Solution of a p-Wave Equation for Pion-Pion Scattering*

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A fixed-momentum-transfer dispersion relation is used in a self-consistent calculation of the ρ meson. An expansion of the scattering amplitude in terms of a new variable, introduced by a conformal transformation, forms the starting point of the present calculation, and this expansion converges throughout the physical region. A self-consistent p-wave resonance is found at 340 MeV. It is concluded that the ρ meson is not capable of producing itself self-consistently and that some other mechanism is responsible for the 750-MeV p-wave resonance observed in the π-π system.

I. INTRODUCTION

WE present some solutions to a set of approximate equations describing the pion-pion scattering amplitude in the low-energy region. This work is based on a method developed by Atkinson,[1] in which partial-wave amplitudes are projected from an expansion of the scattering amplitude which converges throughout the physical region. The partial-wave amplitudes are related to derivatives of the scattering amplitude, evaluated in the forward direction, which can, in turn, be found from forward scattering dispersion relations.

Several years ago, Chew and Mandelstam[2] introduced a theory of low-energy pion-pion scattering, based on the Mandelstam representation.[3] Although the method had some qualitative success, it suffers from the difficulty that it employs an expansion of the scattering amplitude in terms of Legendre polynomials in unphysical regions outside the Lehmann ellipse, where it fails to converge. To circumvent these difficulties, several authors[4,5] have proposed the use of forward or backward scattering dispersion relations in a Taylor series expansion of the scattering amplitude in the cosine of the scattering angle. This expansion converges only for rather small values of energy, due to the existence of the double-spectral functions in the Mandelstam representation.[5,6]

By means of a conformal transformation, Ciulli and Fischer[6] have introduced a new variable, in terms of which an expansion of the scattering amplitude in either the forward or backward direction converges for all physical values of energy and scattering angle. Extending this formalism, Atkinson[1] has derived general expressions for partial-wave amplitudes.

In the present work, Atkinson's results are used to obtain partial-wave dispersion relations. These relations appear to be well suited to a study of low-energy phenomena due to the occurrence of cutoff functions, which emerge in a natural way and which suppress the high-energy behavior of the amplitude. In addition, because only forward or backward scattering dispersion relations are used, the effect of the double spectral functions is minimized, since long-range extrapolations into regions where they fail to vanish are not required.

We apply this method to a self-consistent calculation of the $J=1$, $T=1$ amplitude (ρ meson),[7] in which we employ elastic unitarity and we approximate the left-hand cut by a $J=1$, $T=1$ zero-width resonance. In Sec. II, we describe briefly the work of Atkinson, in Sec. III we derive a partial-wave dispersion relation for the p-wave π-π amplitude, and in Sec. IV we present our numerical results. We conclude in Sec. V with a general discussion.

II. GENERAL EXPRESSIONS FOR PARTIAL-WAVE AMPLITUDES

The Taylor series expansion of the scattering amplitude in z about the point $z=1$ is given by[8]

$$A(\nu,z)=\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}\frac{\partial^n A(\nu,z)}{\partial z^n}\bigg|_{z=1}. \tag{2.1}$$

Partial-wave amplitudes are given by the formula

$$A_l(\nu)=\frac{1}{2}\int_{-1}^{+1}dz\,P_l(z)A(\nu,z)\,, \tag{2.2}$$

and on substituting Eq. (2.1) into Eq. (2.2), we have

$$A_l(\nu)=\sum_{n=l}^{\infty}(-1)^{n+l}\frac{2^n n!}{(n-l)!(n+l+1)!}\frac{\partial^n A(\nu,z)}{\partial z^n}\bigg|_{z=1}. \tag{2.3}$$

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[8] Notation: ν is the momentum squared of one of the pions in the c.m. system; z is the cosine of the scattering angle. We take units $\hbar=c=\mu=1$. Throughout this section, we employ the notation of Atkinson.

