PHYSICAL REVIEW VOLUME 137, NUMBER 2B 25 JANUARY 1965

Relations Between Internal Symmetry and Relativistic Invariance

LOUIS MICHEL*
Summer Institute for Theoretical Physics, University of Wisconsin, Madison, Wisconsin

The problem of combining relativistic invariance and internal symmetry is reviewed, and a critical evaluation of very recent papers on this subject is made. It is proposed that the Poincaré group P is not a subgroup of the group E of invariance of a relativistic quantum theory, but is the quotient $P=E/S$, where S is the internal symmetry group. Pertinent mathematical results are given.

THERE have been recently[1–4] several papers giving theorems on some reasons for the impossibility of "mixing" the connected Poincaré group P with an internal symmetry group S. Indeed, since relativistic theory is invariant under P and since there seems to be an internal symmetry group S (SU_3 being favored, of course) for strong interactions, one cannot avoid the question of how to consider P and S together in strong-coupling physics.

It may seem natural to consider P and S as subgroups of a larger group G, and this was done in the quoted references. The group G could be very large. In the quoted papers, G was taken as the smallest possible group compatible with the condition that no element of S can be identified with a Poincaré transformation. We give a necessary and sufficient mathematical condition for such a choice. It implies that every $x\in G$ is decomposable into a unique product $x=sp$ with $s\in S$ and $p\in P$ (or in a unique way $x=p's'$ with $s'\in S$ and $p'\in P$).

More physics should be injected to establish some property of the group G. The very interesting basic idea of Ref. 1 is: Although the internal quantum numbers due to S (think of SU_3) are Lorentz invariant (the particles in a multiplet have the same spin), they may not be translation invariant, since the particles in a multiplet have different masses and the mass is a function of different quantum numbers of S (i.e., total isospin and hypercharge). However, McGlinn proved[1] that if *all* infinitesimal operators of S, a semisimple Lie group, commute with *every* infinitesimal operator of L, the homogeneous Lorentz group, then they all have to commute with every infinitesimal operator of T, the translation group. In Refs. 2–4, the authors sharpen this theorem and give several theorems of the same type, with a similar negative conclusion for "mixing" S and P (i.e., obtaining noncommutation of some of their elements, which might give rise to relations between quantum numbers of S and P, as, for instance, a mass relation).

The object of Sec. I is to prove a mathematically trivial lemma giving the weakest possible hypothesis which yields this negative conclusion. This lemma is much stronger than all theorems in Refs. 1–4. For instance, a sufficient assumption for the McGlinn theorem is that there is *one* Lorentz transformation Λ such that the commutator $s\Lambda s^{-1}\Lambda^{-1}$ is a Poincaré transformation for *every* $s\in S$.

In Sec. II we try to evaluate the physical implications of these McGlinn-type theorems. In fact, the requirement that the Poincaré group be a subgroup of the invariance group E of a relativistic quantum theory is not a reasonable physical assumption (see, however, Ref. 14). Indeed the existence of half-integer spins shows that $\bar{P}$, the covering of P, has to be considered, and P is not a subgroup of $\bar{P}$. In a formalism with a group S of unobservable transformations (e.g., gauge group, isospin or internal symmetry group, group of the arbitrary phase in the classic Wigner analysis of ray representations) this arbitrariness of degrees of freedom of the formalism appears in the following fashion: The successive application to the formalism of several Poincaré transformations whose product is the identity Poincaré transformation will, in general, yield an unobservable transformation. We show that the mathematical analysis of this situation is: The group S is an invariant subgroup of E, the invariance group of the formalism, and the Poincaré group P is the quotient E/S.

In Sec. III, we give pertinent mathematical results for the problem of finding E, given S and P. For the cases where the mathematical solution is known, the only physically interesting results are summarized by the formula (1).

It should be acknowledged that this last point of view for the relations between relativistic invariance and internal symmetry, although difficult to discard in the present stage of relativistic quantum theories, does not give any help to the very worthwhile purpose of understanding the nature of broken symmetry.

I. STRUCTURE OF THE GROUP $G=S\cdot P$

Given two sets H and K of elements of a group G, we denote by $H\cdot K$ the set of elements of G which are products of one element of H and one element of K (in the given order). A well-known lemma is:

Lemma. If H and K are subgroups of a group and $H\cdot K=K\cdot H$, then $H\cdot K$ is also a subgroup of this group.

