and consider the symmetric tensor operator $\Gamma_{\mu\nu}$ defined by

$$\Gamma_{\mu\nu}=\tfrac{1}{4}g_{\mu\nu}M_{\alpha\beta}M^{\alpha\beta}-\tfrac{1}{2}(M_{\mu}{}^{\rho}M_{\nu\rho}+M_{\nu}{}^{\rho}M_{\mu\rho})\,, \tag{1}$$

where

$$g^{\mu\nu}\Gamma_{\mu\nu}=0\,, \tag{2}$$

and where $g_{\mu\nu}$ is the Lorentz metric tensor: $g_{ij}=-\delta_{ij}$, $g_{44}=1$, $g_{i4}=g_{4i}=0$. The ten components, $\Gamma_{\mu\nu}$, are formally identical to the components of the electromagnetic energy momentum tensor, and together with the six generators $M_{\mu\nu}$ of the Lorentz group, constitute a set of generators for a new group L_0U_4.[4] The commutation relations for the generators of L_0U_4 are as follows:

$$[M_{\mu\nu},M_{\alpha\beta}]=i(g_{\alpha\nu}M_{\mu\beta}+g_{\beta\nu}M_{\alpha\mu}-g_{\alpha\mu}M_{\nu\beta}-g_{\mu\beta}M_{\alpha\nu})\,, \tag{3}$$

$$[\Gamma_{\mu\nu},M_{\alpha\beta}]=i(g_{\alpha\nu}\Gamma_{\mu\beta}+g_{\alpha\mu}\Gamma_{\beta\nu}-g_{\beta\nu}\Gamma_{\alpha\mu}-g_{\mu\beta}\Gamma_{\alpha\nu})\,, \tag{4}$$

$$[\Gamma_{\mu\nu},\Gamma_{\alpha\beta}]=i(g_{\beta\nu}M_{\alpha\mu}+g_{\mu\beta}M_{\alpha\nu}-g_{\alpha\nu}M_{\mu\beta}-g_{\alpha\mu}M_{\nu\beta})\,. \tag{5}$$

Since the generators, $M_{\mu\nu}$, of the Lorentz group are themselves generators of LU_4, the Lorentz group is clearly a subgroup of LU_4. Furthermore, the commutation relations Eqs. (3)–(5) are seen to subsume those of the internal symmetry group SU_3.[5] Thus the group LU_4 may, in some sense, be considered as coupling the Lorentz group and SU_3.

The commutation relations for the momentum operators are obtained by first expanding the relations (3)–(5), making use of the notation $M_{jk}=\epsilon_{jkl}M_l$, $M_{j4}=N_j$. In particular we have

$$[M_j,\Gamma_{k4}]=i\epsilon_{jkl}\Gamma_{l4}\,, \tag{6}$$

$$[M_j,\Gamma_{44}]=0\,, \tag{7}$$

$$[N_j,\Gamma_{44}]=2i\Gamma_{j4}\,, \tag{8}$$

$$[N_j,\Gamma_{k4}]=i(\delta_{jk}\Gamma_{44}+\Gamma_{jk})\,. \tag{9}$$

Equations (6)–(8) are the usual commutation relations among the homogeneous and inhomogeneous subgroups

[1] B. Kurşunoğlu, Phys. Rev. **135**, B761 (1964).
[2] In this paper Greek indices run from 1 to 4 and Latin indices from 1 to 3.
[3] H. S. Snyder, Phys. Rev. **71**, 38 (1947).
[4] The group L_0U_4 is a subgroup of LU_4. The latter group contains as a subgroup a baryon gauge group in addition to L_0U_4. For purposes of the present discussion, commutation relations involving this gauge group are not relevant.
[5] Compare, for example, with J. J. deSwart, Rev. Mod. Phys. **35**, 916 (1963).

of the Lorentz group,

$$[M_{\mu\nu},p_\rho]=i(g_{\rho\nu}p_\mu-g_{\mu\rho}p_\nu)\,, \tag{10}$$

if, following Kurşunoğlu, we identify Γ_{j4} with p_j and Γ_{44} with $2p_4$.

From Eqs. (3)–(5) it follows, however, that the operators $\Gamma_{\mu 4}$ (and hence the operators p_μ) satisfy the commutation relations

$$[\Gamma_{j4},\Gamma_{k4}]=-i\epsilon_{jkl}M_l\,, \tag{11}$$

$$[\Gamma_{j4},\Gamma_{44}]=-2iN_j\,, \tag{12}$$

and not the usual relations $[p_\mu,p_\nu]=0$. Since $p_\mu p^\mu$ represents the mass operator of the group LU_4, the lack of commutativity among the p_μ might be expected to lead to difficulties in the definition of mass in an internal symmetry multiplet. Furthermore, the failure of the momentum operators to commute implies, in addition, that the underlying space-time is curved. Although the introduction of curvature in schemes for coupling space-time symmetries and internal symmetries may be justified on various theoretical grounds,[6] the current emphasis[7] on the group theoretic properties of such coupling schemes makes it desirable to restrict the allowed transformations to those defined over a flat space-time. For these reasons it would be desirable to reinterpret Eqs. (11) and (12) in such a way as to permit the commutation relations $[p_\mu,p_\nu]=0$ to remain intact. This may be accomplished by introducing the notion of a quantized space-time.

