High-Energy Elastic Scattering at Large Angles and the Statistical Model*

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High-energy elastic p-p scattering data have shown, in addition to the forward peak, a large-angle component which is not inconsistent with isotropy in angular distribution but which decreases very rapidly with energy. It has been proposed by several authors that both the precipitous drop with energy and the angular dependence support a statistical model for large-angle scatterings. While it is known that the statistical model actually does not predict isotropy in angular distribution, we argue under plausible assumptions that the precipitous drop with energy is also not a feature of the statistical model; and hence the present data should not be interpreted as strong indications of the presence of a statistical component in high-energy collisions.

RECENT experiments[1] on the high-energy p-p elastic scattering show that besides the dominant forward peak, there is also at large angles a component which decreases very rapidly with energy but which is not inconsistent with isotropy in the angular distribution. It has been proposed by several authors[2] that both the precipitous drop with energy and the angular dependence support a statistical model for large angle scatterings. While it is well known that the statistical model actually does not predict an isotropic angular distribution,[3] we wish to show in this note under some plausible assumptions that the precipitous drop with energy is also not necessarily predicted by the statistical model; and hence the present data should not be interpreted as definite indications of the presence of a statistical component in high-energy collisions.

To describe the statistical model in more detail, we will use the notations of Van Hove[4]:

We label the 2-particle state by its c.m. momentum $|\mathbf{k}, -\mathbf{k}\rangle$. The elastic amplitude is given by

$$E(k,\theta)\delta(P_{\text{out}}{}^{\mu}-P_{\text{in}}{}^{\mu}) = (2\pi k\sqrt{s})\langle\mathbf{k}', -\mathbf{k}'|(1-S)|\mathbf{k}, -\mathbf{k}\rangle, \quad (1)$$

where $k=|\mathbf{k}|=|\mathbf{k}'|$, $\cos\theta=\hat{k}\cdot\hat{k}'$, $\sqrt{s}$ = total c.m. energy, and P^{μ} denotes the total momentum.

In the statistical model that was considered, the final state $(1-S)|\mathbf{k}-\mathbf{k}\rangle$, is supposed to contain a fraction which proceeds via a compound state, $|\text{comp}\rangle$, that subsequently goes into all possible channels in accordance with the phase space. Fast and Hagedorn[3] have shown that the total phase space increases with energy as $e^{3\cdot2\sqrt{s}}$. Thus, the fraction which goes into the elastic channel should decrease as $e^{-3\cdot2\sqrt{s}}$. Furthermore, in this model the fractin of collisions which proceed via a compound intermediate state decreases roughly as $1/s$, while the maximum angular momentum $L_{\max}$ contributing to this fraction is independent of the incident energy.[3] The resulting function $s^{-1}e^{-3\cdot2\sqrt{s}}$ seems to describe well the energy dependence of the large-angle elastic scattering, and this is the basis for the statistical interpretation. As to the angular dependence, Fast and Hagedorn[3] have emphasized that the isotropic "decay" of the compound state is only an assumption. Nevertheless, insofar as the statistical interpretation was motivated by the near isotropy of the large-angle scattering,[5] this assumption is an integral part of the model. With this assumption, the model seems to be consistent with the present experimental data. However, the presence of a compound intermediate state will not only contribute directly to the elastic amplitude as one of the many open channels, but will also indirectly contribute to the elastic amplitude through unitarity. Insofar as the direct contribution is decreasing with energy as $e^{-3\cdot2\sqrt{s}}$, one should investigate whether the indirect contribution becomes more important.

