Angular Correlations in K_{e4} Decays and Determination of Low-Energy π-π Phase Shifts*

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The study of correlations in K_{e4} decays can give unique information on low-energy π-π scattering. To this end we introduce a particularly simple set of correlations. We show that the measurement of these correlations at any fixed π-π c.m. energy allows one to make a model-independent determination of the difference δ_0-δ_1 between the S- and P-wave π-π phase shifts at that energy. Information about the average value of δ_0-δ_1 can be obtained from a measurement of the same correlations averaged over the energy spectrum. Measurement of the average correlations is particularly suited to the testing of any model of low-energy π-π scattering. We discuss in particular two such models: (a) the Chew-Mandelstam effective-range description of S-wave scattering and (b) the Brown-Faier σ-resonance model for the S wave. If the Chew-Mandelstam description is adequate, the suggested measurements should yield a value for the S-wave scattering length in the $I=0$ state. If the σ-resonance model is correct, these measurements should yield a value for the mass of the resonance.

I. INTRODUCTION

THEORETICAL studies[1–4] in the past few years of the decay of a K meson into two pions and a lepton pair have recently been rewarded by the experimental detection of such events in relatively large numbers.[5] This availability of experimental data has raised the hopes of extracting information on the low-energy π-π interaction from such decays.

Several authors have calculated the effects of the final-state π-π interaction on the characteristics of K_{e4} decays.[6–8] This interaction has two different kinds of effects: (a) it determines the phases of the various form factors that appear in the amplitude for the decay; (b) it determines the dominant behavior of these form factors as functions of the c.m. energy of the pions through enhancement effects. The theoretical understanding of the first kind of effects is quite firm (Watson-Fermi theorem); the enhancement effects, however, are not as well understood. To separate the two kinds of effects, we introduce a simple system of angular and energy variables (Sec. II), and define a set of angular correlations in terms of these variables.

Measurement of these angular correlations at a given value of the c.m. energy of the pions can be used to obtain the difference between the S-wave and P-wave π-π scattering phase shifts, $\delta_0-\delta_1$, at that energy, independently of effects of the second kind, as discussed in Secs. III and IV.[9]

In a realistic experiment with limited statistics (say up to 200 to 300 events), it will not be possible to measure the proposed angular correlations as functions of the c.m. energy of the pions, but only their values averaged over the entire energy spectrum. However, such a measurement would still be highly valuable. First of all, it would yield an average value of the quantity $\delta_0-\delta_1$, and secondly, it could be used to test any given model of low-energy π-π scattering.

In Sec. V we consider the particular case in which S-wave scattering in the $I=0$ state is described by a Chew-Mandelstam effective range formula. The measurement of average angular correlations could then yield a value of the S-wave, $I=0$ scattering length. An independent determination of the same quantity could be obtained by a measurement of the spectrum of the c.m. energy of the pions, as discussed by Ciocchetti.[6]

We also discuss briefly (Sec. VII) the model of Brown and Faier,[7] in which S-wave scattering is assumed to be dominated by the postulated σ resonance. Measurement of average correlations could then be used to determine the mass of this resonance.

II. KINEMATICS AND CORRELATIONS

Our approach to the kinematics of the reaction $K^+\rightarrow\pi^+\pi^-e^+\nu$ is the same as that used in analyzing resonances. We visualize this reaction as a two-body decay into a dipion of mass $M_{\pi\pi}$ and a dilepton of mass $M_{e\nu}$. We then consider the subsequent decay of each of these two "resonances" in its own center-of-mass system.

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[2] K. Chadan and S. Oneda, Phys. Rev. Letters **3**, 292 (1959).

[3] V. S. Mathur, Nuovo Cimento **14**, 1322 (1959).

[4] E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **39**, 345 (1960) [English transl.: Soviet Phys.—JETP **12**, 245 (1961)].

[5] R. W. Birge, R. P. Ely, G. Gidal, G. E. Kalmus, A. Kernan, W. M. Powell, U. Camerini, W. F. Fry, J. Gaidos, R. H. March, and S. Natali, Phys. Rev. Letters **11**, 35 (1963). Members of this group have kindly communicated to us that the total of 11 events reported in this paper has now increased to at least 80.

[6] G. Ciocchetti, Nuovo Cimento **25**, 385 (1962).

[7] L. M. Brown and H. Faier, Phys. Rev. Letters **12**, 514 (1964).