Proof: Let $\alpha, \beta, \gamma, \cdots \in H$ and $a, b, c, \cdots \in K$. The hypothesis implies: Given a pair βb, there is at least a

* Permanent address: Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette (S.-et-O.), France.
[1] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964).
[2] O. W. Greenberg, Phys. Rev. **135**, B1447 (1964).
[3] M. E. Mayer, H. J. Schnitzer, E. C. G. Sudarshan, R. A. Acharya, M. Y. Han, Phys. Rev. (to be published).
[4] F. Coester, M. Hamermesh, W. D. McGlinn, Phys. Rev. **135**, B451 (1964).

pair β', b' such that $b\beta=\beta'b'$. Given $\alpha a\in H\cdot K$, $(\alpha a)^{-1}=a^{-1}\alpha^{-1}=(\alpha^{-1})'(a^{-1})'\in H\cdot K$; furthermore, given αa and $\beta b\in H\cdot K$, then $\alpha a\beta b=\alpha\beta'a'b\in H\cdot K$; hence $H\cdot K$ is a group.

Given $x\in H\cdot K$. Is the decomposition $x=\alpha a$ unique? Let us assume two decompositions: $x=\alpha_1 a_1=\alpha_2 a_2$. Then $\alpha_2^{-1}\alpha_1=a_1a_2^{-1}\in K$ and $\alpha_2^{-1}\alpha_1\in H$; so the assumption $H\cap K=\{1\}$ implies $\alpha_1=\alpha_2$, $a_1=a_2$; hence the uniqueness of the decomposition.

Physically, the elements of S and P are given different names and are distinguished. Moreover, if one assumes that $S\cdot P=P\cdot S$, that is, that the product of a Poincaré transformation by an internal symmetry can also be obtained by making first an internal symmetry, and then a Poincaré transformation, the group $S\cdot P=P\cdot S$ is the group G itself and any element of G has a unique decomposition $x=\alpha a$, $\alpha\in S$, $a\in P$. Let H and K be subgroups of a group G with $H\cap K=\{1\}$ and $H\cdot K=G$. For a pair x, y of elements of G, the relation $x^{-1}y\in K$ (or $xy^{-1}\in H$) is an equivalence relation whose classes of equivalence are called "left cosets of K" (right cosets of H); they form a set H' (K'). There is a one-to-one correspondence between the elements of the sets H and H' (K and K'); Every element of H (K) belongs to a coset H' (K') and is called a representative of this coset. Left (right) translations on G are permutations of the left cosets of K (right cosets of H); indeed $x^{-1}y\in K$ $(xy^{-1}\in H)$ implies for every $z\in G$, $(zx)^{-1}(zy)\in K$ $(xz(yz)^{-1}\in H)$. It is also easy to verify that the permutation corresponding to the element zz' of G is the product of the permutations generated by z and z'. This establishes a homomorphism h' (k'):

$$G\xrightarrow{h'}\mathcal{P}(H')\quad(G\xrightarrow{k'}\mathcal{P}(K')),$$

where $\mathcal{P}(X)$ is the permutation group of the elements of the set X. The one-to-one natural mapping between H and H' (K and K') transfers this homomorphism to

$$G\xrightarrow{h}\mathcal{P}(H)\quad(G\xrightarrow{k}\mathcal{P}(K)).$$

These well-known properties—summarized in mathematics by the statements: H' (K') is the G-homogeneous space G/K $(H\backslash G)$—make the proof of our main lemma very simple.

Lemma: Let H and K be subgroups of G such that every element $x\in G$ is a unique product $x=\alpha a$, $\alpha\in H$, $a\in K$. If for *one* a of K and every $\alpha\in H$, $\alpha^{-1}a\alpha\in K$, and if K is the only invariant subgroup of K containing a, then G is a semidirect product $G=K\times H$ with K as invariant subgroup. (Of course one can permute the role of K and H, a and α, left and right, in order to get a syntactically symmetric lemma.)

Indeed, $\alpha^{-1}a\alpha\in K$ implies that α and $a\alpha$ are in the same left coset of K and this for all $\alpha\in H$. So $a\in\text{Ker}\,h$, the kernel of h, which is an invariant subgroup of G. Moreover, Ker $h\cap K$ is an invariant subgroup of K which contains a; from our hypothesis it is K itself, hence $K=K\cap\text{Ker}\,h\subset\text{Ker}\,h$. This implies, for every $b\in K$ and every $z\in G$, that z and bz are in the same left coset of K, i.e., $z^{-1}bz\in K$. This proves that K is an invariant subgroup of G and, by definition of the semidirect product, $G=K\times H$.