Consider now the elastic unitarity equation

$$\begin{aligned}2\,\mathrm{Re}\langle\mathbf{k}', -\mathbf{k}'|T|\mathbf{k}, -\mathbf{k}\rangle &= \sum_{\text{elastic}}\langle\mathbf{k},' -\mathbf{k}'|T^{\dagger}|\text{elastic}\rangle\langle\text{elastic}|T|\mathbf{k}, -\mathbf{k}\rangle \\ &\quad+\sum_{\text{inelastic}}\langle\mathbf{k}', -\mathbf{k}'|T^{\dagger}|\text{inelastic}\rangle \\ &\quad\times\langle\text{inelastic}|T|\mathbf{k}, -\mathbf{k}\rangle. \quad (2)\end{aligned}$$

We denote the second term on the right-hand side by $F(k,\theta)$; or, more precisely, we denote

$$\begin{aligned}F_1(k,\theta)\delta(P_{\text{out}}{}^{\mu}-P_{\text{in}}{}^{\mu}) &= (\pi k\sqrt{s})\sum_{\text{inelastic}}{}'\langle\mathbf{k}', -\mathbf{k}'|T^{\dagger}|\text{inelastic}\rangle \\ &\quad\times\langle\text{inelastic}|T|\mathbf{k}-\mathbf{k}\rangle, \quad (3\text{a})\end{aligned}$$

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[1] G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubinstein *et al.* Phys. Rev. Letters **11**, 499 (1963); and W. F. Baker, E. W. Jenkins, A. L. Read, G. Cocconi, V. T. Cocconi *et al.*, *ibid.* **12**, 132 (1964).

[2] L. W. Jones, Phys. Letters **8**, 287 (1964); G. Cocconi (quoted by Orear, Ref. 8). The statistical interpretation was further extended to the process $p+p\rightarrow d+\pi$ based on the same considerations of angle and energy dependences at large angles; see O. E. Overseth, R. M. Heinz, L. W. Jones, M. J. Longo, D. E. Pelett *et al.*, Phys. Rev. Letters **13**, 59 (1964). See also A. Biales and V. F. Weisskopf, CERN report, 1964 (unpublished).

[3] G. Fast and R. Hagedorn, Nuovo Cimento **27**, 208 (1963).

[4] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).

[5] It is to be noted that the minimal amplitude model of T. Kinoshita [Phys. Rev. Letters **12**, 257 (1964)] gives values which are too small near 90°, and the optical model of R. Serber [Rev. Mod. Phys. **36**, 649 (1964)] also gives values too small near 90° for laboratory momenta below 25 BeV/c, although the discrepancy seems to improve for still higher energies.

$$F_2(k,\theta)\delta(P_{\rm out}{}^\mu-P_{\rm in}{}^\mu)=(\pi k\sqrt{s})\langle\mathbf{k}',\,-\mathbf{k}'|T^\dagger|\text{comp}\rangle\times\langle\text{comp}|T|\mathbf{k},\,-\mathbf{k}\rangle,\quad(3\text{b})$$

$$F(k,\theta)=F_1(k,\theta)+F_2(k,\theta)\,,$$

where $\sum'$ denotes summation over all inelastic states except those which proceed via the compound state, and where we have neglected the elastic part in $|\text{comp}\rangle$ in Eq. (3b) because it is a negligible fraction of $|\text{comp}\rangle$.

In terms of $E(k,\theta)$ and $F(k,\theta)$, Eq. (2) becomes

$$2F(k,\theta)=2\,\mathrm{Re}E(k,\theta)-\frac{1}{4\pi}\int d\cos\theta_1 d\cos\theta_2\frac{E^*(k,\theta_1)E(k,\theta_2)\Theta(K)}{K^{1/2}},\quad(4)$$

where $K=1-\cos\theta_1{}^2-\cos\theta_2{}^2-\cos\theta^2+2\cos\theta_1\cos\theta_2\cos\theta$.

As was discussed by Van Hove,[4] at high energies $E(k,\theta)$ is predominantly real. For simplicity in the discussion, we will assume for the following that $E(k,\theta)$, and hence also $F(k,\theta)$, are real, although the validity of the discussion is independent of this assumption.

