calculated were somewhat narrower, and peaked at lower energies than the experimental excitation functions. Even more striking was the error in the magnitude of the calculated excitation functions for nuclides near the doubly magic Ni^{56} nucleus. This error has been noted for a large number of target-projectile systems yielding the same product nuclides measured in this work; a consistent explanation is that the level densities of these nuclides are influenced by shell structure, even at reasonably high excitation energies.

The second set of calculations employed a level-density expression in which the influence of shell structure on level densities was taken into consideration in the manner suggested by Rosenzweig. The agreement between calculated and experimental peak cross sections was generally improved in the second set of calculations, as is summarized in Table III.

In the third set of calculations two simplifying assumptions were made in considering the influence of large values of angular momentum on the decay of highly excited nuclei. The excitation functions calculated in this fashion were broadened, and shifted to higher excitation energies compared with those of the first set of calculations.

On the basis of the comparisons of the calculated and experimental excitation functions of this work, we find no reason to abandon the concept of the statistical model up to the highest energies encountered in this work.

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Effect of β Vibrations on Multiple Coulomb Excitation Within the Ground-State Band*

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Alder and Winther's theory of multiple Coulomb excitation is applied to the rotation-vibration model of axially deformed nuclei. It is shown that in the transition region of deformed nuclei one can account in this way for the differences between values obtained from experiments and the theoretical excitation probabilities of the rotational model.

I. INTRODUCTION

MANY levels of the nuclei in the region $150<A<190$ can be classified as rotational bands built on collective vibrations with the two shape parameters β and γ.[1] A considerable amount of experimental information on these bands was obtained[2] in Sm^{152}. Theoretical investigations which confirm the view that the deformed nuclei have predominantly a prolate-spheroidal equilibrium shape were carried out by Yamazaki[3] and by Gupta and Preston.[4] The so-called "rotation-vibration-interaction" for axially symmetric nuclei was investigated in detail by Faessler and Greiner[5–7] and by Preston and Kiang[8]; according to these views this interaction can cause mixing different rotational bands. Nielson[9] as well as Greenberg *et al.*[10] found that the gamma-band admixture to the ground-

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[1] R. K. Sheline, Rev. Mod. Phys. **32**, 1 (1960).
[2] J. S. Greenberg, D. A. Bromley, G. C. Seaman, and E. V. Bishop, *Proceedings of the Third Conference on Reactions Between Complex Nuclei, Asilomar, California*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963), p. 295.
[3] T. Yamazaki, Nucl. Phys. **49**, 1 (1963).
[4] S. Das Gupta and M. A. Preston, Nucl. Phys. **49**, 401 (1963).
[5] A. Faessler and W. Greiner, Z. Physik **168**, 425 (1962).
[6] A. Faessler and W. Greiner, Z. Physik **170**, 105 (1962).
[7] A. Faessler and W. Greiner, Z. Physik **177**, 190 (1964).
[8] M. A. Preston and D. Kiang, Can. J. Phys. **41**, 742 (1963).
[9] O. B. Nielson, *Proceedings of the Rutherford Jubilee International Conference* (Heywood and Company, Ltd., London, 1961), p. 317.
[10] J. S. Greenberg, G. C. Seaman, E. V. Bishop, and D. A. Bromley, Phys. Rev. Letters **11**, 211 (1963).

state band was small in the nuclei whose de-excitation transition ratios they had measured. The latter authors, however, could account for the deviations from a purely rotational level scheme by additional beta-band admixtures. Recently, Stephens *et al.*[11] were able to account for the level spacings in the ground-state band of Hf^{170} up to very high spins surprisingly well by β-ground mixing alone.

The purpose of the present paper is to examine the effect of beta vibrations on the multiple Coulomb excitation within the ground state band. This effect should occur partly on account of an additional term in the quadrupole operator and partly through the admixture to the final-state wave function. The complete quadrupole operator is needed only for evaluating the matrix elements between the undisturbed wave functions while for the admixture the quadrupole operator of a pure rotator may be used.

The amplitudes of the admixtures are taken over from the rotation-vibration model.[5–8] The work is restricted to an interaction Hamiltonian linear in the vibration amplitudes. Only the excitation of the ground-state band will be dealt with. The ξ correction[12] is not so decisive in this case and need not be accounted for accurately.