Let $H=P$, the Poincaré group ($P=T\times L$, the translation group T being the only proper invariant subgroup of P). Let $S=K$ be *any* group. In the physical interpretation, S is the internal symmetry group. Instead of the requirement[1–3] that elements of S be Lorentz invariant (i.e., for every $s\in S$ and every $\Lambda\in L$, $s\Lambda=\Lambda s$), our lemma, worded for P and S groups, yields the minimum hypothesis to obtain the conclusion of the theorems of Refs. 1, 2 and Theorem 1 of Ref. 3.

Lemma for P. Let S be any group and S and P subgroups of G such that any $z\in G$ has a unique decomposition into a product $z=sp$, with $s\in S$ and $p\in P$. If there is *one* $q\in P$ which is not a translation and such that for all $s\in S$, $s^{-1}qs\in P$, then G is a semidirect product $G=P\times S$.

Let us make precise which semidirect product is meant by $G=P\times S$. For every group R, the group Int$R$ of inner automorphisms is an invariant subgroup of Aut$R$, the group of automorphisms, and the quotient is Aut$R$/Int$R$=Out$R$, the group of outer automorphisms of H. If $G=K\times H$, every element of G induces (by inner automorphisms) an automorphism of the invariant subgroup K; this can be translated by the existence of a homomorphism $G\xrightarrow{f}\text{Aut}K$. The image of f is the semidirect product $f(K)\times f(H)$, since $f(K)=\text{Int}K$, $f(H)\subset\text{Out}K$. In order to make precise which semidirect product is $G=K\times H$, one has therefore to make precise the corresponding homomorphism $H\xrightarrow{f'}\text{Out}K$ (where f' is the restriction of f to H).

The Poincaré group has no center. Its group of automorphisms is $\text{Aut}P=(P\times Z_2)\otimes R^{\times}$, where the non-unit element of the two-element group Z_2 corresponds to space symmetries; $R^{\times}$, the multiplicative group of the real numbers, corresponds to the automorphisms $(a,\Lambda)\to(\alpha a,\Lambda)$ of P (α is a real number, $a\in T$, $\Lambda\in L$); and $\otimes$ means direct product.[5]

So the semidirect product $G=P\times S$ is completely defined by the homomorphism $S\xrightarrow{f'}Z_2\otimes R^{\times}$, or, since the image of f' is an Abelian group, by $S/S'\xrightarrow{f''}Z_2\otimes R^{\times}$, where S' is the group generated by the commutators of S. Since T is invariant under all outer automorphisms of P, the structure of G can be written more explicitly: $G=T\times(L\times S)$. By the (well-known to physicists) Frobenius-Wigner method for the characterization of unitary irreducible representations of semidirect pro-

[5] The direct product is a special case of the semidirect product when f' is the trivial homomorphism. It might be worth remarking that all algebraic automorphisms of the Poincaré group P are continuous automorphisms of the topological group P. So Aut$P$ given here is the group of automorphisms of P even when its topology is neglected.

ducts, one immediately establishes that physical representations of G (for $m \geqslant 0$) are labeled by mass, spin, possibly parity, and the invariants of $\mathrm{Ker} f' \supset S'$. Hence all masses are equal in a multiplet.

Theorem 2 of Ref. 3 and that in Ref. 4, are superseded by the application of the lemma for S.

Lemma for S. Let S and P be subgroups of G such that $G = S \cdot P = P \cdot S$ and $S \cap P = \{1\}$. If there is *one* element $s \in S$ such that for all $p \in P$, $psp^{-1} \in S$, and if no proper invariant subgroup of S contains s, then G is a semidirect product $G = S \times P$.

If, furthermore, there are no nontrivial homomorphisms of P into $\mathrm{Out}S$—e.g.,[6] S is a finite group, a compact Lie group, or a semisimple Lie group on the real field (e.g., SU_3)—then G is the direct product $S \otimes P$. Of course, if one applies the hypothesis of the lemma for both $S = H$, $P = K$ and $S = K$, $P = H$, then, whatever the arbitrary group S, $G = S \otimes P$ (generalization of Theorems 3–5 of Ref. 3).

II. PHYSICAL DISCUSSION OF THE PROBLEM

I should like to emphasize first that the knowledge that G is a semidirect (or direct) product is not enough for its physical interpretation. Indeed, in the hypothesis of our lemma, the position of the subgroups H or K is not canonical (invariant with respect to automorphisms of G); it is given by physical interpretation. To take a very simple and historical example: Let H be SU_2, the isospin invariance group, and let $K = Z_2$, the two-element group generated by charge conjugation. What is the group $G = H \cdot K$ generated by them? That Z_2 has only two elements implies SU_2 is an invariant subgroup of G, and since SU_2 has no outer automorphisms, G is the direct product $SU_2 \otimes Z_2$. This structure of G does not mean that charge conjugation commutes with isospin transformations. And it is physically very important to know their relations. However, the fact that G has to be a direct product must also be interpreted by the physicist; indeed, the element of G which is not an element of SU_2 and which commutes with every element of SU_2 has a physical meaning, namely, its eigenvalue is the isotopic parity (often called G parity).[7]

To summarize: Physicists are not essentially interested in abstract groups (that is, groups up to an isomorphism) but in groups whose elements have names which often distinguish them (even if these elements can be transformed into each other by a group automorphism).