[8] B. A. Arbuzov, Nguyen Van Hieu, and R. N. Faustov, Zh. Eksperim. i Teor. Fiz. **44**, 329 (1963) [English transl.: Soviet Phys.—JETP **17**, 225 (1963)].

[9] The usefulness of angular correlations in the determination of $\delta_0-\delta_1$ was first recognized by E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **44**, 765 (1963) [English transl.: Soviet Phys.—JETP **17**, 517 (1963)]. See also erratum, Zh. Eksperim. i Teor. Fiz. **45**, 2085 (1963).

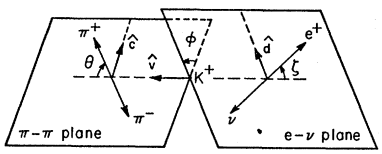

FIG. 1. Angular variables and unit vectors used in the kinematical description of the reaction $K^+ \rightarrow \pi^+\pi^-e^+\nu$.

The total decay is described by the following five variables[10]:

(1) $R^2 \equiv M_{\pi\pi}^2$, the effective mass squared of the dipion system or the square of the c.m. energy of the pions,

(2) $K^2 \equiv M_{e\nu}^2$, the effective mass squared of the dilepton system,

(3) θ, the angle of the π^+ in the c.m. system of the pions with respect to the direction of flight of the dipion in the K^+ rest system,

(4) ζ, the angle of the e^+ in the c.m. system of the leptons with respect to the direction of flight of the dilepton in the K^+ rest system, and

(5) ϕ, the angle between the plane formed by the pions in the K^+ rest system and the corresponding plane formed by the leptons.

Angles θ and ζ are polar; ϕ is azimuthal. Our choice of angles is illustrated in Fig. 1.

To specify the above variables more precisely, let p_+, p_-, p_e, and p_ν be the four-momenta of the π^+, π^-, e^+, and ν, and let $\mathbf{p}_+$ be the three-momentum of the π^+ in the c.m. system of the pions and $\mathbf{p}_e$ the three-momentum of the e^+ in the c.m. system of the leptons. Further, let $\hat{v}$ be a unit vector along the direction of flight of the dipion in the K^+ rest system; $\hat{c}$, a unit vector along the projection of $\mathbf{p}_+$ perpendicular to $\hat{v}$; and $\hat{d}$, a unit vector along the projection of $1\mathbf{p}_e$ perpendicular to $\hat{v}$. (The vectors $\hat{v}$, $\hat{c}$, and $\hat{d}$ are shown in Fig. 1.) We then have:

$$R^2 = (p_+ + p_-)^2; \quad K^2 = (p_e + p_\nu)^2;$$
$$\cos\theta = \hat{v}\cdot\mathbf{p}_+/|\mathbf{p}_+|; \quad \cos\zeta = -\hat{v}\cdot\mathbf{p}_e/|\mathbf{p}_e|;$$
$$\cos\phi = \hat{c}\cdot\hat{d}; \quad \sin\phi = (\hat{c}\times\hat{v})\cdot\hat{d}.$$

The ranges of the angular variables are $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \zeta \leqslant \pi$, and $-\pi < \phi \leqslant \pi$.

We consider the following four correlations:

(a) the distribution in $\cos\theta$ or the forward-backward asymmetry of the π^+ in the c.m. system of the pions;

(b) the distribution in $\sin\phi$ or the up-down asymmetry of the positron with respect to the plane formed by the two pions in the K^+ rest system;

(c) the distribution in $\cos\phi$ or the right-left asymmetry of the positron with respect to the plane formed by the line of flight of the dipion in the K^+ rest system and the normal to the plane of the pions;

(d) the distribution in R^2 or the effective-mass spectrum of the pions.

The term "forward" implies $\cos\theta > 0$, "backward" $\cos\theta < 0$; "up" implies $\sin\phi > 0$, "down" implies $\sin\phi < 0$; "right" implies $\cos\phi > 0$, "left" $\cos\phi < 0$.

Each of these correlations is related to a definite term (or terms) in the expression for the angular distribution of the decay. (See Sec. IV.) Standard correlations, on the other hand, such as an energy spectrum of a particle or the distribution in the angle between two particles, have a more complicated relation to the angular distribution.