III. THE QUANTIZATION OF FLAT SPACE-TIME

Several proposals have been advanced[3,8,9] for quantizing space-time, all having in common the introduction of a fundamental length a in some manner or other. Of special interest in the present context is a quantization scheme first proposed by Snyder.[3]

Let x, y, z, t and x', y', z', t' be the space-time coordinates of an event as viewed from two systems K and K', respectively. Special relativity requires that the indefinite quadratic form

$$s^2=c^2t^2-x^2-y^2-z^2\,, \tag{13}$$

be invariant under transformations that take one from the observations of the system K to those of K'. To introduce the notion of quantization of space-time, it is first assumed that x, y, z, and t are Hermitian operators. The requirement of Lorentz invariance is then stated as follows: If x', y', z', and t' are Hermitian operators formed by taking linear combinations of the operators x, y, z, and t in such a manner as to leave the quadratic form (13) invariant, then the spectra of the primed operators shall be the same as that of the unprimed operators.[3]

An explicit representation of the operators x, y, z, and t, may be obtained by considering the quadratic form

$$-\eta^2=\eta_4^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_0^2\,, \tag{14}$$

where the η's are real variables which may be viewed as the homogeneous projective coordinates of a real four-dimensional space-time of constant curvature.[10] The operators x_μ are defined as follows:

$$x_j=ia(\eta_0\partial/\partial\eta_j-\eta_j\partial/\partial\eta_0)\,, \tag{15}$$

$$x_4=ict=-a(\eta_0\partial/\partial\eta_4+\eta_4\partial/\partial\eta_0)\,, \tag{16}$$

where a is the previously mentioned fundamental length, and c is the velocity of light. The spatial coordinate operators x_j satisfy the eigenvalue equation

$$x_j|\psi\rangle=ma|\psi\rangle\,, \tag{17}$$

where m is a positive or negative integer or zero. The operator x_4 possesses a continuous spectrum extending from minus infinity to plus infinity. Since the question of whether the spatial coordinate operators possess continuous or discrete spectra is merely one of satisfying appropriate boundary conditions, the preceding definition of these operators is clearly consistent with the axioms of quantum mechanics. One can only ask whether or not a specific choice of the constant a is consistent with observation. Furthermore, the construction of the coordinate operators in such a way as to leave the quadratic form (13) invariant is sufficient to guarantee that the definition of these operators is consistent with the requirements of special relativity.

In addition to the four operators x_μ, one may define a set of six operators which are formally equivalent to the generators of the Lorentz group:

$$\begin{aligned}M_1&=i(\eta_3\partial/\partial\eta_2-\eta_2\partial/\partial\eta_3)\,,\\ M_2&=i(\eta_1\partial/\partial\eta_3-\eta_3\partial/\partial\eta_1)\,,\\ M_3&=i(\eta_2\partial/\partial\eta_1-\eta_1\partial/\partial\eta_2)\,,\\ N_j&=i(\eta_4\partial/\partial\eta_j+\eta_j\partial/\partial\eta_4)\,.\end{aligned} \tag{18}$$

IV. QUANTIZATION OF SPACE-TIME IN LU_4

From the definitions (15) and (16) the following commutation relations among the operators x_μ may be derived:

$$[x_j,x_k]=ia^2\epsilon_{jkl}M_l\,, \tag{19}$$

$$[x_4,x_j]=-a^2N_j\,. \tag{20}$$

From Eqs. (19) and (20) we note that if the identi-

[6] See, for example, R. L. Ingraham, Phys. Rev. **106**, 595 (1957); and D. W. Joseph, *ibid.* **126**, 319 (1962).

[7] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964); F. Coester, M. Hamermesh, and W. D. McGlinn, Phys. Rev. **135**, B451 (1964); A. O. Barut, *ibid.* **135**, B839 (1964); O. W. Greenberg, *ibid.* **135**, B1447 (1964); M. E. Mayer, H. J. Schnitzer, E. C. G. Sudarshan, R. Acharya, and M. Y. Han, *ibid.* **136**, B888 (1964); L. Michel, (preprint).

[8] C. N. Yang, Phys. Rev. **72**, 874 (1947).