The result obtained differs from Alder's[13] mainly because of the β vibrations which are not included in his calculation. The fact that Alder uses Dayvdov's nuclear model with a constant γ deformation[14] is believed to be less significant.[3]

II. DESCRIPTION OF THE CALCULATION

The excitation amplitude for a transition from the ground state $|0\rangle$ to a final state $|f\rangle$ follows from Ref. 12 and is

$$a_f=\langle f|\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}H_{\text{int}}(t)dt\right)|0\rangle \tag{1}$$

in the sudden approximation. H_{int} is the instantaneous Coulomb field seen by the target nucleus, but without the monopole term. Only the quadrupole part of this field is dealt with and a quadrupole operator expanded to first order in γ and $(\beta-\beta_0)/\beta_0$ is used.[5] Here $\beta=\beta_0$ corresponds to the equilibrium deformation. For an even-even nucleus it is thus found that

$$\begin{aligned}H_{\text{int}}(t)=(\pi/5)^{1/2}Z_1e^2Q_0\Big[&(1+(\beta-\beta_0)/\beta_0)\\ &\times\sum_{\mu=-2}^{+2}D_{\mu0}{}^2(\varphi,\theta,\psi)\bar{S}_{2,\mu}(t)+(\gamma/\sqrt{2})\\ &\times\sum_{\mu=-2}^{+2}\left[D_{\mu-2}{}^2(\varphi,\theta,\psi)+D_{\mu-2}{}^2(\varphi,\theta,\psi)\right]\bar{S}_{2,\mu}(t)\Big],\end{aligned} \tag{2}$$

where eZ_1 is the nuclear charge of the projectile. The quantity Q_0 is the intrinsic quadrupole moment of the deformed nucleus and can be obtained from the experimental $B(E2)$ values. It differs slightly from the corresponding value of the original Bohr-Mottelson theory.[15] The "collision functions" $\bar{S}_{2,\mu}(t)$ are used following the definition of Alder and Winther.[12]

The basic functions of Greiner and Faessler[7] are used for describing the deformed nuclei. The β vibrations are thus assumed to be harmonic and are treated in the second quantization. As for the γ-vibrational wave functions, the only difference between Ref. 7 and Ref. 8 comes from a different choice of the volume element. The complete basis wave functions shall be represented by $|IKn_2n_0\rangle$, where I is the nuclear spin, K its projection on the nuclear symmetry axis, n_2 is related to the quantum number n_γ of Preston and Kiang[8] by $n_\gamma=2n_2+\frac{1}{2}K$ and n_0 is the number of excited β phonons. The unperturbed energy[6] is

$$\begin{aligned}E_{n_2,n_0}(K)=(n_0+\tfrac{1}{2})E_\beta+(2n_2+\tfrac{1}{2}K+1)E_\gamma&\\ +[I(I+1)+K^2]\epsilon/2\,,&\end{aligned} \tag{3}$$

where E_β, E_γ represent the excitation of the lowest member of the beta and gamma band, respectively, and ϵ is essentially the reciprocal of the moment of inertia J_0, $\epsilon=\hbar^2/J_0$.

Instead of simply using the perturbation expansion of Greiner and Faessler[6] for a ground-state band disturbed by the rotation-vibration interaction, two alterations are made as follows. In the first place in the matrix element

$$\begin{aligned}\zeta_\gamma\equiv\langle n_2=0,\,K=0|\gamma/\sqrt{2}|n_2=0,\,K=2\rangle&\\ =((3\epsilon/2E_\gamma)b_\gamma)^{1/2}&\end{aligned} \tag{4}$$

$b_\gamma=\frac{1}{2}$ rather than $b_\gamma=1$ is used. Good agreement with experiment is then obtained for the ratios $B(E2,\,22\rightarrow0)/B(E2,\,20\rightarrow0)$ without having to expand the quadrupole operator to second order in the vibration amplitudes.[7] This can be seen e.g. from Fig. 2 of Yamazaki.[3] It should be mentioned that the choice of b_γ used here leads to the same $B(E2,\,22\rightarrow0)/B(E2,\,20\rightarrow0)$ ratios as follow from Davydov's model.[3]

Secondly, in addition to (4) the relation

$$\zeta_\beta=\langle n_0=1|(\beta-\beta_0)/\beta_0|n_0=0\rangle=\left(\frac{3\epsilon}{2E_\beta}b_\beta\right)^{1/2} \tag{5}$$

[11] F. S. Stephens, N. Lark, and R. M. Diamond, Phys. Rev. Letters **12**, 225 (1964).