III. MATRIX ELEMENT

If we set the positron mass equal to zero and assume a local V-A coupling for the leptons, we can write the matrix element for the decay $K^+ \rightarrow \pi^+\pi^-e^+\nu$ to first order in perturbation theory as

$$(2\pi)^4\delta^{(4)}(p_K - p_+ - p_- - p_e - p_\nu)(G/\sqrt{2}) \times[\bar{\nu}\gamma^\lambda(1+\gamma_5)e]\langle\pi^+\pi^-|J_\lambda{}^V + J_\lambda{}^A|K^+\rangle. \quad (1)$$

In this expression, G is the universal Fermi coupling constant for the weak interactions, $J_\lambda{}^V$ is the vector current of the strongly interacting particles, and $J_\lambda{}^A$ is the axial current. The rest of the notation is obvious.

From invariance considerations we obtain

$$\langle\pi^+\pi^-|J_\lambda{}^V|K^+\rangle = (ih/M_K{}^3)\epsilon_{\lambda\mu\nu\sigma}p_K{}^\mu(p_+ + p_-)^\nu(p_+ - p_-)^\sigma, \quad (2a)$$

and

$$\langle\pi^+\pi^-|J_\lambda{}^A|K^+\rangle = (f/M_K)(p_+ + p_-)_\lambda + (g/M_K)(p_+ - p_-)_\lambda, \quad (2b)$$

where the form factors f, g, and h are, in general, functions of R^2, (p_Kp_+), and (p_Kp_-). The factor $M_K{}^{-3}$ in Eq. (2a) and the factor $M_K{}^{-1}$ in (2b) have been inserted to make f, g, and h dimensionless. Due to the opposite relative intrinsic parities of the K^+ and $\pi^+\pi^-$ states, the matrix element of the vector current between these two states, Eq. (2a), transforms as an axial vector, whereas that of the axial current, Eq. (2b), transforms as an ordinary vector.

Since the singularities in the variables (p_Kp_+) and (p_Kp_-) are far from the physical region, it is reasonable to neglect the dependence of f, g, and h on these two variables. This is equivalent to assuming that the two pions are emitted only in $l=0,1$ relative angular-momentum states.[11,11a] If we further assume that the $|\Delta I| = \frac{1}{2}$ rule is valid, then, from symmetry considerations, the f term is just the amplitude for the pions to be

[10] Our metric is $(AB) = A_0B_0 - \mathbf{A}\cdot\mathbf{B}$.

[11] Nguyen Van Hieu, Zh. Eksperim. i Teor. Fiz. 44, 162 (1963) [English transl.: Soviet Phys.—JETP 17, 113 (1963)].

[11a] *Note added in proof*. A more detailed discussion of the structure of the K_{e4} matrix element, including the general dependence of the form factors on all three variables, has been given in a recent unpublished paper by C. Kacser, P. Singer, and T. N. Truong.

emitted in an $I=0$, $l=0$ state, while the g and h terms give the amplitude for emission in an $I=1$, $l=1$ state.

The final-state interaction of the pions manifests itself in the matrix element (1) in two ways. First, by the Watson-Fermi theorem,[12] it determines the phases of f, g, and h as follows:

$$f=\tilde{f}e^{i\delta_0(R^2)},\quad g=\tilde{g}e^{i\delta_1(R^2)},\quad h=\tilde{h}e^{i\delta_1(R^2)},\qquad (3)$$

where $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ are real. The quantities δ_0 and δ_1 are the $I=0$, $l=0$, and $I=1$, $l=1$ $\pi-\pi$ scattering phase shifts; they are functions only of the total energy of the pions in their center-of-mass system. Secondly, the π-π interaction gives rise to enhancement effects which are assumed to determine the dominant R^2 dependence of $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$.

IV. ANGULAR DISTRIBUTION

We integrate the decay probability obtained from expressions (1), (2), and (3) over the variables K^2 and $\cos\zeta$ to obtain an expression for the angular distribution in θ and ϕ as a function of $x^2=R^2/M_K^2$. Details of this integration are given in Appendix A.

$$\begin{aligned}d\Gamma(x^2)=&[G^2\pi^2M_K^5/(32(2\pi)^8)]dx^2d\cos\theta d\phi\{\tilde{f}^2A(x^2)\\&+\tilde{g}^2[B(x^2)+C(x^2)\cos^2\theta+D(x^2)\sin^2\theta\sin^2\phi]\\&+\tilde{h}^2E(x^2)\sin^2\theta(1+2\cos^2\phi)\\&+\tilde{f}\tilde{g}\cos(\delta_0-\delta_1)S(x^2)\cos\theta\\&+\tilde{f}\tilde{g}\sin(\delta_0-\delta_1)T(x^2)\sin\theta\sin\phi\\&+\tilde{f}\tilde{h}\cos(\delta_0-\delta_1)U(x^2)\sin\theta\cos\phi\\&+\tilde{g}\tilde{h}V(x^2)\cos\theta\sin\theta\cos\phi\}.\end{aligned}\qquad (4)$$