[9] A. Schild, Phys. Rev. **73**, 414 (1948).

[10] It should be noted that the hypersurfaces η_0=const are Minkowski spaces.

fications

$$x_j = ia\Gamma_{j4},$$
$$x_4 = (a/2)\Gamma_{44}, \tag{21}$$

are made, these commutation relations become identical to (11) and (12). This establishes one of the objectives of this paper, namely, the reinterpretation of the operators $\Gamma_{\mu 4}$ as coordinate rather than momentum operators, thereby eliminating the necessity for introducing a curved space-time. It will now be demonstrated that the commutation relations $[p_\mu, p_\nu] = 0$ follow from the commutation relations for LU_4 if the identifications (21) are made. To establish this result, it must be noted that, in the present case, the most general form that the commutator of the coordinate and the momentum operators can assume consistent with the requirements of the limit $a \to 0$ is

$$[x_\mu, p_\mu] = i\hbar\{1 + \text{const} \times a^\beta f_\mu(x_j, x_4, p_j, p_4)\}, \tag{22}$$

where $\beta > 0$ and the constant so adjusted to make the entire term dimensionless. However, the commutator as given above is not covariant under the transformations $x_\mu \to -x_\mu$ or $p_\mu \to -p_\mu$, except for the trivial choice $f_\mu = 0$. This implies that the usual commutation relations between conjugate coordinate and momentum operators remain unaltered, a result which is to be expected on other grounds. To find an explicit expression for p_j, we assume that p_j may be written in the form $p_j = (\hbar/a)g_j$, where g_j is a dimensionless function of the η's and their derivatives. Using the definition of x_j, Eq. (15), we have:

$$[x_j, p_j] = i\hbar(\eta_0 \partial g_j/\partial \eta_j - \eta_j \partial g_j/\partial \eta_0) = i\hbar, \tag{23}$$

from which is follows that g_j is a solution of the equation

$$\eta_0 \partial g_j/\partial \eta_j - \eta_j \partial g_j/\partial \eta_0 = 1. \tag{24}$$

The general solution to Eq. (24) is of the form

$$\Phi\{\eta_0^2 + \eta_j^2, g_j + \arctan(\eta_0/\eta_j)\} = 0. \tag{25}$$

If we further assume that p_4 may be expressed in the form $p_4 = (i\hbar/a)g_4$, then g_4 may be seen to be a solution of the equation

$$\eta_0 \partial g_4/\partial \eta_4 + \eta_4 \partial g_4/\partial \eta_0 = -1, \tag{26}$$

the general solution of which is of the form:

$$\Phi\{\eta_0^2 - \eta_4^2, g_4 - \ln(\eta_0 - \eta_4)\} = 0. \tag{27}$$

In particular, from Eqs. (25) and (27) we may write:

$$p_j = \hbar/a\{(\eta_0^2 + \eta_j^2)/a^2 - \arctan(\eta_0/\eta_j)\}, \tag{28}$$

$$p_4 = i\hbar/a\{(\eta_0^2 - \eta_4^2)/a^2 + \ln(\eta_0 - \eta_4)\}. \tag{29}$$

From Eqs. (28) and (29) it then follows that $[p_\mu, p_\nu] = 0$, which establishes the desired result.[11]

Having made the identifications contained in Eqs. (21), it is natural to inquire into the physical significance that underlies the introduction of a quantized space-time into the coupling scheme derived by Kurşunoğlu. The commutation relations Eqs. (19) and (20) provide a link between the space-time variables x_μ and some parameter a, whose nature has not yet been specified. If we identify a with the Compton wavelength $\hbar/(Mc)$, where M is the mass of an (initially degenerate) internal symmetry multiplet, then the commutation relations directly link the quantal properties of space-time to those of the internal symmetry multiplet. In particular, if the multiplet is subjected to a perturbation which lifts the initial mass degeneracy, the resulting shift in the values of M will directly alter the properties of space-time, through the commutation relations (19) and (20). These results are of interest from an operationalist point of view since they again emphasize how the properties of space-time are intimately bound up with the characteristics of the matter it contains. This, then, is the sense in which space-time symmetries and internal symmetries may be thought of as being coupled via the group LU_4: The commutation relations of the generators of LU_4 through the identifications (21) couple these two symmetries by the statement that the quantal properties of space-time are determined by the characteristics of the matter it contains.

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[11] It should be noted that the relations $[p_\mu, p_\nu] = 0$ do not imply that a flat quantized space-time is translationally invariant. Since the spatial coordinate operators obey Eq. (17), the only transformations consistent with quantization are of the form $x_j \to x_j + ma$, where m is an integer. These transformations, however, do not preserve the form of s^2. The absence of translational degrees of freedom does not, however, introduce any difficulties into the present work.