[12] K. Alder and A. Winther, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **32**, No. 8 (1960).

[13] K. Alder, *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963), p. 253.

[14] A. S. Davydov and G. F. Filippov, Nucl. Phys. **8**, 237 (1958).

[15] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **27**, No. 16 (1953).

will be used with $b_\beta=\frac{1}{2}$ instead of $b_\beta=1$. In this way there is introduced essentially the same strength parameter $b=b_\beta^{-1}=b_\gamma^{-1}$ as that of Sheline.[1] This leads to reasonable values for the $I^2(I+1)^2$ terms of the ground state energy spectra.

Using (4) and (5), one obtains the following wave function of the ground state band[16]:

$$\Psi=A_1|I000\rangle+A_2|I200\rangle+A_3|I001\rangle, \tag{6}$$

where

$$A_1=1,\quad A_2=\zeta_\gamma^3(2/3b_\gamma)[(I+2)(I+1)I(I-1)/3]^{1/2},$$
$$A_3=\zeta_\beta^3(2/3b_\beta)I(I+1). \tag{7}$$

This wave function is renormalized to approach the result of a diagonalization procedure. The contribution of the first term in (6) to the excitation amplitude (1) for backward scattering is [apart from a factor $(2I+1)^{1/2}$]

$$\begin{aligned}\frac{1}{4\pi}\int_0^{2\pi}d\psi\int_0^{\pi}&d\theta\sin\theta\, P_I(\cos\theta)\\ &\times\exp[-i(4q/3)P_2(\cos\theta)]\\ &\times\exp\{-\tfrac{1}{2}[(4q/3)\zeta_\beta P_2(\cos\theta)]^2\}\\ &\times\{1-c_2\times\tfrac{1}{2}[\tfrac{4}{3}q\zeta_\gamma(4\pi/5)^{1/2}\\ &\times(Y_{22}^*(\theta,\psi)+Y_{2-2}^*(\theta,\psi))]^2\\ &+c_4\times 1/24[\tfrac{4}{3}q\zeta_\gamma(4\pi/5)^{1/2}\\ &\qquad\times(Y_{22}^*(\theta,\psi)+Y_{2-2}^*(\theta,\psi))]^4+\cdots\}.\end{aligned} \tag{8}$$

The dimensionless parameter q is defined in Ref. 12:

$$q=\frac{Z_1e^2Q_0}{4\hbar va^2}. \tag{9}$$

Here v is the velocity of the projectile and a is half the distance of closest approach in a head-on collision. The Legendre polynomial of order 2 is denoted by P_2, and the Y_{lm} are spherical harmonics in standard notation. For $\zeta_\beta=\zeta_\gamma=0$ Eq. (8) gives the familiar expression for a pure rotator.

The second exponential function in (8) results from applying the exponential operator in (1) to the β-phonon vacuum after having replaced $(\beta-\beta_0)/\beta_0$ by $\zeta_\beta(\hat{\beta}_0+\hat{\beta}_0^+)$ in (2), where $\hat{\beta}_0$, $\hat{\beta}_0^+$ are destruction and creation operators respectively. The expression inside the curly braces in (8) can be obtained by a diagonalizating procedure for the γ-vibrational part of the exponential operator in (1). Then the actual values of the coefficients c_2, c_4 depend on the specific γ-vibrational states which are taken into account. Employment of five states gave $c_2=1$, $c_4=4$. The integral in (8) was treated in the following way. The second exponential function was first replaced by its power series. Then all terms $\zeta_\beta^n\zeta_\gamma^m$ with $n+m>4$ were dropped. The remaining part of (8) was expressed in terms of the functions $A_{l,0}(\pi,q)$ for backward scattering introduced by Alder and Winther.[12] The calculation of β and γ band admixtures was carried out only in a first-order perturbation treatment of the vibrational part of the quadrupole operator, the expansions in terms of the $A_{l,0}(\pi,q)$'s listed in Ref. 12 having been used. The matrix element resulting from the second term in (6) is proportional to ζ_γ^4 and the one resulting from the last term is proportional to ζ_β^4. Finally, an expression for the Coulomb excitation amplitude divided by $(2I+1)^{1/2}$ of the following type