We will show first of all, that if $E(k,\theta)$ drops precipitously with energy for all fixed θ outside the forward peak, as is indicated by the experiments for the energies measured so far, then $F(k,\theta)$ must also decrease precipitously with energy for fixed θ away from the forward direction. The elastic data are consistent with $E(k,\theta)\sim e^{-kR}$, where R depends on θ but has a finite minimum value R_0 outside the forward peak. We will use e^{-kR_0} to replace the phrase "precipitous drop with energy"; the precise form of the function is, of course, irrelevant since we only wish to avoid the lengthy wording. We write

$$E(k,\theta)=A(k,\theta)\,,\qquad \theta\leq\theta_m\,,$$
$$E(k,\theta)\leq B(\theta)e^{-kR_0},\quad \theta>\theta_m\,.$$

Here θ_m denotes the angle of the forward peak. The constancy of the total cross section implied that $A(k,0)\sim k^2$. For the integral on the right-hand side of Eq. (4), clearly only the contribution from A^*A is not decreasing as e^{-kR_0}. But the Θ function limits the contribution of $A^*(k,\theta_1)A(k,\theta_2)$ to values of θ of the same order of magnitude as θ_m. Thus for all $\theta\gg\theta_m$, the right-hand side of Eq. (4) is decreasing with energy as e^{-kR_0}.

Let us now consider the left-hand side of Eq. (4). When $\mathbf{k}'=\mathbf{k}$, it follows from the definition Eq. (3) that $(4\pi/k^2)F(k,0)$ is the total inelastic cross section, and $(4\pi/k^2)F_2(k,0)$ is that fraction comes from inelastic states that proceed via a compound intermediate state. From what was said before, this fraction goes like $1/s$, that is $F_2(k,0)\sim$const, whereas $F(k,0)\sim k^2$. For $\theta\neq0$, we consider $F_1(k,\theta)$ and $F_2(k,\theta)$ separately. Regarding the angular dependence of $F_1(k,\theta)$, some assumptions will have to be made. Data of high-energy collisions have indicated that in the inelastic final states, most particles emerge within a small cone around the axis of the incident momentum $\mathbf{k}$, the angle of the cone θ_0 decreasing as $s^{-1/2}$. Since an isotropic component has already been subtracted from $\sum'$, it is reasonable to assume that the inelastic states contained in the summation $\sum'$ are even more purely of the type with particles emerging within the forward and backward cones. It follows that $F_1(k,\theta)$ is strongly damped at large angles as the energy increases. More quantitatively, Van Hove[4] has shown from considerations of uncorrelated jet models that the corresponding $F_1(k,\theta)\sim e^{-\theta^2k^2C}$, where C is at most weakly dependent on the energy. Thus, $F_1(k,\theta)$ for $\theta\neq0$ decreases very rapidly with energy, if we assume it has features of these models.

Next, we consider $F_2(k,\theta)$. From the defining relation Eq. (3b), one sees that if we interpret the isotropy of $|\text{comp}\rangle$ as a statement on the amplitude itself, then $\langle\mathbf{k},\,-\mathbf{k}|T|\text{comp}\rangle$ is independent of the direction of $\mathbf{k}$, and $F_2(k,\theta)$ is independent of θ. It would then follow that for finite θ, $F_2(k,\theta)\sim$independent of energy, and this would lead to an immediate contradiction with the behavior required by the right-hand side of Eq. (4). However, this will be an unreasonable stretching of the isotropy requirement. The requirement need only be that the square of the matrix element shall be isotropic, but the individual constituents of the compound state can have contributions which undergo different changes in phase as $\mathbf{k}$ is rotated to $\mathbf{k}'$; and thus $F_2(k,\theta)$ will not be independent of θ.[6] Put in different words, one can expand $\langle\mathbf{k},\,-\mathbf{k}|T|\text{comp}\rangle$ in products of spherical waves; and for an N-particle final state, for instance, after averaging over all $(N-1)$ particles except one, the wave function for the remaining particle need not correspond to $Y_l{}^m$ with $l=0$, but can have any value of l as long as the different m states are equally populated to give a resulting isotropic distribution. Once such an expansion is given, $F_2(k,\theta)$ as a function of θ is determined. Clearly one can arrange by destructive interference to have $F_2(k,\theta)$ decrease in magnitude when θ is outside the forward region, but the amount of destructive interference depends on the maximum value of l in $Y_l{}^m$. As was mentioned before, in the model considered the maximum total angular momentum $L_{\max}$ entering into a compound state is independent of the incident energy, while the dimension of the interaction region contributing to the compound state decreases with energy as $s^{-1/2}$.[7] Thus, the maximum value of l does not increase with energy. Consequently, $F_2(k,\theta)$ at large angles cannot as a function of energy decrease much faster than $F_2(k,0)$. Since $F_2(k,0)\sim$const, at some energy when $F_2(k,\theta)$ eventually dominates over $F_1(k,\theta)$ at large angles, Eq. (4) becomes inconsistent if $E(k,\theta)$ goes like e^{-kR_0} for $\theta\geq\theta_m$. In this sense, the present statistical model does not give rise to an $E(k,\theta)$ that decreases precipitously with energy at large angles.