In this equation the quantities $A(x^2)$, $B(x^2)$, $C(x^2)$, etc. are well-defined functions of x^2. If we let $\beta=(1-4m_\pi^2/R^2)^{1/2}$, they are given by

$$\begin{aligned}A(x^2)&=(\beta/96)(1-8x^2+8x^6-x^8-12x^4\ln x^2)\\&=S\text{-wave } R^2 \text{ correlation},\\B(x^2)&=(\beta^3/72)[x^2+9x^4-9x^6-x^8+6x^4(1+x^2)\ln x^2]\\&=P\text{-wave } R^2 \text{ correlation},\\C(x^2)&=\beta^2A(x^2)+B(x^2)\\&=P\text{-wave } \cos^2\theta \text{ correlation},\\D(x^2)&=2B(x^2)\\&=P\text{-wave } \sin^2\theta\sin^2\phi \text{ correlation},\\E(x^2)&=(\beta^3/192)[\tfrac{1}{5}x^2-3x^4-16x^6+16x^8+3x^{10}\\&\qquad-\tfrac{1}{5}x^{12}-12x^6(1+x^2)\ln x^2]\\&=P\text{-wave } \sin^2\theta(1+2\cos^2\phi) \text{ correlation},\\S(x^2)&=(\beta^2/144)[3-20x^2+90x^4-128x^5+60x^6-5x^8]\\&=S\text{–}P\text{-wave interference } \cos\theta \text{ correlation},\\T(x^2)&=(\pi/105)\beta^2[x-14x^3+35x^4-35x^5+14x^6-x^8]\\&=S\text{–}P\text{-wave interference } \sin\theta\sin\phi \text{ correlation},\end{aligned}$$

[12] K. M. Watson, Phys. Rev. **95**, 228 (1954); E. Fermi, Nuovo Cimento **2**, Suppl. 1, 17 (1955).

$$\begin{aligned}U(x^2)=&(\pi/2)\beta^2\Big[x^3\int_x^{(1+x^2)/2}d\xi(\xi^2-x^2)^{1/2}(1+x^2-2\xi)^{1/2}\\&-x\int_x^{(1+x^2)/2}d\xi\,\xi^2(\xi^2-x^2)^{1/2}(1+x^2-2\xi)^{1/2}\Big]\\=&S\text{–}P\text{-wave interference } \sin\theta\cos\phi \text{ correlation},\end{aligned}$$

and

$$\begin{aligned}V(x^2)=&(\pi/16)\beta^3x\{4x^2[\tfrac{1}{3}(1+x^2)(1-x)^3-\tfrac{1}{5}(1-x)^5]\\&-[\tfrac{1}{3}(1+x^2)^3(1-x)^3-\tfrac{3}{5}(1+x^2)^2(1-x)^5\\&+(3/7)(1+x^2)(1-x)^7-\tfrac{1}{9}(1-x)^9]\}-\beta x^2T(x^2).\\=&P\text{-wave-vector–axial-vector interference}\\&\cos\theta\sin\theta\cos\phi \text{ correlation}.\end{aligned}$$

[*Note added in proof.* An earlier version of this paper, Laurence Radiation Laboratory Report No. UCRL-11590, incorrectly defined the function $T(x^2)$ with an over-all minus sign.]

We can now indicate the simple connection, mentioned in Sec. II, between our correlations and the angular distribution. An examination of Eq. (4) shows that the π^+ forward-backward asymmetry arises from the $S(x^2)$ term; the positron up-down asymmetry from the $T(x^2)$ term; the positron right-left asymmetry from the $U(x^2)$ term; and the π-π effective mass spectrum from the $A(x^2)$, $B(x^2)$, $C(x^2)$, $D(x^2)$, and $E(x^2)$ terms. For fixed values of $W=(R^2)^{1/2}$, the total c.m. energy of the pions, the correlations are given explicitly by the following expressions:

