Density-Operator Theory of Harmonic Oscillator Relaxation

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The reduced density operator for a driven electromagnetic field mode relaxing to a bath of oscillators is formally found in closed form. In the limit of weak coupling, the density operator which maximizes the entropy subject to the constraints of a given average energy and average electric and magnetic field remains invariant under interaction with the bath. The quantum characteristic function is found in the weak-coupling case and agrees with previous results. The method may be generalized to many modes relaxing to a bath of oscillators.

I. INTRODUCTION

IN the quantum theory of relaxation or dissipation one considers a single system such as a single spin coupled weakly to an ensemble of lossless systems called a bath. The bath might be made up of all the other spins in the specimen, for example. Only a small amount of energy is transferred from the single spin to a single bath system but since the bath has a large number of systems, there can be a finite net transfer of energy from the spin to the bath. Of course, the model is not valid for infinite times since the energy will ultimately return to the spin. An extreme example consists of only two spins weakly coupled which obviously would make a poor model for an attenuator.

Similar models apply to radiation damping of an atom. The atom radiates or relaxes to its ground state by coupling its energy weakly into a continuum of normal modes of a radiation field in free space. In the usual density operator formulation of such problems,[1–10] perturbation theory is used by necessity due to the complexity of the model.

Lax[11] has given a systematic method of improving the results by a self-consistent approximation which becomes more complex with each stage of improvement. In the present paper we consider a simple model of a single system (e.g., a single mode of a radiation field) coupled linearly to a bath of oscillators. Such a model might represent loss in a cavity[12,13] or dissipation by phonons in a solid. One virtue of the model is that the reduced density operator which describes the statistical properties of the radiation field mode may be obtained exactly in closed form in terms of several time dependent parameters. In order to obtain these parameters, approximations must be made. Such an exact solution is of value in comparing various approximate methods for more complicated systems.

The linear coupling assumed implies that the coupling is weak so that the exact solutions are exact only in so far as the model is exact.

The model also assumes a driving term so that steady state conditions can be obtained. The quantum characteristic function is found for weak coupling from which all statistical properties of the cavity mode may be found. It is in agreement with previous results.[13] It might be noted as a general comment that it is easier to find quantum expectation values in the Heisenberg picture as in Ref. 13 than it is to use the density matrix formulation in the Schrödinger picture.

In the theory of relaxation one is usually interested in the statistical properties of the system which is relaxing and not in the behavior of the bath.[14] If we let a represent the dynamical variables of the system under study and b represent all the bath variables, the ensemble average of a function of the system variables, $f(a)$, is given by

$$\langle f(a)\rangle = \mathrm{Tr}[\rho(a,b,t)f(a)], \tag{1}$$

where $\rho(a,b,t)$ is the density operator and all quantities are given in the Schrödinger picture. The trace is taken with respect to the a and b variables. Since f is a function of a only, we may define[5,14] a reduced density operator by means of

$$R(a,t) = \mathrm{Tr}_b\rho(a,b,t) \tag{2}$$

in which we take the trace over the bath variables. Then (1) becomes

$$\langle f(a)\rangle = \mathrm{Tr}_a[R(a,t)f(a)]. \tag{3}$$

The immediate goal of the present work is to evaluate $R(a,t)$ for the model in which the density operator satisfies the equation of motion

$$i\hbar(\partial\rho/\partial t)(a,b,t) = [H(a,b,t),\rho(a,b,t)], \tag{4}$$

where H is the Hamiltonian and ρ is subject to the usual

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[1] N. Bloemberger, E. M. Purcell, and R. V. Pound, Phys. Rev. **73**, 679 (1948).
[2] R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 278 (1953).
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[11] M. Lax, Bull. Am. Phys. Soc. **9**, 82 (1964).
[12] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961); **131**, 2827 (1963).
[13] J. P. Gordon, L. R. Walker, and W. H. Louisell, Phys. Rev. **130**, 806 (1963) and references contained therein.
[14] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N. Y.), **24**, 118 (1963). These authors consider problems similar to the one contained here and use Feynman's space-time formulation.

conditions that

$$\rho(t)=\rho^\dagger(t)\,,\quad \mathrm{Tr}\rho(t)=1\,. \tag{5}$$

The method we shall use seems simpler for the particular model we shall consider than that used by Feynman and Vernon.[14] In principle, the present method could be adapted to fermions although this appears tedious.

II. THE MODEL

The Hamiltonian for an electromagnetic field mode weakly coupled linearly to a bath of oscillators may be taken as[15]

$$H=\hbar\{\omega a^\dagger a+\sum_{j=1}^{N}\omega_j b_j^\dagger b_j+\sum_{j=1}^{N}\kappa_j^* b_j^\dagger(a+q^*a^\dagger)+(a^\dagger+qa)\sum_{j=1}^{N}\kappa_j b_j+e(t)a^\dagger+e^*(t)a\}\,. \tag{6}$$

In this Hamiltonian, the a, $a^\dagger$, b_j, and $b_j^\dagger$ are boson annihilation and creation operators which satisfy the usual commutation rules

$$\begin{aligned}&[a,a^\dagger]=1\,,\qquad [a,a]=[a^\dagger,a^\dagger]=0\,,\\&[b_j,b_k^\dagger]=\delta_{jk}\,,\quad [b_j,b_k]=[b_j^\dagger,b_k^\dagger]=0\,,\\&[a,b_j]=[a,b_j^\dagger]=[a^\dagger,b_j]=[a^\dagger,b_j^\dagger]=0\,.\end{aligned} \tag{7}$$

The first term in the Hamiltonian is the energy of the field mode of frequency ω in the absence of interaction and the term $\hbar\omega_j b_j^\dagger b_j$ gives the uncoupled energy of the jth bath mode (loss oscillator) of frequency ω_j which is summed over the $N(\sim 10^{23})$ bath modes. One assumes that the bath modes are continuously distributed with a density $g(\omega_j)$ over the frequency spectrum about the frequency ω.