$$\mathcal{A}_{I0}(\pi,q)=A_{I0}(\pi,q)+a_\beta^{(2)}\zeta_\beta^2a_\gamma^{(2)}\zeta_\gamma^2$$
$$+a_\beta^{(4)}\zeta_\beta^4+a_\gamma^{(4)}\zeta_\gamma^4+a_{\beta\gamma}^{(2)}\zeta_\beta^2\zeta_\gamma^2. \tag{10}$$

was obtained. This result is equivalent to a fourth-order perturbation expansion in the parameters ζ_β and ζ_γ, provided they are of the same order of magnitude. The A's are tabulated in Refs. 12 and 17. The a's have been calculated as described above.

The excitation probability for backward scattering is given by

$$P_I=(2I+1)|\mathcal{A}_{I0}(\pi,q)|^2 \tag{11}$$

III. RESULTS

The a's of Eq. (10) depend on the nuclear spin I of the excited level of the ground-state band. They are complicated oscillating functions of q. The error in (8) resulting from the restriction to fourth order terms in the ζ's is believed not to exceed 1% by much for low nuclear spins and q smaller than 4. Above $q=4$ projectile and target nucleus collide with certainty and the theory loses its validity.

In Figs. 1 and 2 the multiple Coulomb excitation probabilities are plotted for $\zeta_\gamma=0.125$ and $\zeta_\gamma=0.175$.[18] The first of these two values of the parameter should be adequate for Hf^{178} and the second for the (Sm^{152},Gd^{154}) region. The curves shown in Figs. 1 and 2 are labeled by the values of the nuclear spin I and those of ζ_β. The values of ζ_β following from Eq. (5) are 0.140 and 0.215 for Hf^{178} and Sm^{152}, respectively. The curves for $\zeta_\beta=0$ and small ζ_γ values may be expected to be nearly in agreement with interpolations from Alder's paper,[13] where Davydov's model is used. Deviations should occur only for high q values and may be caused in addition to (4) by the γ-vibrational matrix elements which are involved in the diagonalization for the γ vibrations. Such deviations, however, would not be very significant as they have to compete with the inaccuracy resulting from the limited number of γ-vibrational states used in diagonalizing procedure.

[16] As for the γ-band admixtures to the final-state wave function, one has only to account for the admixture of the $K=2$ band, because for $K\leq 4$ the only nonvanishing matrix element $\langle n_2K|\gamma|00\rangle$ is the one with $n_2=0$, $K=2$.

[17] R. Graetzer, R. Hooverman, and E. M. Bernstein, Nucl. Phys. **39**, 124 (1962).

[18] These values correspond to a nonaxiality parameter $\gamma_0=10.13°$ rsp. 14.18° in Davydov's theory, Ref. 14.

The effect of interest in the present work, namely the influence of sizable ζ_β values, is especially evident for large q values. However, for $I=6$ it is quite large even for small q values.

It is instructive to multiply the excitation probabilities in the sudden approximation for the pure rotator by the de-excitation ratios $(R_4)_{\text{exp}}/(R_4)_{\text{theory}}$ obtained by de Boer *et al.* for Sm^{152} (cf. Fig. 6 of Ref. 19, solid points).