However, elastic thresholds in the crossed channels give rise to branch points in the z plane, located at

$$z_1=1+2/\nu\,,\quad z_2=-1-2/\nu\,, \tag{2.4}$$

and the cuts are chosen to run along the real axis to $\pm\infty$, respectively. From Fig. 1, it can be seen that the expansion (2.1) will not converge for physical z near $z=-1$, whenever $z_1\leqslant 3$; i.e., for $\nu\geqslant 1$. Consequently, Eq. (2.3) would not be expected to give reliable results for $\nu\geqslant 1$. By expanding the scattering amplitude in both the forward and backward directions, one can extend the region of convergence to $\nu\leqslant 2$. These expansions can be introduced by rewriting Eq. (2.2) in the form

$$A_l(\nu)=\frac{1}{2}\int_{-1}^{0}dz\,A(\nu,z)P_l(z)+\frac{1}{2}\int_{0}^{1}dz\,A(\nu,z)P_l(z)\,,$$

where in the first term we use the expansion in the backward direction, and in the second the expansion in the forward direction.

In order to obtain similar expressions for the partial-wave amplitudes based on a convergent series, Atkinson introduced the new variable $\omega(z)$, where

$$\omega(z)=\frac{[(z_2-z)/(z_2-1)]^{1/2}-[(z_1-z)/(z_1-1)]^{1/2}}{[(z_2-z)/(z_2-1)]^{1/2}+[(z_1-z)/(z_1-1)]^{1/2}}\,, \tag{2.5}$$

$$\omega(1)=0\,.$$

This transformation conformally maps the entire z plane into the unit circle, the cuts in the z plane being mapped onto the boundary of the circle. The Taylor series expansion of $A(\nu,z)$ in terms of ω about the point $\omega=0$,

$$A(\nu,z)=\sum_{n=0}^{\infty}\frac{\omega^n}{n!}\frac{\partial^n A(\nu,z)}{\partial\omega^n}\bigg|_{\omega=0}\,, \tag{2.6}$$

now converges for all physical ν and z. Atkinson has shown that, when this expansion is truncated after N terms, the first N partial-wave amplitudes are given by the formulas

$$A_l(\nu)=\sum_{n=0}^{N}U_{ln}{}^{N}(\nu)\frac{\partial^n A(\nu,z)}{\partial z^n}\bigg|_{z=1}\,, \tag{2.7}$$

where

$$U_{ln}{}^{N}(\nu)=\frac{\beta^n}{n!}\sum_{r=n}^{N}C_{r-n}{}^{n}(-\alpha)\frac{1}{2}\int_{-1}^{+1}dz\,P_l(z)\omega^r\,,$$

and $\alpha=\nu/(\nu+2)$, $\beta=(8/\nu)[(\nu+1)/(\nu+2)]$; $C_r{}^n(x)$ are Gegenbauer polynomials defined by the series

$$\frac{1}{(1-2x\omega+\omega^2)^n}=\sum_{r=0}^{\infty}\omega^r C_r{}^n(x)\,,$$

and $P_l(z)$ are the Legendre polynomials. It is hoped that the use of Eq. (2.7) instead of Eq. (2.3) will give a better approximation for the partial-wave amplitudes. The question of how many terms we should keep in the expansion of Eq. (2.6) is discussed later.

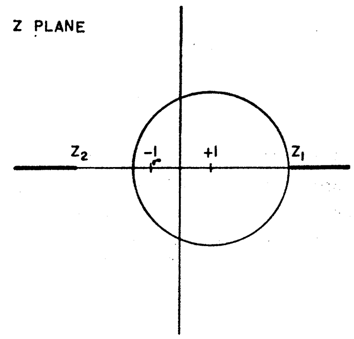

FIG. 1. The z plane, showing the two cuts of the scattering amplitude, starting at z_1 and z_2, and the circle of convergence of a Taylor series expansion of the scattering amplitude about the point $z=1$.