The terms involving κ_j and κ_j^* give the coupling between the field and the bath oscillators. These coupling coefficients are assumed small compared with the frequency $\omega\approx\omega_j$. This is a necessary assumption unless we are prepared to consider nonlinear interactions involving cubic and higher order terms in H.

The q and q^* appearing in (6) are parameters of order unity. Their presence allows for either dipole-dipole type coupling or for the field to be coupled to electric dipole moments of the loss oscillators. We shall show later that their presence is not important under the weak coupling approximation.

The last two terms in (6) involving $e(t)$ and $e^*(t)$ allow for the presence of a driving term for the field mode so that a steady state may be reached.

[15] We shall show that under the assumption of weak coupling the terms involving q and q^* will play a very minor role on our results since they involve high-frequency terms which will average to zero for times of interest. They would become important for stronger coupling but in that case cubic terms should be added to the Hamiltonian.

III. THE DENSITY OPERATOR. ENTROPY

The density operator describes the statistical properties of the system at time t in terms of its properties at $t=0$. Accordingly, in order to solve (4) we must specify the initial value of the density operator. We assume that for $t<0$, the field and loss oscillators are uncoupled and there are no driving terms. The loss oscillators are assumed to be in thermal equilibrium with a Maxwell-Boltzmann distribution at temperature T so the density operator for the bath at $t=0$ is given by

$$\begin{aligned}\rho_L(0)&=\prod_j[1-\exp(-\lambda_j)]\exp[-\lambda_j b_j^\dagger b_j]\,,\\&\equiv\prod_j[1-\exp(-\lambda_j)]\exp[-\sum_k\lambda_k b_k^\dagger b_k]\,,\end{aligned} \tag{8}$$

where

$$\lambda_j=\hbar\omega_j/kT\,. \tag{9}$$

It is well-known that we may derive this density operator if we maximize the entropy of the bath S_L defined by

$$S_L=-k\,\mathrm{Tr}[\rho_L(0)\ln\rho_L(0)]\,, \tag{10}$$

where k is Boltzmann's constant, subject to the constraints that

$$\mathrm{Tr}\rho_L(0)=1 \tag{11}$$

and the only knowledge we have about the bath, viz., its average energy or temperature

$$\langle H_L\rangle=\mathrm{Tr}\{\rho_L(0)H_L\}\,, \tag{12}$$

where

$$H_L=\sum\hbar\omega_j b_j^\dagger b_j\,. \tag{13}$$

This procedure suggests[16,17] how to choose the density operator $\rho_a(0)$ which describes an ensemble of field modes in the cavity at $t=0$ subject to the available information. The entropy of the field is given by

$$S_a=-k\,\mathrm{Tr}[\rho_a(0)\ln\rho_a(0)]\,. \tag{14}$$

If we maximize this subject to the constraints

$$\mathrm{Tr}\rho_a(0)=1\,, \tag{15}$$

$$\langle H_a\rangle=\mathrm{Tr}[\rho_a(0)H_a]\,, \tag{16}$$

where

$$H_a=\hbar\omega a^\dagger a \tag{17}$$

and the additional constraints

$$\langle p\rangle=\mathrm{Tr}[\rho_a(0)p]\,, \tag{18}$$

$$\langle q\rangle=\mathrm{Tr}[\rho_a(0)q]\,, \tag{19}$$

where q and p are the electric and magnetic fields in the cavity defined by

$$\begin{aligned}q&=(\hbar/2\omega)^{1/2}(a^\dagger+a)\,,\\p&=i(\hbar\omega/2)^{1/2}(a^\dagger-a)\end{aligned} \tag{20}$$

[16] J. P. Gordon, in *Proceedings of the Third International Conference on Quantum Electronics* (Columbia University Press, New York, 1963), p. 55.

[17] E. T. Jaynes, Phys. Rev. **106**, 620 (1957); **108**, 171 (1957).

we find by the usual method of Lagrange multipliers that

$$\rho_a(0)=(1-e^{-\lambda})\exp[-\lambda(a^\dagger-w^*)(a-w)], \quad (21)$$

where

$$\langle H_a\rangle=\hbar\omega\{[1/(e^{\lambda}-1)]+|w|^2\} \quad (22)$$

and

$$\begin{aligned}\langle p\rangle&=i(\hbar\omega/2)^{1/2}(w^*-w),\\ \langle q\rangle&=(\hbar\omega/2)^{1/2}(w^*+w).\end{aligned} \quad (23)$$

Physically, we see that the constraints (18) and (19) allow for the presence of a signal or state of excitation of the electric and magnetic fields in the cavity mode at $t=0$ while the constraint (16) allows for the presence of signal energy, $\hbar\omega|w|^2$, in (22) in addition to noise energy given by the first term in (22). One sees that if $w=w^*=0$, (21) reduces to the Boltzmann distribution with $\lambda=\hbar\omega/kT_c$ with T_c the ensemble cavity temperature. We shall accordingly assume throughout that the ensemble of cavity modes is initially described by (21). The density operator at $t=0$ before the systems are coupled is given by the simple product

$$\rho(a,b,0)=\tau\prod_j\tau_j \times\exp[-\lambda(a^\dagger-w^*)(a-w)-\sum_k\lambda_k b_k{}^\dagger b_k], \quad (24)$$

where

$$\begin{aligned}\tau&=1-e^{-\lambda},\\ \tau_j&=1-e^{-\lambda_j}.\end{aligned} \quad (25)$$

We may combine the exponentials since the a and $a^\dagger$ commute with the b_j and $b_j{}^\dagger$ by (7).