(a) Forward-backward asymmetry:

$$\frac{d(N_F-N_B)}{dW}=\tilde{f}\tilde{g}\cos(\delta_0-\delta_1)\frac{G^2\pi^3M_K^4}{8(2\pi)^8}[xS(x^2)];\qquad (5)$$

(b) Up-down asymmetry:

$$\frac{d(N_\uparrow-N_\downarrow)}{dW}=\tilde{f}\tilde{g}\sin(\delta_0-\delta_1)\frac{G^2\pi^3M_K^4}{8(2\pi)^8}[xT(x^2)];\qquad (6)$$

(c) Right-left asymmetry:

$$\frac{d(N_R-N_L)}{dW}=\tilde{f}\tilde{h}\cos(\delta_0-\delta_1)\frac{G^2\pi^3M_K^4}{8(2\pi)^8}[xU(x^2)];\qquad (7)$$

(d) Effective-mass spectrum of the pions:

$$\frac{d\Gamma(x^2)}{dW^2}=\frac{G^2\pi^3M_K^3}{8(2\pi)^8}\Big\{\tilde{f}^2A(x^2)+\tilde{g}^2[B(x^2)+\tfrac{1}{3}C(x^2)+\tfrac{1}{3}D(x^2)]+\tilde{h}^2\tfrac{4}{3}E(x^2)\Big\}.\qquad (8)$$

By measuring angular correlations at a fixed value of W, one can obtain $\tan(\delta_0-\delta_1)$ at that energy from the ratio of the up-down to forward-backward asymmetries without having to know the values of $\tilde{f}$ and $\tilde{g}$. Furthermore, it is clear from Eqs. (5) through (8) that the various correlations can, in principle, also be used to determine $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ at any given value of W.

In view of the poor experimental statistics likely to be available in the foreseeable future, it is more practical to look at the angular correlations averaged over W. In this manner one should still be able to get an idea of the sign and magnitude of the average value of $(\delta_0-\delta_1)$ in the energy region under consideration.

V. EFFECTIVE RANGE MODEL

Since a direct determination of the phase shifts is likely to prove difficult, an alternative approach is to attempt a fit of the experimental data to a given model for f and g. (To a very good approximation, the contribution of the h term is expected to be negligible,[13] and we shall ignore it in this section. In any case, this contribution does not affect the determination of the phase shifts, since it gives rise to an angular correlation different from those correlations used for this purpose.)

For example, one could assume that the low-energy π-π scattering in the $I=0$, $l=0$ state can be described by a large scattering length,[14] and that the scattering in the $I=1$, $l=1$ state is dominated by the ρ resonance. Since the ρ singularity is rather distant from the physical region, one sets g equal to a constant value g_0. One assumes that the R^2 dependence of f is given by the relativistic Watson enhancement factor[15]:

$$f\propto(1/\beta)\sin\delta_0(R^2)e^{i\delta_0(R^2)}, \qquad (9)$$

where $\beta=(1-4m_\pi^2/R^2)^{1/2}$, as in Sec. IV. The R^2 dependence of the phase shift is assumed to be given by the Chew-Mandelstam effective-range formula[16]:

$$\cot\delta_0=1/(\beta a_0)+(2/\pi)\ln\{[(R^2)^{1/2}/2m_\pi](1+\beta)\}, \qquad (10)$$

where a_0 is the π-π scattering length for the $I=0$, $l=0$ state in units of the pion Compton wavelength. One then proceeds to find that value of the parameter a_0 which gives the best fit to the experimental data.

Using this model, we write the angular distribution as a function of $x^2=R^2/M_K^2$ in the form

$$d\Gamma(x^2)=dx^2d\cos\theta d\phi\frac{10^{-5}G^2\pi^2M_K^5f_0^2}{32(2\pi)^8}\times\{A_1(x^2)+\eta A_2(x^2)\cos\theta+\eta A_3(x^2)\sin\theta\sin\phi+\eta^2A_4(x^2)\cos^2\theta+\eta^2A_5(x^2)(1+2\sin^2\theta\sin^2\phi)\}, \qquad (11)$$

where $f_0=f(R^2=4m_\pi^2)$, and $\eta=g_0/f_0$. The correlation coefficients $A_1(x^2)$, $A_2(x^2)$, and $A_3(x^2)$ are plotted in Figs. 2, 3, and 4 for various values of the scattering length, a_0. In plotting these curves, we have normalized the Watson enhancement factor to unity at $R^2=4m_\pi^2$. Thus no significance should be attached in Fig. 2 to the relative magnitudes of the plots of $A_1(x^2)$ for the various values of a_0. A similar statement hold for Figs. 3 and 4.