III. DERIVATION OF p-WAVE DISPERSION RELATION

In this section, we discuss the application of Atkinson's equations to a self-consistent calculation of a p-wave resonance in the pion-pion system. In terms of the Mandelstam representation, the three channels of the π-π problem give rise to the same scattering and can be written in terms of the energy momentum and isotopic spin of the incoming and outgoing particles as[2]

$$\begin{aligned}
&(P_1,\alpha)+(P_2,\beta)\rightarrow(-P_2,\gamma)+(-P_4,\delta)\,, \quad \text{(I)}\\
&(P_1,\alpha)+(P_3,\gamma)\rightarrow(-P_2,\beta)+(-P_4,\delta)\,, \quad \text{(II)}\\
&(P_1,\alpha)+(P_4,\delta)\rightarrow(-P_2,\beta)+(-P_3,\gamma)\,. \quad \text{(III)}
\end{aligned} \tag{3.1}$$

The usual invariant variables are

$$\begin{aligned}
s&=(P_1+P_2)^2=(P_3+P_4)^2=4(\nu+1)\,,\\
t&=(P_1+P_3)^2=(P_2+P_4)^2=-2\nu(1-\cos\theta)\,,\\
u&=(P_1+P_4)^2=(P_2+P_3)^2=-2\nu(1+\cos\theta)\,.
\end{aligned} \tag{3.2}$$

The invariant amplitude is written as

$$\begin{aligned}
T(s,t,u)=A(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta}&+B(s,t,u)\delta_{\alpha\gamma}\delta_{\beta\delta}\\
&+C(s,t,u)\delta_{\alpha\delta}\delta_{\beta\gamma}\,,
\end{aligned} \tag{3.3}$$

and the isotopic spin amplitudes are given in terms of the amplitudes A, B, C, by the equations

$$\begin{aligned}
A^0&=3A+B+C\,,\\
A^1&=B-C\,,\\
A^2&=B+C\,.
\end{aligned} \tag{3.4}$$

For $N=3$, our basic equation for the p-wave isotopic-spin-one amplitude is, from Eq. (2.7),

$$\begin{aligned}
A_1{}^1(\nu)=U_{11}{}^3(\nu)A^{1(1)}(\nu)&+U_{12}{}^3(\nu)A^{1(2)}(\nu)\\
&+U_{13}{}^3(\nu)A^{1(3)}(\nu)\,,
\end{aligned} \tag{3.5}$$

where we use the abbreviation

$$A^{I(n)}=\partial^n A^I(\nu,z)/\partial z^n\big|_{z=1}\,. \tag{3.6}$$

Each of the amplitudes A, B, C obeys the fixed momentum-transfer dispersion relation

$$A(s,t)=\frac{1}{\pi}\int_4^{\infty}ds'\frac{A_s(s',t)}{s'-s}+\frac{1}{\pi}\int_4^{\infty}du'\frac{A_u(u',t)}{u'-u}\,, \tag{3.7}$$

and it can readily be shown that the isotopic-spin-one amplitude obeys the relation

$$A^1(s,t)=\frac{1}{\pi}\int_4^\infty ds'\left[\frac{A_s^1(s',t)}{s'-s}+\frac{-\frac{1}{3}A_s^0(s',t)+\frac{1}{2}A_s^1(s',t)+A_s^2(s',t)}{s'-4+s+t}\right], \tag{3.8}$$

in which crossing symmetry, implied by the kinematics of the problem, has been used in the second term on the right. In line with the p-wave approximation, we keep in Eq. (3.8) only the isotopic-spin-one amplitude; we further represent the absorptive part of the amplitude in the physical region by the lowest angular-momentum states, in this case, the p wave, and we have

$$A_s^0(s,t)=A_s^2(s,t)=0, \quad [\partial^n A_s^1(s,t)/\partial t^n]=0, \quad n\geqslant 2. \tag{3.9}$$

To evaluate the derivates $A^{1(n)}$, we note that

$$A^{1(n)}(\nu)=(2\nu)^n[\partial^n A^1(\nu,t)/\partial t^n]|_{t=0}, \tag{3.10}$$

and then, differentiating Eq. (3.8), and using Eq. (3.9), we obtain

$$\frac{\partial A^1(s,t)}{\partial t}=\frac{1}{\pi}\int_4^\infty ds'\left[\frac{\partial A_s^1/\partial t}{s'-s}+\frac{1}{2}\frac{\partial A_s^1/\partial t}{s'-4+s+t}-\frac{1}{2}\frac{A_s^1}{(s'-4+s+t)^2}\right], \tag{3.11}$$

and in general, for $n\geqslant 2$,

$$\frac{\partial^n A^1(s,t)}{\partial t^n}=-(-1)^n\frac{n!}{2\pi}\int_4^\infty ds'\left[\frac{\partial A_s^1/\partial t}{(s'-4+s+t)^n}-\frac{A_s^1}{(s'-4+s+t)^{n+1}}\right]. \tag{3.12}$$