A second point to emphasize is that the hypothesis that the Poincaré group is a subgroup of the invariance group of a relativistic quantum theory is unphysical. Indeed the existence of half-integer spins shows that $\bar{P}$, the covering of P, has to be considered, and P is not a subgroup of $\bar{P}$. In a relativistic quantum theory with a gauge group or internal symmetry group S (the elements of S are essentially not observable), the plausible physical requirement seems to me: Given some Poincaré transformations whose product is the identity, their successive action on the formalism can be at most an unobservable transformation (i.e., a gauge or internal-symmetry transformation). In mathematical terms, that means that E, the invariance group of the formalism, is an extension of P, with kernel S, i.e., S is an invariant subgroup of E and P is the quotient $P = E/S$. This is just the case in the classic analysis by Wigner[8] of the ray representations of P (then S is the group of the phase). It has also to be the case when one uses an algebra $\mathcal{A}$ of linear operators on a Hilbert space $\mathcal{H}$ for describing the theory (e.g., algebra of observables, algebra of quantum local fields, von Neumann envelope of the C^* algebra of local operations by Haag and Kastler[9]), and the requirement of relativistic invariance is that P be a subgroup of the group of implementable automorphisms of $\mathcal{A}$ (that is, those automorphisms of $\mathcal{A}$ which can be realized by linear operators on $\mathcal{H}$). Then the bounded linear operators on $\mathcal{H}$ which induce on $\mathcal{A}$ an automorphism corresponding to P form an extension E of the Poincaré group (see, for instance, Ref. 10, Chap. III, or Ref. 14), and the kernel S of the extension E is the group of bounded regular operators of $\mathcal{A}'$, the commutant of $\mathcal{A}$ (i.e., $\mathcal{A}'$ is the set of operators which commute with every operator of $\mathcal{A}$). In all proposed physical theories, $\mathcal{A}$ is a *-algebra; then $\mathcal{A}'$ is a von Neumann algebra, and one can restrict S to the group of unitary operators of $\mathcal{A}'$.

So physicists are interested in the question, "What is known about the group extensions of P?"

III. GROUP EXTENSIONS OF THE POINCARÉ GROUP

What are the extensions E of P which have S as kernel? The question we have already met, to make precise which semidirect product was $G = P \times S$, arises here again, in the problem of extensions: A homomorphism $P \xrightarrow{g} \mathrm{Out}S$ should be chosen. For some g there might be no solutions.[10,11] For the other nontrivial

[6] Because, for those groups, the set $\mathrm{Out}S$ is at most countable. If one considers only topological automorphisms, this is also the case for any finite dimensional semisimple Lie group. On the other hand, if G is to be a connected topological group with S as a topological subgroup, then for any compact group S we obtain the direct product $S \otimes P$; see K. Iwasawa, Ann. Math. **50**, 507 (1949).

[7] L. Michel, Nuovo Cimento **10**, 319 (1953). I am very grateful to T. D. Lee and C. N. Yang for the great advertisement they gave to this new quantum number [*ibid.* **13**, 749 (1956), Footnote 3]. However, I disagree with them for having changed the name "isotopic parity" into the unexpressive G parity.

[8] E. P. Wigner, Ann. Math. **40**, 149 (1939), and the earlier book on group theory and quantum mechanics: English translation, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).

[9] R. Haag and D. Kastler, J. Math. Phys. **5**, 848 (1964).

[10] For a mathematical criterion satisfied by the homomorphism g for which solutions exist, see S. Eilenberg and S. MacLane, Ann. Math. **48**, 326 (1947).