[6] The author wishes to thank Professor C. N. Yang for stressing this point to him.

[7] See Ref. 3; also, reference to a more detailed description of the model can be found there.

Finally, we consider numerically, but in a very crude approximation, when $F_2(k,\theta)$ is expected to dominate over $F_1(k,\theta)$ for nonforward angles. At $\theta=0$, $F_1(k,0)$ goes like k^2, and dominates over $F_2(k,0)$ which goes like a constant. At 25 BeV, Fast and Hagedorn[3] estimated the ratio $F_2(k,0)/F_1(k,0)$ to be ~ 0.1. The rate of decrease of $F_1(k,\theta)$ with θ is determined in the uncorrelated jet models of Van Hove, by the constant A in $e^{-A\theta^2}$ and $A \geq (k^2/4\pi)\sigma^{\text{inelastic}}$. At 25 BeV, $A \geq 60$. To get an order of magnitude of the variation of $F_2(k,\theta)$ with θ, we consider the very simple case of an N-particle final state of spinless bosons all in the same single-particle state $\sum_{l,m} C_l Y_l^m(\theta_i,\varphi_i)$. From the group property of the rotation matrices, it is easy to see that

$$F_2(k,\theta) \sim \left[\sum_{l=0}^{l_{\max}} |C_l|^2 P_l(\cos\theta)\right]^N.$$

In the model considered, $Nl_{\max}$ as a function of energy is constant. To get a very crude idea of the numbers, we take a fixed value for both N and $l_{\max}$; taking $C_0=0$ to accentuate the variation with θ, and in compensation also neglecting all C_l with $l \geq 2$. This gives a monotonically decreasing function of θ from 0° to 90°; and for small θ, this behaves like $e^{-N\theta^2/2}$. At 25 BeV the average multiplicity is not very high so that $(N/2) \ll A$; consequently at angles of the order of $\theta \geq 1/10$, $F_2(k,\theta)$ should already begin to dominate over $F_1(k,\theta)$.

In conclusion, we have attempted to show that if a compound intermediate state is assumed for high-energy collisions, the indirect contribution to the elastic amplitude through unitarity will most likely dominate over the direct contribution; and that furthermore the indirect contribution is not decreasing precipitously with the energy. The arguments are obviously nonrigorous, particularly with respect to $F_1(k,\theta)$ where assumptions have been made that it possesses features which were deduced from considerations of uncorrelated jet models. This amounts to assuming that for the inelastic final states, we take one part to go via the compound intermediate state and be more or less isotropic, and the other part to be sharply peaked in the forward and backward directions. Conceivably, the other part, which gives rise to $F_1(k,\theta)$, can have different behaviors. However, in order to invalidate our arguments, $F_1(k,\theta)$ must cancel the contributions from $F_2(k,\theta)$. This seems unlikely because the cancellation must occur over a range of angles and energies. Furthermore, in order for this to happen, $F_1(k,\theta)$ must have contributions at large angles which are not rapidly decreasing with energy, and in fact comparable in strength to the statistical contributions. In such a case, the two parts must be intimately interwoven, so that attempts to extract out a statistical component is no longer very meaningful or useful. It will be perhaps more fruitful to attempt to understand the large angle and small angle elastic scatterings in an integrated manner, rather than to treat the two separately, as has already been suggested by Orear.[8]

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[8] J. Orear, Phys. Rev. Letters **12**, 112 (1964).