The solution of the density operator equation is most easily accomplished if we put all operators in normal form.[18] We therefore have for (24)

$$\rho(a,b,0)=\tau\prod_j\tau_j N\{\exp[-\tau(\bar{a}^\dagger-w^*)(\bar{a}-w) -\sum_k\tau_k\bar{b}_k{}^\dagger\bar{b}_k]\}, \quad (26)$$

where the normal ordering operator N means that the daggered operators stand to the left of the undaggered operators when the expression on which N operates is expanded in a power series. In practice, the variables in this expression behave as c numbers until after the final ordering step is performed. A bar is used to indicate this explicitly. The operator $\rho(a,b,0)$ above is in normal form by definition and this may be indicated with a small superscript n. Thus,

$$\rho(a,b,0)=\rho^{(n)}(a,b,0). \quad (27)$$

Next we proceed to solve the density operator equation (4) subject to the initial condition (26). We note first that H in (6) is automatically in normal form since in each term in the sum all annihilation operators are to the right of all creation operators. Assume for the time being that $\rho(t)$ has been put into normal form also. We then have from (4)

$$i\hbar(\partial\rho^{(n)}/\partial t)=H^{(n)}\rho^{(n)}-\rho^{(n)}H^{(n)}. \quad (28)$$

Although the left side and each of the factors $H^{(n)}$ and $\rho^{(n)}$ are separately in normal form, the result on the right is not in normal form. We may use a well-known result from quantum mechanics[18] to put the right side in normal form, viz., the commutation relations

$$[a,f^{(n)}(a,a^\dagger)]=\partial f^{(n)}/\partial a^\dagger \quad (29a)$$

$$[a^\dagger,f^{(n)}(a,a^\dagger)]=-\partial f^{(n)}/\partial a, \quad (29b)$$

where a and $a^\dagger$ are annihilation and creation operators, respectively. (These commutators are actually valid whether the function f is in normal form or not.) Accordingly, we see that if $f^{(n)}$ is in normal form, $af^{(n)}$ is not, but by (29a), we have

$$af^{(n)}=f^{(n)}a+(\partial f^{(n)}/\partial a^\dagger) \quad (30)$$

and the right side is now in normal form. Similarly, we have by (29b)

$$f^{(n)}a^\dagger=a^\dagger f^{(n)}+(\partial f^{(n)}/\partial a). \quad (31)$$

We may use (30) and (31) in order to put the right side of (28) into normal form. The terms involving q and q^* will require two applications of (30) and (31). Once (28) is in normal form, we may treat all operators as c numbers as noted before and it may be written in matrix form as

$$i\frac{\partial\rho^{(n)}}{\partial t}=\frac{1}{2}\left(\frac{\partial}{\partial\bar{a}^\dagger}\ \frac{\partial}{\partial\bar{a}}\ \frac{\partial}{\partial\bar{b}^\dagger}\ \frac{\partial}{\partial\bar{b}}\right)\begin{bmatrix}0&0&q\kappa\cdot&0\\0&0&0&-q^*\kappa^*\cdot\\ \cdot q\kappa&0&0&0\\0&-\cdot q^*\kappa^*&0&0\end{bmatrix}\begin{bmatrix}\partial/\partial\bar{a}^\dagger\\ \partial/\partial\bar{a}\\ \partial/\partial\bar{b}^\dagger\\ \partial/\partial\bar{b}\end{bmatrix}\rho^{(n)}$$
$$+(\bar{a}^\dagger\bar{a}\bar{b}^\dagger\bar{b})\begin{bmatrix}\omega&0&\kappa\cdot&-q^*\kappa^*\cdot\\0&-\omega&q\kappa\cdot&-\kappa^*\cdot\\ \cdot\kappa^*&-\cdot q^*\kappa^*&\Omega&0\\ \cdot q\kappa&-\cdot\kappa&0&-\Omega\end{bmatrix}\begin{bmatrix}\partial/\partial\bar{a}^\dagger\\ \partial/\partial\bar{a}\\ \partial/\partial\bar{b}^\dagger\\ \partial/\partial\bar{b}\end{bmatrix}\rho^{(n)}+[e^*(t)\quad -e(t)\quad 0\quad 0]\begin{bmatrix}\partial/\partial\bar{a}^\dagger\\ \partial/\partial\bar{a}\\ \partial/\partial\bar{b}^\dagger\\ \partial/\partial\bar{b}\end{bmatrix}\rho^{(n)}. \quad (32)$$

[18] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Company, Inc., New York, 1964), Chap. 3. R. Kubo, J. Phys. Soc. Japan, **17**, 1100 (1962); F. Coester and H. Kümmel, Nucl. Phys. **17**, 477 (1960).

A word should be said about the matrix notation. In the two "square" 4×4 matrices in (32), the "11, 12, 21, and 22" elements are all 1×1. The dimensions of the "13, 14, 23, and 24" are $1\times N$, the "31, 32, 41, and 42" are $N\times 1$ while the "33, 34, 43, and 44" elements are $N\times N$. Ω is diagonal with elements ω_j, and $\kappa\cdot$ has elements κ_j as does $\cdot\kappa$. The dots indicate the directions of operation. The row and column vectors have dimensionality $(1,1,N,N)$ for the four respective elements. In the future similar resolutions will be made and their dimensionalities should be clear from the context. It should also be noted that the vector space used in (32) is different from that of $\rho^{(n)}$ so that the ρ "matrix" commutes with the matrices involved in (32).