After gathering the results so far, we find that our p-wave amplitude becomes

$$A_1^1(\nu)=\frac{2\nu U_{11}^3(\nu)}{\pi}\int_4^\infty ds'\left[\frac{\partial A_s^1/\partial t}{s'-s}+\frac{1}{2}\frac{\partial A_s^1/\partial t}{s'-4+s}-\frac{1}{2}\frac{A_s^1}{(s'-4+s)^2}\right]+\frac{(2\nu)^2U_{12}^3(\nu)}{\pi}$$
$$\times\int_4^\infty ds'\left[-\frac{\partial A_s^1/\partial t}{(s'-4+s)^2}+\frac{A_s^1}{(s'-4+s)^3}\right]+\frac{3(2\nu)^3U_{13}^3(\nu)}{\pi}\int_4^\infty ds'\left[\frac{\partial A_s^1/\partial t}{(s'-4+s)^3}-\frac{A_s^1}{(s'-4+s)^4}\right]. \tag{3.13}$$

The next step is to find expressions for the numerators occurring in the integrands of these equations, $\partial A_s^1(s,t)/\partial t|_{t=0}$ and $A_s^1(s,t)|_{t=0}$. Equation (3.9) yields

$$\mathrm{Im}A^{1(2)}(\nu)=\mathrm{Im}A^{1(3)}(\nu)=0, \tag{3.14}$$

and we have that

$$\left.\frac{\partial A_s^1(s,t)}{\partial t}\right|_{t=0}=\frac{1}{2\nu}\mathrm{Im}A^{1(1)}(\nu)=\frac{1}{2\nu U_{11}^3(\nu)}\mathrm{Im}A_1^1(\nu). \tag{3.15}$$

From the third-order isotopic-spin-one s-wave equation,

$$A_0^1(\nu)=0=A^{1(0)}(\nu)+U_{01}^3(\nu)A^{1(1)}(\nu)+U_{02}^3(\nu)A^{1(2)}(\nu)+U_{03}^3(\nu)A^{1(3)}(\nu), \tag{3.16}$$

we find

$$A_s^{1(0)}(\nu)=-[U_{01}^3(\nu)/U_{11}^3(\nu)]\,\mathrm{Im}A_1^1(\nu). \tag{3.17}$$

Finally, we introduce, as a matter of convenience, a new amplitude $a(\nu)$, defined by the equation:

$$a(\nu)=A_1^1(\nu)/U_{11}^3(\nu). \tag{3.18}$$

The third-order p-wave equation of our theory is now written:

$$a(\nu)=\frac{\nu}{\pi}\int_0^\infty d\nu'\frac{\mathrm{Im}a(\nu')}{\nu'(\nu'-\nu)}+\frac{\nu}{2\pi}\int_0^\infty d\nu'\frac{\mathrm{Im}a(\nu')}{\nu'(\nu'+\nu+1)}+\frac{\nu}{4\pi}\int_0^\infty d\nu'\frac{U_{01}^3(\nu')\,\mathrm{Im}a(\nu')}{(\nu'+\nu+1)^2}$$
$$-\frac{\nu^2}{2\pi}\frac{U_{12}^3(\nu)}{U_{11}^3(\nu)}\int_0^\infty d\nu'\frac{\mathrm{Im}a(\nu')}{\nu'(\nu'+\nu+1)^2}-\frac{\nu^2}{4\pi}\frac{U_{12}^3(\nu)}{U_{11}^3(\nu)}\int_0^\infty d\nu'\frac{U_{01}^3(\nu')\,\mathrm{Im}a(\nu')}{(\nu'+\nu+1)^3}$$
$$+\frac{3\nu^3}{4\pi}\frac{U_{13}^3(\nu)}{U_{11}^3(\nu)}\int_0^\infty d\nu'\frac{\mathrm{Im}a(\nu')}{\nu'(\nu'+\nu+1)^3}+\frac{3\nu^3}{8\pi}\frac{U_{13}^3(\nu)}{U_{11}^3(\nu)}\int_0^\infty d\nu'\frac{U_{01}^3(\nu')\,\mathrm{Im}a(\nu')}{(\nu'+\nu+1)^4}. \tag{3.19}$$