We have also calculated the correlation coefficients in the angular distribution averaged over R^2:

$$d\Gamma=d\cos\theta d\phi 10^{-5}\frac{G^2\pi^2M_K^5f_0^2}{32(2\pi)^8}\times\{A_1+\eta A_2\cos\theta+\eta A_3\sin\theta\sin\phi+\eta^2A_4\cos^2\theta+\eta^2A_5(1+2\sin^2\theta\sin^2\phi)\}. \qquad (12)$$

The values of A_1, A_2, A_3, A_4, and A_5 are given in Table I as functions of the scattering length, a_0. Again the relative magnitudes of A_1 for different values of a_0

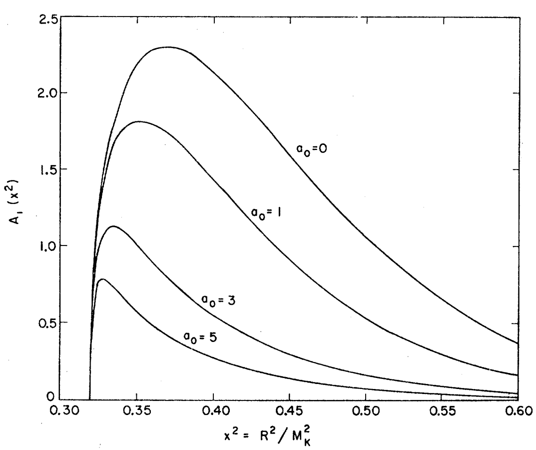

FIG. 2. The correlation coefficient $A_1(x^2)$ for various values of a_0. All the curves go to zero at $x^2=1$.

TABLE I. Coefficients in the angular distribution for the decay $K^+\to\pi^+\pi^-e^+\nu$ as functions of the $I=0$, $l=0$, π-π scattering length, a_0.

a_0	A_1	A_2	A_3	A_4	A_e
0	4.16	4.67	0	1.26	0.23
1	2.64	3.28	0.73	1.26	0.23
2	1.67	2.22	0.83	1.26	0.23
3	1.13	1.58	0.77	1.26	0.23
4	0.81	1.19	0.69	1.26	0.23
5	0.61	0.94	0.62	1.26	0.23

[13] The extra factor M_K^{-2} in the h term [Eq. (2a)] as compared to the f and g terms [Eq. (2b)] ensures that the contribution to the decay probability of terms involving h is small compared to that of terms involving only f and g, unless h is much larger than f or g. A rough estimate of h based on the dominance of nearest-pole terms and on unitary symmetry considerations shows that h is probably small compared to f.

[14] A large $I=0$, $l=0$ π–π scattering length is suggested by the results of N. E. Booth, A. Abashian, and K. M. Crowe, Phys. Rev. Letters **7**, 35 (1961).

[15] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[16] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

have no significance. The same holds true for A_2 and A_3. The ratio A_3/A_2 depends uniquely on a_0, and can be used to determine this quantity.

For $a_0=0$, we have $A_3=\int dx^2 \tilde{f}\tilde{g}\sin(\delta_0-\delta_1)T(x^2)=0$, since $\sin(\delta_0-\delta_1)=0$.

VI. RESONANCE MODEL

Another model for describing final-state effects in K_{e4} decays is that proposed by Brown and Faier,[7] who assumed that the $I=0$, $l=0$ π-π scattering is dominated by a postulated resonance, the σ, with mass about 400 MeV and width about 75 MeV. They used this model to calculate various spectra for the decay $K^+\to\pi^+\pi^-e^+\nu$ as functions of the resonance parameters.

The existence of such a resonance could be decided on the basis of the effective-mass spectrum of the pions and from the peculiar behavior which it would cause in angular correlations in the vicinity of the resonant energy. If δ_1 is indeed small, the up-down asymmetry of the positron should have a peak at this energy, while the forward-backward asymmetry of the π^+ should go through zero, since δ_0 would pass through $\pi/2$.

ACKNOWLEDGMENTS

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APPENDIX A. PHASE-SPACE INTEGRATION

In this appendix we present an outline of the phase-space integration used to obtain Eq. (4).