The next problem involves the solution of (32). With an obvious change in notation, we see that (32) is of the form

$$i\frac{\partial\rho^{(n)}}{\partial t}=\left[\frac{\partial}{\partial\zeta_i}\Lambda_{ij}\frac{\partial}{\partial\zeta_j}+\zeta_i M_{ij}\frac{\partial}{\partial\zeta_j}+N_i\frac{\partial}{\partial\zeta_i}\right]\rho^{(n)}, \quad (33)$$

where we use the repeated index summation convention on i and j. All quadratic Hamiltonians will lead to a density operator equation of this type. In the particular case involved here $\Lambda_{ij}=\Lambda_{ji}{}^t$, the trace of M is zero, $M_{ii}=0$, and N_i is a function of t.

In order to solve (33) it is convenient to introduce the Fourier transform of $\rho^{(n)}(\bar{a}^\dagger,\bar{a},\bar{b}^\dagger,\bar{b},t)\equiv\rho^{(n)}(\zeta,t)$. If we let

$$\begin{aligned}\bar{a}&=x_0+iy_0, & \bar{b}_j&=x_j+iy_j,\\ \bar{a}^\dagger&=x_0-iy_0, & \bar{b}_j{}^\dagger&=x_j-iy_j,\end{aligned} \quad (34)$$

we have

$$\rho^{(n)}(x_0,y_0,x_j,y_j,t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}e^{i[\alpha_0x_0+\beta_0y_0+\alpha_lx_l+\beta_ly_l]}\times F(\alpha_0,\beta_0,\alpha_j,\beta_j,t)\frac{d\alpha_0d\beta_0}{4\pi}\prod_{j=1}^{N}\frac{d\alpha_jd\beta_j}{4\pi}, \quad (35)$$

where we again use the summation convention in the exponential. The inverse relation is

$$F(\alpha_0,\beta_0,\alpha_j,\beta_j,t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}e^{-i[\alpha_0x_0+\beta_0y_0+\alpha_lx_l+\beta_ly_l]}\times\rho(x_0,y_0,x_j,y_j,t)\frac{dx_0dy_0}{\pi}\prod_{j=1}^{N}\frac{dx_jdy_j}{\pi}. \quad (36)$$

The transform is introduced since the transform of (33) is easier to solve than (33) directly. Furthermore, the transform allows us to take the trace over the bath variables and thereby obtain the reduced density operator in an almost trivial way. We may see this as follows.

Consider a function $f(c,c^\dagger)$ where $[c,c^\dagger]=1$, $[c,c]=[c^\dagger,c^\dagger]=0$ whose trace is desired. It is given by

$$\begin{aligned}\mathrm{Tr}f(c,c^\dagger)&=\sum_{n=0}^{\infty}\langle n|f(c,c^\dagger)|n\rangle\\ &=\sum_{n,m=0}^{\infty}\langle n|f(c,c^\dagger)|m\rangle\delta_{nm}.\end{aligned} \quad (37)$$

We may write δ_{nm} as

$$\delta_{nm}=\int\int\frac{d^2c'c'^nc'^{*m}e^{-|c'|^2}}{\pi(n!m!)^{1/2}}, \quad (38)$$

where $d^2c'\equiv|c'|d|c'|d\varphi$ and $c'=|c'|e^{i\varphi}$. If we put (38) into (37), note that $(m!)^{1/2}|m\rangle=c^{\dagger m}|0\rangle$, $\langle n|(n!)^{1/2}=\langle 0|c^n$ where $|0\rangle$ is the vacuum state; we may carry out the sums over n and m and obtain

$$\mathrm{Tr}f(c,c^\dagger)=\int\int\frac{d^2c'}{\pi}e^{-|c'|^2}\langle 0|e^{c'c}f(c,c^\dagger)e^{c'^*c^\dagger}|0\rangle. \quad (39)$$

If f is next put in normal form, it may be written as

$$f(c,c^\dagger)=f^{(n)}(c,c^\dagger)=\sum_{k,l}f_{kl}c^{\dagger k}c^l \quad (40)$$

and by means of (29), we see that

$$e^{-|c'|^2}\langle 0|e^{c'c}f^{(n)}(c,c^\dagger)e^{c'^*c^\dagger}|0\rangle=f^{(n)}(c',c'^*). \quad (41)$$

Thus,[19]

$$\mathrm{Tr}f(c,c^\dagger)=\int_{-\infty}^{\infty}\int\frac{d^2c'}{\pi}f^{(n)}(c',c'^*). \quad (42)$$

If we let $c'=x+iy$ and $c'^*=x-iy$ we see that the Fourier transform of $f^{(n)}(c',c'^*)$ is [compare with (36)]

$$F(\alpha,\beta)=\int_{-\infty}^{\infty}\int e^{-i(\alpha x+\beta y)}f^{(n)}(x,y)\frac{dxdy}{\pi}. \quad (43)$$