For brevity, we rewrite Eq. (3.19) as

$$a(\nu)=\frac{\nu}{\pi}\int_0^\infty d\nu'\frac{\mathrm{Im}a(\nu')}{\nu'(\nu'-\nu)}+B(\nu). \tag{3.20}$$

This equation has the same structure as the p-wave equation of Chew and Mandelstam,[2] with cuts along the real axis for $-\infty<\nu<-1$ and $0<\nu<\infty$. The crossing relation for $\mathrm{Im}a(\nu)$ on the left-hand cut is different, however. We note that, in terms of $a(\nu)$, the unitarity condition reads

$$\mathrm{Im}a(\nu)=U_{11}{}^3(\nu)\left(\frac{\nu}{\nu+1}\right)^{1/2}|a(\nu)|^2, \tag{3.21}$$

and

$$\mathrm{Im}[a(\nu)]^{-1}=-U_{11}{}^3(\nu)\left(\frac{\nu}{\nu+1}\right)^{1/2}. \tag{3.22}$$

$\mathrm{Im}A_1{}^1(\nu)$ is now approximated on the left-hand cut by a zero-width resonance:

$$\mathrm{Im}A_1{}^1(\nu)=\pi R\delta(\nu-\nu_R), \tag{3.23}$$

where

$$R=\tfrac{1}{12}(g^2/4\pi)\nu_R, \tag{3.24}$$

and

$$\nu_R=\tfrac{1}{4}(m_R{}^2-4),$$

m_R is the mass of the resonance, and g is the $\rho\pi\pi$ coupling constant, defined by its relation to the width of the resonance Γ:

$$\Gamma=\frac{1}{12}\frac{g^2}{4\pi}\frac{(m_R{}^2-4)^{3/2}}{m_R{}^2}. \tag{3.25}$$

Substituting Eq. (3.23) into Eq. (3.19) gives for the driving term $B(\nu)$ the expression

$$\begin{aligned}B(\nu)=&\frac{R}{2\nu_R U_{11}{}^3(\nu_R)}\frac{\nu}{\nu+\nu_L}+\frac{R}{4}\frac{U_{01}{}^3(\nu_R)}{U_{11}{}^3(\nu_R)}\frac{\nu}{(\nu+\nu_L)^2}-\frac{R}{2\nu_R U_{11}{}^3(\nu_R)}\frac{U_{12}{}^3(\nu)}{U_{11}{}^3(\nu)}\frac{\nu^2}{(\nu+\nu_L)^2}\\&-\frac{R}{4}\frac{U_{01}{}^3(\nu_R)}{U_{11}{}^3(\nu_R)}\frac{U_{12}{}^3(\nu)}{U_{11}{}^3(\nu)}\frac{\nu^2}{(\nu+\nu_L)^3}+\frac{3R}{4\nu_R U_{11}{}^3(\nu_R)}\frac{U_{13}{}^3(\nu)}{U_{11}{}^3(\nu)}\frac{\nu^3}{(\nu+\nu_L)^3}+\frac{3R}{8}\frac{U_{01}{}^3(\nu_R)}{U_{11}{}^3(\nu_R)}\frac{U_{13}{}^3(\nu)}{U_{11}{}^3(\nu)}\frac{\nu^3}{(\nu+\nu_L)^4},\end{aligned} \tag{3.26}$$

where $\nu_L=\nu_R+1$.