Here, in accordance with Eq. (7) of the reference just cited the de-excitation probability R_I is the number of de-excitations of a level with spin I divided by the number of backscattered particles and the subscripts "exp" and "theory" refer, respectively, to experimental and theoretical values of $R4$. Assuming that the ξ correction is not too strongly affected by the nuclear vibrations[20] and would almost drop out in these ratios, the products under discussion should be approximately the excitation ratios expected in the sudden approximation from a more realistic nuclear model. Fig. 2 shows that these values lie fairly close to the curve $\zeta_\beta=0.2$. This

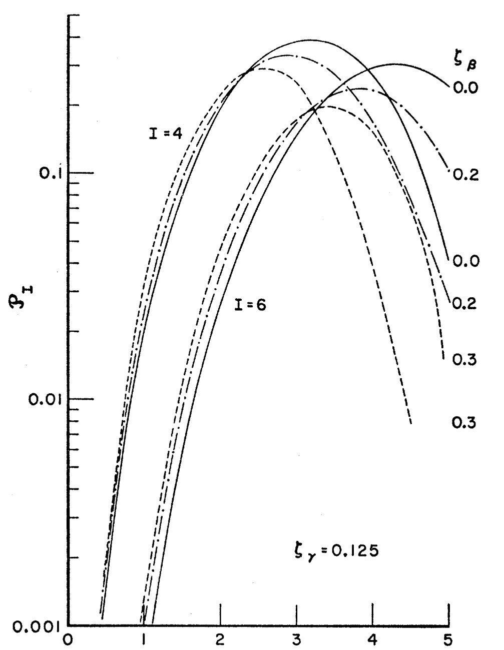

FIG. 1. Multiple Coulomb excitation probabilities for backward scattering in the sudden approximation; ζ_γ is chosen to be 0.125. The various curves are labeled by the nuclear spin of the excited level and by ζ_β. (For Hf^{178}, $\zeta_\gamma=0.126$ and $\zeta_\beta=0.140$.)

[19] J. de Boer, G. Goldring, and H. Winkler, Phys. Rev. **134**, B1032 (1964).

[20] The β vibration does not affect the linear ξ correction term to first order in the vibration amplitude.

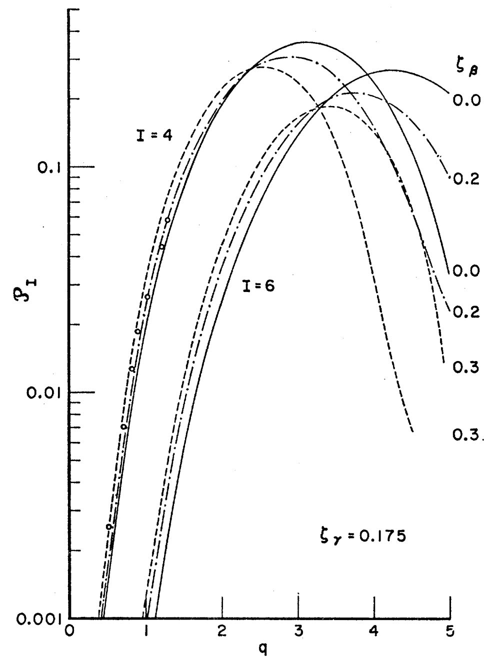

FIG. 2. A set of curves similar to those in Fig. 1 but with $\zeta_\gamma=0.175$. (For Sm^{152}, $\zeta_\gamma=0.171$ and $\zeta_\beta=0.215$.) The plotted points were obtained by multiplying the excitation probabilities of the pure rotator by the ratios $(R_4)_{\text{experiment}}/(R_4)_{\text{theory}}$ for Sm^{152} from Ref. 19.

supports the rotation-vibration model. Another indication for the influence of the β vibrations seems to be that for $q\approx 1.35$, i.e., $\chi\approx 0.82$, the ratio $(R_6)_{\text{exp}}/(R_6)_{\text{theory}}$ (cf. Table III, Ref. 19) is larger for Sm^{152} than for Sm^{154}. For $I=4$ and $q\approx 1.3$, Adams *et al.*[21] have in fact found a larger yield for Sm^{152} than for Sm^{154}. In favor for the predicted effect is also the fact that the double ratios D_I defined by de Boer *et al.* in Ref. 19 and plotted there in Fig. 16 show a slight increase for nuclei in the transition region. For the data in Table III of Ref. 19 which seem to contradict the results of the present paper the quoted experimental error is large in most cases. It seems worth while to improve the experiments and to extend them to higher energies.

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[21] B. M. Adams, D. Eccelshall, and M. J. L. Yates, *Proceedings of the Second Conference on Reactions Between Complex Nuclei*, edited by A. Zucker, F. T. Howard, and E. C. Halbert (John Wiley & Sons, Inc., New York, 1960), p. 95.