When we compare (42) and (43) it is clear that we may take the trace of f by letting α and $\beta=0$ in its Fourier transform. That is,

$$\mathrm{Tr}f(c,c)=F(0,0). \quad (44)$$

Therefore, in our particular case, the reduced density matrix is

$$R^{(n)}(x_0,y_0,t)=\int\cdots\int\rho^{(n)}(x_0,y_0,x_j,y_j,t)\prod_j\frac{dx_jdy_j}{\pi}, \quad (45)$$

so that the Fourier transform of the reduced density matrix is

$$F(\alpha_0,\beta_0,0,0,t)=\int\int e^{-i(\alpha_0x_0+\beta_0y_0)}R^{(n)}(x_0,y_0,t)\frac{dx_0dy_0}{\pi} \quad (46)$$

[19] R. J. Glauber, Phys. Rev. **131**, 2766 (1963); J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).

and

$$R^{(n)}(x_0,y_0,t)=\frac{1}{4\pi}\int\int e^{i(\alpha_0 x_0+\beta_0 y_0)}\times F(\alpha_0,\beta_0,0,0,t)d\alpha_0 d\beta_0. \quad (47)$$

We therefore proceed to solve for the Fourier transform of (33) and take the trace over the bath variables by letting the α_j and β_j corresponding to these variables go to zero.

Rather than change from the $\bar{a}^\dagger$, $\bar{a}$, $\bar{b}_j^\dagger$, and $\bar{b}_j$ complex variables to the real variables x_0, y_0, x_j, y_j we may symbolically write the Fourier transform of $\rho^{(n)}$ as

$$\rho^{(n)}(\zeta,t)=\int e^{i\sigma\cdot\zeta}F(\sigma,t) \quad (48)$$

where we have ordered the variables as

$$\zeta=[\bar{a}^\dagger,\bar{a},\bar{b}_j^\dagger,\bar{b}_j], \quad \sigma=[\sigma_0^*,\sigma_0,\sigma_j^*,\sigma_j] \quad (49)$$

and

$$\sigma_0=\tfrac{1}{2}(\alpha_0-i\beta_0), \quad \sigma_j=\tfrac{1}{2}(\alpha_j-i\beta_j), \quad \sigma_0^*=\tfrac{1}{2}(\alpha_0+i\beta_0), \quad \sigma_j^*=\tfrac{1}{2}(\alpha_j+i\beta_j). \quad (50)$$

If we put (48) into (33) we obtain (using matrix notation)

$$i\int e^{i\sigma\cdot\zeta}\frac{\partial F}{\partial t}=\int[-\sigma_i\Lambda_{ij}\sigma_j+iN_j\sigma_j]\times e^{i\sigma\cdot\zeta}F+i\zeta_i M_{ij}\sigma_j e^{i\sigma\cdot\zeta}F. \quad (51)$$

The last term may be integrated by parts with the integrated portion vanishing and we have

$$i(\partial F/\partial t)(\sigma,t)=[-\sigma_i\Lambda_{ij}\sigma_j+iN_j\sigma_j-\mathrm{Tr}M]F(\sigma,t)-(\partial F/\partial\sigma_i)M_{ij}\sigma_j. \quad (52)$$

But $\mathrm{Tr}M=0$ in our case. This equation may be solved easily if we let

$$F(\sigma,t)=\exp[-\sigma_i P_{ij}(t)\sigma_j-i\sigma_j Q_j(t)+R(t)]. \quad (53)$$

When we equate equal powers of σ, we see that P, Q and R satisfy

$$(\partial P/\partial t)-i(PM+M^tP)=-i\Lambda, \quad (54a)$$

$$(\partial Q/\partial t)-iQM=iN(t), \quad (54b)$$

$$\partial R/\partial t=+i\,\mathrm{Tr}M=0, \quad (54c)$$

since $\Lambda_{ij}=\Lambda_{ji}{}^t$ and $P_{ij}=P_{ji}$. M^t is the transpose of M. Formal exact solutions of (54) may be written down immediately if needed in terms of $P(0)$, $Q(0)$, and $R(0)$. These initial values must be obtained from the Fourier transform of $\rho^{(n)}(\bar{a},\bar{a},\bar{b}_j,\bar{b}_j,0)$ given in (26). One finds easily from (36) that (with obvious change in notation)

$$F(\sigma,0)=\exp\left[-\frac{\sigma_0^*\sigma_0}{\tau}-\sum_k\frac{\sigma_k^*\sigma_k}{\tau_k}-iw^*\sigma_0^*-iw\sigma_0\right] \quad (55)$$

so that

$$P(0)=\tfrac{1}{2}\begin{bmatrix}0&\tau^{-1}&0&0\\ \tau^{-1}&0&0&0\\ 0&0&0&L^{-1}\\ 0&0&L^{-1}&0\end{bmatrix};\quad Q(0)=[w^*\quad w\quad 0\quad 0] \quad (56)$$

and $R(0)=0$. L^{-1} is diagonal with the jth element given by τ_j. The dimensionality of the matrices is the same as those in (32).

We take the trace over the bath variables by letting the σ_j^* and σ_j corresponding to the bath vanish in (53). We then have

$$F(\sigma_0^*,\sigma_0,0,0,t)=\exp-(\sigma_0^*\sigma_0)\begin{pmatrix}u&v\\ l&r\end{pmatrix}\begin{pmatrix}\sigma_0^*\\ \sigma_0\end{pmatrix}-i(z\quad w)\begin{pmatrix}\sigma_0^*\\ \sigma_0\end{pmatrix}, \quad (57)$$

where we have let

$$P(t)=\begin{bmatrix}P_{11}(t)&P_{12}(t)\\ P_{21}(t)&P_{22}(t)\end{bmatrix};\quad Q(t)=[Q_1(t)\quad Q_2(t)]. \quad (58)$$