We solve Eqs. (3.19) and (3.26) by the N/D technique,[2] setting

$$a(\nu)=N(\nu)/D(\nu). \tag{3.27}$$

Then

$$N(\nu)=B(\nu)+\frac{\nu}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\times U_{11}{}^3(\nu')N(\nu')\frac{B(\nu')-B(\nu)}{\nu'(\nu'-\nu)}, \tag{3.28}$$

and

$$D(\nu)=1-\frac{\nu}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}U_{11}{}^3(\nu')\frac{N(\nu')}{\nu'(\nu'-\nu)}, \tag{3.29}$$

i.e., we solve first a linear integral equation for $N(\nu)$ and use this result to find $D(\nu)$. It should be noted that this procedure requires quantities in the physical region only.

In the foregoing development we have taken $N=3$; i.e., we kept three terms in the expansion of the scattering amplitude. In order to determine the optimum number of terms to be kept in this expansion, we must first consider the significance of the parameter N. In general, the larger N is, the better will be the representation of an assumed amplitude. However, making N larger also has the effect of minimizing the effect of the cuts in the z plane shown in Fig. 1. In fact, as $N\to\infty$, the effect of the cuts in the z plane would entirely disappear and we would produce the Chew-Mandelstam equation which does completely ignore these cuts. For example, in the present calculation we are assuming that the amplitude is a pure p wave. We know that this cannot be correct because a pure p-wave amplitude would not have the cuts of Fig. 1 that the amplitude must have. Using the expansion given by Eq. (2.6), with finite N to approximate the amplitude, guarantees that the result will have the proper cuts in z (although, of course, it says nothing about the discontinuity across the cuts). As we make N larger to extend the assumed amplitude to higher energies it will also be extended further towards the region $z>1$ and look less like a function with the z cut than it should. As ν increases, the cut in the z plane approaches the physical region at $z=1$, and we would expect a smooth interpolation between the region $z>1$ where there is a cut and the region $z<1$ where we want to approximate the assumed amplitude.

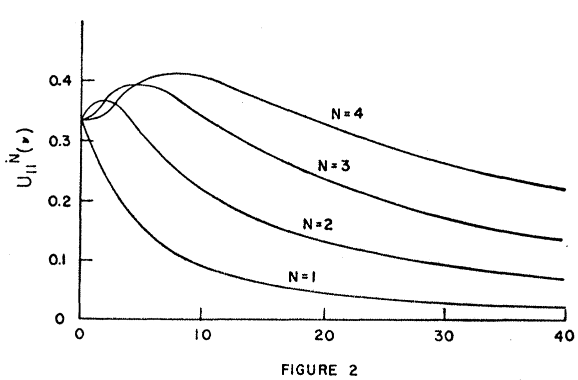

FIG. 2. Plot of the functions $U_{11}{}^N(\nu)$ for $N=1, 2, 3, 4$.

TABLE I. Calculated values at R_{out} versus assumed values of R_{in} as a function of resonance position, for first, third, and fourth orders.

N	ν_R	R_{in}	R_{out}
1	0.35 (326 MeV)	3.10	3.10
3	0.38 (330 MeV)	1.57	1.28
	0.56 (350 MeV)	2.24	2.96
4	0.38 (330 MeV)	1.09	0.90
	0.56 (350 MeV)	1.55	1.72

From the foregoing discussion, we conclude that N should be large enough to approximate the amplitude up to that energy below which we can believe our assumptions, and no larger. A measure of what value of N to use is given by the dependence of the $U_{ln}{}^N(\nu)$ functions on ν. As long as the $U_{ln}{}^N(\nu)$ are close to their threshold values, the assumed amplitude will be well represented, so that we pick a value of N that will approximately preserve the $U_{ln}{}^N(\nu)$ at their threshold values up to the appropriate energy. In the present calculation we are assuming that the elastic p wave dominates the total π-π cross section up to energies just beyond the ρ mass. We base this assumption on experimental information and thus use the experimental value for the ρ mass. In Fig. 2 we have plotted $U_{11}{}^N(\nu)$, which is the most important U function for the p wave, for $N=1$, 2, 3, 4. $N=3$ keeps $U_{11}{}^N(\nu)$ at approximately its threshold value throughout the energy region of the ρ meson and has therefore been used in the present calculation.