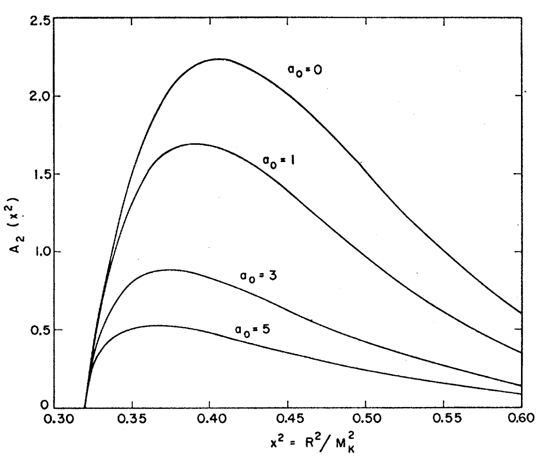

FIG. 3. The correlation coefficient $A_2(x^2)$ for various values of a_0. All the curves go to zero at $x^2=1$.

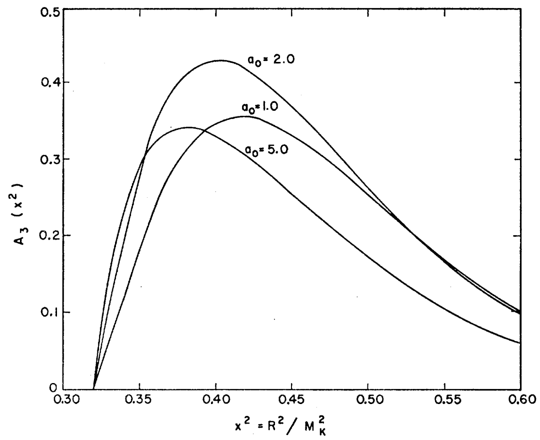

FIG. 4. The correlation coefficient $A_3(x^2)$ for various values of a_0. All the curves go to zero at $x^2=1$.

We define the following variables:

$$R=p_++p_-;\quad K=p_e+p_\nu;$$
$$Q=p_+-p_-;\quad L=p_e-p_\nu.$$

The square of the matrix element [Eq. (1)] summed over the lepton polarizations can then be written as

$$(2\pi)^4\delta^{(4)}(R+K-p_K)(G^2/2)T^{\mu\lambda}V_\mu V_\lambda^*, \qquad \text{(A1)}$$

where

$$T^{\mu\lambda}=(K^\mu K^\lambda-L^\mu L^\lambda-K^2g^{\mu\lambda}-i\epsilon_{\mu\lambda\alpha\beta}K_\alpha L_\beta)$$

and

$$V_\mu=(f/M_K)R_\mu+(g/M_K)Q_\mu+(ih/M_K{}^3)\epsilon_{\mu\nu\rho\sigma}R^\nu Q^\rho K^\sigma.$$

The total decay rate in the K^+ rest system is given by

$$\Gamma=[G^2/(32(2\pi)^8M_K)]I, \qquad \text{(A2)}$$

where

$$I=\tfrac{1}{2}\int\frac{d^3p_+}{E_+}\frac{d^3p_-}{E_-}\frac{d^3p_e}{E_e}\frac{d^3p_\nu}{E_\nu}\times\delta^{(4)}(R+K-p_K)T^{\mu\lambda}V_\mu V_\lambda^*. \qquad \text{(A3)}$$

We rewrite I as

$$I=\tfrac{1}{2}\int d^4Rd^4K\,\delta^{(4)}(R+K-p_K)\times\int\frac{d^3p_+}{E_+}\frac{d^3p_-}{E_-}\delta^{(4)}(p_++p_--R)V_\mu V_\lambda^*\times\int\frac{d^3p_e}{E_e}\frac{d^3p_\nu}{E_\nu}\delta^{(4)}(p_e+p_\nu-K)T^{\mu\lambda}.$$

The integrations over the three δ functions can be performed separately:

$$\int d^4R \int d^4K\, \delta^{(4)}(R+K-p_K)$$
$$=\int dM_{\pi\pi}^2 dM_{e\nu}^2 \int d^4R d^4K$$
$$\times \delta^{(4)}(R+K-p_K)\delta(M_{\pi\pi}^2-R^2)\delta(M_{e\nu}^2-R^2)$$
$$=\tfrac{1}{4}\int dR^2 dK^2 \int \frac{d^3R}{R_0}\frac{d^3K}{K_0}\delta^{(4)}(R+K-p_K)$$
$$=\pi \int dR^2 dK^2 (P/M_K)\,, \quad \text{(A5)}$$

where $P=|\mathbf{R}|$ in the K^+ rest system; P can be expressed in terms of R^2 and K^2.