The dimensions of the P_{ij} matrices are: P_{11} is a 2×2, P_{12} is a $2\times2N$, P_{21} is a $2N\times2$, and P_{22} is a $2N\times2N$. Q_1 and Q_2 are vectors with $N+1$ columns. When the bath σ's vanish only P_{11} and Q_1 remain. We have therefore let

$$P_{11}(t)=\begin{bmatrix}u(t)&v(t)\\ l(t)&r(t)\end{bmatrix};\quad Q_1(t)=[z(t)\quad w(t)]. \quad (59)$$

The reduced density matrix is given by the transform of (57) according to (47) and we find the result

$$R^{(n)}(a,a^\dagger,t)=\Delta^{-1/2}N\{\exp\Delta^{-1}[-(v+l)(a^\dagger-z)(\bar{a}-w)+u(\bar{a}-w)^2+r(\bar{a}^\dagger-z)^2]\}, \quad (60)$$

where

$$\Delta=(v+l)^2-4ur \quad (61)$$

which is the desired answer in closed form. It contains six complex time-dependent parameters u, v, l, r, z, and w. We shall show later that these are related by

$$u(t)=r^*(t);\quad v(t)=l(t);\quad z(t)=w^*(t). \quad (62)$$

To obtain these parameters we must resort to approximate techniques but the gross statistical properties for the field mode are given by the dependence of $R^{(n)}$ on a and $a^\dagger$ rather than the parameters. Under the assumption of weak coupling, we shall show that

$$u(t)=r^*(t)\cong0.$$

In that case $R^{(n)}$ reduces to

$$R^{(n)}(a,a^\dagger,t)\cong y^{-1}N\{-y^{-1}(\bar{a}^\dagger-w^*)(\bar{a}-w)\}, \quad (63)$$

where

$$y(t)=v(t)+l(t)=2v(t)$$

and we see by comparing with (21) that the form of the reduced density operator for the field which maximizes the entropy subject to a given average energy and electric and magnetic field remains invariant when interacting with the bath even in the presence of a driving term. We shall show that $y(t)$ is related to the average energy in the field at time t while $w(t)$ and $w^*(t)$ are related to the average field. The exact connection is provided by the characteristic function which will be derived in the next section.

It is straightforward to show that the trace of the reduced density operator (60) or (63) equals unity as it must. The entropy given by (63) is easily seen to be

$$S(t)=k[y\ln y-(y-1)\ln(y-1)] \tag{64}$$

and depends only on y and not on w and w^*, the field in the cavity. It is also independent of the driving term as we shall see.

It may occasionally be important to know the matrix elements of $R(t)$ in (63) in the representation in which $a^\dagger a$ is diagonal. They are given by

$$\langle n|R(t)|m\rangle=\frac{(y-1)^n}{y^{n+1}}\left(\frac{w^*}{y}\right)^{m-n}\left(\frac{n!}{m!}\right)^{1/2} \times\exp\left(-\frac{|w|^2}{y}\right)L_n^{m-n}\left(-\frac{|w|^2}{y(y-1)}\right) \tag{65}$$

if $m\geq n$, and

$$\langle n|R(t)|m\rangle=\frac{(y-1)^m}{y^{m+1}}\left(\frac{w}{y}\right)^{n-m}\left(\frac{m!}{n!}\right)^{1/2} \times\exp\left(-\frac{|w|^2}{y}\right)L_m^{n-m}\left(-\frac{|w|^2}{y(y-1)}\right) \tag{66}$$

if $n\geq m$, where $L_m^\alpha(x)$ is the generalized Laguerre polynomial. Again it may be noted that these are exact in terms of the parameters y, w, and w^* if the coupling is weak. For simplicity, the derivation is omitted although it is straightforward.

IV. THE QUANTUM CHARACTERISTIC FUNCTION

From (20) we see that either p or q may be expressed as $\delta a+\delta^*a^\dagger$ so the characteristic function[13,18] defined by

$$C(\mu)=\mathrm{Tr}\{R(t)\exp i\mu[\delta a+\delta^*a^\dagger]\}, \tag{67}$$

where μ is a parameter, is a moment-generating function for all moments of p or q. If we introduce the operators d and $d^\dagger$ defined by

$$d=a-w(t),\quad d^\dagger=a^\dagger-w^*(t), \tag{68}$$

which satisfy the same commutation relations as a and $a^\dagger$, since w and w^* are c numbers, we may write (67) as

$$C(\mu)=y^{-1}\exp i\mu[\delta w(t)+\delta^*w^*(t)] \times\mathrm{Tr}\{\exp[\ln(1-y^{-1})d^\dagger d]\exp i\mu[\delta d+\delta^*d^\dagger]\}. \tag{69}$$

where we have used (63). The trace is evaluated in Appendix I and the result is[18]

$$C(\mu)=\exp\{i\mu[\delta w(t)+\delta^*w^*(t)] -\tfrac{1}{2}\mu^2|\delta|^2[2y(t)-1]\}. \tag{70}$$

With the proper choice of δ [compare (20)], we see that the average electric and magnetic fields in the cavity with weak coupling to the bath are given by

$$\begin{aligned}\langle p(t)\rangle&=\partial C_p/\partial(i\mu)|_{\mu=0}=i(\hbar\omega/2)^{1/2}[w^*(t)-w(t)],\\ \langle q(t)\rangle&=(\hbar/2\omega)^{1/2}[w^*(t)+w(t)]\end{aligned} \tag{71}$$

while the average energy in the ω mode is

$$\tfrac{1}{2}\langle[p^2+\omega^2q^2]\rangle=\tfrac{1}{2}[\langle p(t)\rangle^2+\omega^2\langle q(t)\rangle^2]+\hbar\omega[y(t)-\tfrac{1}{2}]. \tag{72}$$

The first term is the average coherent or signal energy while $\hbar\omega y(t)$ is the noise energy and $\hbar\omega/2$ is the familiar zero-point energy. We see that the detailed statistical properties are determined by the parameters $y(t)$, $w(t)$, and $w^*(t)$.