[11] L. Michel, lectures on the theory of group extensions at the Istanbul Summer School in Theoretical Physics, July 1962 (to be published by Gordon and Breach).

homomorphisms g for which there is a solution (for example the gauge group of quantum electrodynamics), I do not know of any general results (see, however, Ref. 12). In the case where g is the trivial homomorphism (and we have seen a list of groups S for which the only possible homomorphism g is the trivial one), the only extensions of P by S useful for physics[11,13] are of the form

$$E_\alpha = (S \otimes \bar{P})/Z_2(\alpha), \tag{1}$$

where $\bar{P}$ is the covering of P; $Z_2(\alpha)$ is a two-element group, a subgroup of the center of $S\otimes\bar{P}$, generated by (α,ω), where ω is the nontrivial element of the center of $\bar{P}$ ("rotation by 2π"); and α is an element of the center of S such that $\alpha^2=1$. (For $\alpha=1$, E is the direct product $S\otimes P$.) When P and S are considered as abstract groups, it is proved[11,13] that the E_α of (1) are the only solutions when the center of S is a reduced Abelian group, i.e., it has no nontrivial divisible subgroup (a group is divisible if every element of it has at least one nth root, for any n), e.g., if the center of S is finite (as is the case for SU_3). For arbitrary S, one can give a purely algebraic condition[13] which is satisfied only by the solutions E_α of (1); this implies for the other solutions a very pathological behavior which makes them unphysical.

If S is a (finite-dimensional) Lie group and E is assumed to be a Lie group, then (1) gives again all solutions.[14] The case where P is a topological subgroup (see Ref. 15 for the precise topology required) of the group of implementable automorphisms of a *-algebra $\mathfrak{A}$ realized as operators on a Hilbert space is treated in a recent paper of Kadison[15] (in fact, several groups are studied, including the homogeneous Lorentz group L but not P; to pass from L to P one can use some results of Ref. 13). Again, all solutions are given by (1).

[12] When S is a finite dimensional Lie group and E is then assumed to be a Lie group, it is natural to translate the problem in terms of Lie algebra. From the semisimplicity of L and the last theorem of G. Hochshild and J. P. Serre, Ann. Math. **57**, 591 (1953), the only solution is the semidirect product of Lie algebras. How to go back from Lie algebras to Lie groups is explained and illustrated, for instance, in Ref. 10, Chap. I.

[13] L. Michel, Nucl. Phys. **57**, 356 (1964). There exists a precise definition of equivalent solutions of the extension problem. Solutions of (1) for different α are distinct, although the different corresponding E_α might be isomorphic (example: $S=Z_2\otimes Z_2$). Physically, the E_α are certainly to be distinguished. It is therefore worthwhile to remark that the concept of inequivalent extensions (which might happen to be isomorphic) is, in this case, more physical than the concept of nonisomorphic extensions.

[14] This case covers exactly Ref. 4. Indeed, although in Refs. 1, 3, and 4 the authors give their theorems in terms of groups, they work with Lie algebra. This is more physical and, of course, does not exclude $\bar{P}$.

[15] R. V. Kadison (unpublished).

CONCLUSION

Equation (1) gives a very general way of "mixing" relativistic invariance with internal symmetry of other symmetry groups of a relativistic theory. Although the corresponding "mixing" is slight, it has to be physically interpreted. One interpretation which has already been proposed[16] is the relation "$b+l+2j$ is even" for any physical state (b = baryonic charge, l = sum of all leptonic charges, j = spin).

When the same considerations are extended to the complete Poincaré group (including P, T, and PT) and the charge conjugation C, there are many more extensions by S, and their physical interpretation is more subtle and richer.[17]

Although the existence of an extension of the Poincaré group seems to be a feature of any relativistic theory, this has no bearing on the nature of broken symmetries. There does not seem to be much meaning in combining directly a group of perfect invariance (as P) with that of a broken symmetry outside the frame of an approximate physical theory where the partial symmetry is supposed to be exact. If one could study the relation between, say, the algebra $\mathfrak{A}$ of the refined physical theory (where the invariance S is broken) and the algebra $\mathfrak{B}$ of the approximate theory, where S is an exact symmetry, then a relation would appear between $\mathrm{Aut}\mathfrak{B} \supset S \cup P$ and $\mathrm{Aut}\mathfrak{A} \supset P$ and this would be the relation between P and S looked for in Refs. 1–4. However, up to now, physicists have been able to compare $\mathfrak{A}$ and $\mathfrak{B}$ only in the framework of perturbation theory. Although this approach is quite satisfactory, for example, for the study of the breaking of isospin in nuclear level spectroscopy, and although it has also been useful for guessing the mass relation in SU_3 multiplets, it is surely unsatisfactory for the latter situation.

ACKNOWLEDGMENTS

The author benefited very much from a short visit at the Boulder symposium on Lorentz groups, and from the very stimulating atmosphere of the Summer Institute for Theoretical Physics at the University of Wisconsin.

[16] F. Lurçat and L. Michel, Nuovo Cimento **21**, 574 (1961) and Comptes Rendus Conference, Aix-en-Provence (C.E.N. Saclay, France, 1962), p. 183.

[17] See Ref. 10 and F. Lurçat, and L. Michel (to be published).