We perform the integrations over the pion and lepton δ functions in the respective rest frames of these particles:

$$\int \frac{d^3p_+}{E_+}\frac{d^3p_-}{E_-}\delta^{(4)}(p_+ + p_- - R)V_\mu V_\lambda^*$$
$$=\pi\beta \int d\cos\theta\; V_\mu V_\lambda^*\,, \quad \text{(A6)}$$

where

$$\beta=(1-4m_\pi^2/R^2)^{1/2};$$

and

$$\int \frac{d^3p_e}{E_e}\frac{d^3p_\nu}{E_\nu}\delta^{(4)}(p_e+p_\nu-K)T^{\mu\lambda}$$
$$=\tfrac{1}{2}\int d\phi \int d\cos\zeta\; T^{\mu\lambda}\,, \quad \text{(A7)}$$

where we have set $m_e=0$. The integral I can now be written in the form:

$$I=\pi^2 \int dR^2 \int d\cos\theta \int d\phi \frac{1}{2M_K}$$
$$\times \int dK^2 \beta P V_\mu V_\lambda^* \tfrac{1}{2}\int d\cos\xi T^{\mu\lambda}\,. \quad \text{(A8)}$$

Integration over $\cos\zeta$ gives

$$\langle T^{\mu\lambda}\rangle = \tfrac{1}{2}\int d\cos\zeta\; T^{\mu\lambda}$$
$$=\tfrac{2}{3}(K^\mu K^\lambda - K^2 g^{\mu\lambda})+\tfrac{1}{3}K^2(n^\mu n^\lambda - d^\mu d^\lambda)$$
$$-i\frac{\pi}{4}(K^2)^{1/2}\epsilon_{\mu\lambda\alpha\beta}K_\alpha d_\beta\,, \quad \text{(A9)}$$

where $d=(0;\hat{d})$ and $n=(0;\hat{d}\times\hat{v})$ in the c.m. system of the leptons. The unit vectors $\hat{d}$ and $\hat{v}$ are defined in Sec. II.

It is convenient to change from the variable K^2 to $R_0=(R^2+P^2)^{1/2}$. In the K^+ rest frame, we have

$$K^2=(p_K-R)^2=M_K^2+R^2-2MR_0$$

and $dR_0=(-1/2M_K)dK^2$. The range of integration is given by $(R^2)^{1/2}<R_0<(1/2M_K)(M_K^2+R^2)$.

Contracting $\langle T^{\mu\lambda}\rangle$ with $V_\mu V_\lambda^*$ and evaluating the resulting expression in the c.m. system of the pions, we obtain Eq. (4) for the decay rate. The coefficients $A(x^2)$, $B(x^2)$, etc. in that equation are to be evaluated from the following integrals:

$$A(x^2)=M_K^{-4}\tfrac{2}{3}\beta \int dR_0 P^3\,,$$

$$B(x^2)=M_K^{-6}\tfrac{1}{3}\beta^3 R^2 \int dR_0 K^2 P\,,$$

$$E(x^2)=M_K^{-8}\tfrac{1}{3}\beta^3 R^2 \int dR_0 K^2 P^3\,,$$

$$S(x^2)=M_K^{-5}\tfrac{4}{3}\beta^2\left(M_K \int dR_0 P^2 R_0 - R^2 \int dR_0 P^2\right),$$

$$T(x^2)=M_K^{-5}\frac{\pi}{2}\beta^2 (R^2)^{1/2}\int dR_0 P^2 (K^2)^{1/2}\,,$$

$$U(x^2)=-M_K^{-6}\frac{\pi}{2}\beta^2 (R^2)^{1/2}\int dR_0 P^3 (K^2)^{1/2}\,,$$

$$V(x^2)=-M_K^{-7}\frac{\pi}{2}\beta^3 (R^2)^{1/2}$$
$$\times\left(M_K \int dR_0 P^2 R_0 (K^2)^{1/2} - R^2 \int dR_0 P^2 (K^2)^{1/2}\right).$$

The only integral which cannot be evaluated in terms of elementary functions is that occurring in the expression for $U(x^2)$; it can be evaluated in terms of elliptic functions, if desired.