V. THE WIGNER-WEISSKOPF APPROXIMATION

The exact formal solutions in matrix form of (54a) and (54b) are not useful. We must therefore resort to an approximate solution of the Wigner-Weisskopf type.[20] For simplicity we shall carry out this procedure only for (54b) and merely give the result for (54a) which is more tedious.

If we decompose Q according to (58) and similarly decompose M as

$$M=\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix};\quad M_{11}=\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix};\quad M_{22}=\begin{pmatrix}\Omega & 0\\ 0 & -\Omega\end{pmatrix};$$

$$M_{12}=\begin{pmatrix}\kappa\cdot & -q^*\kappa^*\cdot\\ q\kappa\cdot & -\kappa^*\cdot\end{pmatrix};\quad M_{21}=\begin{pmatrix}\cdot\kappa^* & -\cdot q^*\kappa^*\\ \cdot q\kappa & -\cdot\kappa\end{pmatrix}; \tag{73}$$

and decompose $N(t)$ as

$$N(t)=(N_1(t)0);\quad N_1(t)=[e^*(t)-e(t)] \tag{74}$$

we see that (54b) becomes

$$\begin{aligned}i(\partial Q_1/\partial t)+Q_1M_{11}+Q_2M_{21}&=-N_1(t),\\ i(\partial Q_2/\partial t)+Q_1M_{12}+Q_2M_{22}&=0.\end{aligned} \tag{75}$$

If we take the Laplace transform of these equations and designate the transform of all quantities by a bar, we have

$$\bar{Q}_1(s)[is+M_{11}]+\bar{Q}_2(s)M_{21}=-\bar{N}_1(s)+Q_1(0), \tag{76a}$$

$$\bar{Q}_2(s)[is+M_{22}]+\bar{Q}_1(s)M_{12}=0 \tag{76b}$$

since $Q_2(0)=0$ and $Q_1(0)=[w^*w]$ according to (56). Since M_{11} and M_{22} are diagonal, we may solve (76b) for

[20] V. Weisskopf and E. Wigner, Z. Physik **63**, 54 (1930); G. Källen, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 5, p. 274.

$\bar{Q}_2(s)$ and put it into (76a) with the result

$$\bar{Q}_1(s)\{is+M_{11}-M_{12}[1/(is+M_{22})]M_{21}\} = -\bar{N}_1(s)+Q_1(0). \quad (77)$$

If we use the notation of (59), let

$$K=\sum \frac{|\kappa_j|^2}{s-i\omega_j}-|q|^2\sum_i \frac{|\kappa_j|^2}{s+i\omega_j}, \qquad O=q\sum|\kappa_j|^2\left[\frac{1}{s-i\omega_j}+\frac{1}{s+i\omega_j}\right]. \quad (78)$$

Equations (77) become

$$[s-i\omega+K]\bar{z}(s)+O\bar{w}(s)=w^*+i\bar{e}^*(s), \qquad O^*\bar{z}(s)+(s+i\omega+K^*)\bar{w}(s)=w-i\bar{e}(s) \quad (79)$$

from which we see immediately that

$$z(t)=w^*(t).$$

We may solve (79) for $\bar{w}(s)$ with the result

$$\bar{w}(s)=\frac{w-i\bar{e}(s)+\{O^*[w^*+i\bar{e}^*(s)]/(s-i\omega+K)\}}{s+i\omega+K^*-[|O|^2/(s-i\omega+K)]}. \quad (80)$$

The Wigner-Weisskopf approximation here consists in neglecting the term in the numerator multiplying O^* since it is of order $|\kappa|^2$ in the coupling compared with $w-i\bar{e}(s)$. Also the term $|O|^2$ is of order $|\kappa|^4$ and may be neglected in the denominator. The inverse transform of the remaining terms in (80) is

$$w(t)=\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{e^{zt}[w-i\bar{e}(z)]}{z+i\omega+K^*(z)}dz, \quad (81)$$

where $\epsilon>0$ is real and the integration is in the complex z plane. We may let $z=\epsilon+iy$ and (81) becomes

$$w(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{\epsilon t}e^{iyt}[w-i\bar{e}(\epsilon+iy)]dy}{y+\omega-i\epsilon-iK^*(\epsilon+iy)}. \quad (82)$$

Since[21]

$$\lim_{\epsilon\to 0}\frac{1}{a-i\epsilon}=P\frac{1}{a}+i\pi\delta(a) \quad (83)$$

where P means the Cauchy principal part, (82) becomes as $\epsilon\to 0$

$$w(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{iyt}[w-i\bar{e}(iy)]dy}{y+\omega-P\sum|\kappa_j|^2\{[1/(y+\omega_j)]-|q|^2[1/(y-\omega_j)]\}-i\pi\sum|\kappa_j|^2[\delta(y+\omega_j)-|q|^2\delta(y-\omega_j)]}. \quad (84)$$

We have assumed small coupling so as a zeroth approximation the denominator has a simple pole at $y=-\omega$. We assume

$$\sum_j|\kappa_j|^2\delta(\omega_j-\omega)\ll\omega, \qquad P\sum|\kappa_j|^2/(\omega_j-\omega)\ll\omega. \quad (85)$$

We see that the terms proportional to $|q|^2$ have very little effect on (84). We may therefore define a frequency shift $\Delta\omega$ and attenuation parameter γ as

$$\Delta\omega=-P\sum_j|\kappa_j|^2/(\omega_j-\omega), \qquad \gamma=2\pi\sum_j|\kappa_j|^2\delta(\omega_j-\omega) \quad (86)$$

so that

$$w(t)\cong we^{-i(\omega+\Delta\omega)t-(\gamma t/2)} -i\int_0^t e(t-t')e^{-i(\omega+\Delta\omega)t'-(\gamma t'/2)}dt'. \quad (87)$$

The coupling causes a small frequency shift, $\Delta\omega$, as well as damping. The relaxation time for the system is γ^{-1}. The terms due to $|q|^2$ are smaller still when the coupling is weak.

A similar although more tedious procedure for (54a) shows that

$$v(t)=l(t)\cong\frac{e^{-\gamma t}}{2(1-e^{-\lambda})}+\frac{(1-e^{-\gamma t})}{2[1-\exp(-\kappa\omega/kT)]} \quad (88)$$

with $u(t)=r^*(t)$ of order κ^2 smaller than v and l.

We may define an effective temperature, T_e, for the ensemble of cavity ω modes by

$$y=\frac{1}{1-\exp(-\hbar\omega/kT_e)} =\frac{\exp(-\gamma t)}{1-\exp(-\lambda)}+\frac{1-\exp(-\gamma t)}{1-\exp(-\hbar\omega/kT)}.$$

At $t=0$, T_e is determined by the signal and noise energy initially in the cavity. After a time long compared with the relaxation time, the effective cavity temperature approaches thermal equilibrium with the bath, $T_e\to T$. If at $t=0$ the cavity and bath temperatures are equal ($\lambda=\hbar\omega/kT$), the system is always in thermodynamic equilibrium with the bath and $T_e=T$ for all time since y is independent of time.

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[21] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., p. 66.

APPENDIX I

We evaluate here the trace occurring in (69). Let

$$\Delta(\epsilon)=\exp\{\epsilon\ln(1-y^{-1})d^\dagger d\}\exp\{i\mu[\delta d+\delta^* d^\dagger]\}, \quad \text{(A1)}$$

where ϵ is a parameter. Although there are several methods for evaluating the trace of Δ with $\epsilon=1$, we shall evaluate it by putting Δ into normal form.

We note that when $\epsilon=0$,

$$\Delta(0)=\exp\{i\mu[\delta d+\delta^* d^\dagger]\}=\exp(i\mu\delta^* d^\dagger)\times\exp(i\mu\delta d)\exp(-\mu^2|\delta|^2/2), \quad \text{(A2)}$$

where we have used the special case of the Baker-Hausdorff theorem[18,19]

$$e^{A+B}=e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{(A3)}$$

provided $[A,[A,B]]=[B,[A,B]]=0$. The latter form of writing $\Delta(0)$ is in normal form.

We find from (A1) that

$$\partial\Delta^{(n)}/\partial\epsilon=\ln(1-y^{-1})d^\dagger d\Delta^{(n)} =\ln(1-y^{-1})d^\dagger[\Delta^{(n)}d+(\partial\Delta^{(n)}/\partial d^\dagger)], \quad \text{(A4)}$$

where we assumed $\Delta^{(n)}$ is in normal form and we used (30). We may now assume a solution of the form

$$\Delta^{(n)}=N\{e^G\}, \quad \text{(A.5a)}$$

where

$$G=+\{(A(\epsilon)-1)\bar{d}^\dagger\bar{d}+B(\epsilon)\bar{d}+C(\epsilon)\bar{d}^\dagger+D(\epsilon)\}. \quad \text{(A.5b)}$$

If we compared (A.5) and (A.2), we see that

$$A(0)=1,\quad B(0)=i\mu\delta,\quad C(0)=i\mu\delta^*,\quad D(0)=-\tfrac{1}{2}\mu^2|\delta|^2. \quad \text{(A.6)}$$

If we now substitute (A.5) into (A.4) and equate equal powers of d, $d^\dagger$, we find that

$$\frac{dA}{d\epsilon}=\ln(1-y^{-1})A;\quad \frac{dC}{d\epsilon}=\ln(1-y^{-1})C;$$
$$\frac{dB}{d\epsilon}=0;\quad \frac{dD}{d\epsilon}=0.$$

The solutions of these equations which satisfy (A.6) are readily obtained, from which we see that (A.5) becomes when $\epsilon=1$

$$\Delta(1)=N\{\exp[-y^{-1}\bar{d}^\dagger\bar{d}+i\mu\delta\bar{d}+i\mu\delta^*(1-y^{-1})\bar{d}^\dagger-\tfrac{1}{2}\mu^2|\delta|^2]\}. \quad \text{(A.7)}$$

We now may easily take the trace of $\Delta(1)$. According to (4.2) we have

$$\begin{aligned}\mathrm{Tr}\Delta(1)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\int d^{(2)}d'\exp[-y^{-1}|d'|^2+i\mu\delta d'+i\mu\delta^*(1-y^{-1})d'^*]\exp(-\tfrac{1}{2}\mu^2|\delta|^2)\\&=y\exp[-\mu^2|\delta|^2(y-1)]\exp(-\tfrac{1}{2}\mu^2|\delta|^2)\\&=y\exp[-\tfrac{1}{2}\mu^2|\delta|^2(2y-1)],\end{aligned} \quad \text{(A8)}$$

which is the required result for (70).