It should also be noticed that there is some internal consistency to the method in that, once a value of N has been selected, the cutoff character of the $U_{ln}{}^N(\nu)$ would tend to suppress the effects of high-energy contributions. This is especially important in any calculation which neglects inelastic contributions which would be expected to come in at high energies.

IV. NUMERICAL RESULTS

Following the procedure of Zachariasen and Zemach[9,10] we found self-consistent solutions for $N=1$, 3, and 4. The self-consistent conditions are given by the equations

$$\mathrm{Re}D(\nu_R)=0\,, \tag{4.1}$$

$$\frac{\mathrm{Im}D(\nu_R)}{\frac{1}{4}(d\,\mathrm{Re}D(\nu)/d\nu)\big|_{\nu=\nu_R}}=2(\nu_R+1)^{1/2}\Gamma\,. \tag{4.2}$$

From Eq. (3.29), one obtains

$$\mathrm{Im}D(\nu_R)=U_{11}{}^3(\nu_R)[\nu_R/(\nu_R+1)]^{1/2}N(\nu_R)\,, \tag{4.3}$$

so that Eq. (4.2) can be rewritten as

$$R=\frac{U_{11}{}^3(\nu_R)N(\nu_R)}{d\,\mathrm{Re}D(\nu)/d\nu\big|_{\nu=\nu_R}}\,. \tag{4.4}$$

Our procedure was essentially the following: For certain ν_R, R was varied until one R was found, R_{in}, which satisfied Eq. (4.1); then, from Eq. (4.4), R_{out} was computed. This was repeated for several ν_R, the result being a series of points lying on two curves, one R_{in} versus ν_R, and the other R_{out} versus ν_R. The point at which these two curves crossed determined the self-consistent solution, i.e., at that point $R_{in}=R_{out}$ for the same ν_R.

For the case $N=1$, it was possible to write the numerator and denominator functions in closed form, enabling us to determine the self-consistent coupling constant and resonance position with good accuracy. Since, for $N=3$ and 4, it was impossible for all practical purposes to get exact analytic solutions, we had to resort to a computer. We found solutions by approximating the linear integral equation for $N(\nu)$ by a set of simultaneous algebraic equations, which were then solved by matrix inversion. $D(\nu)$ was then evaluated by numerical integration. In order to test the accuracy of this procedure, the computer solutions were checked for the case in which $B(\nu)$ is given by a single pole—a case known to have an exact analytical solution. The two solutions agreed to within 2%.

For the three cases $N=1$, 3, 4, we found that a self-consistent solution existed, and that, for the computer solutions of $N=3$ and $N=4$, the self-consistent resonance position fell between two adjacent mesh points, $0.38<\nu_R<0.56$. The results are shown in Table I. According to the discussion of the preceding section, we regard $N=3$ as giving the best solution to this problem. This case yielded a resonance at 340 MeV with a coupling constant of about 50.

V. GENERAL DISCUSSION AND CONCLUSIONS

We conclude that a zero-width p-wave resonance in the crossed channel, along with the assumption of elastic unitarity, does not yield the experimental rho meson (mass of 750 MeV), although a self-consistent position and coupling constant are found. It will be noted that a coupling constant of 50 corresponds to a width of about 280 MeV by Eq. (3.25), whereas the position of the resonance is only 60 MeV above threshold. This resonance is anomalously wide and therefore does not really yield a self-consistent solution, since a very narrow (zero-width) resonance is assumed in the crossed channel. We should like to emphasize that the only purely mathematical assumption made in this problem is in the truncation of the initial expansion of the scattering amplitude, Eq. (2.6); the equations which result are, in principle, soluble.

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These facts indicate that a narrow p-wave resonance is not the dominant feature driving the pion-pion system and that inelastic and other high-energy effects must be included, as has been pointed out by several other authors.[10–12] In particular, the effect of the π-ω channel has been emphasized and is being investigated along the lines developed in this paper.

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Asymptotic Decrease of Scattering Amplitudes*

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The recent analysis by Orear of the large-angle p-p scattering data at high energies indicates that the scattering amplitude outside the diffraction peak falls off asymptotically like $\exp[-c(\theta)s^{1/2}]$ for fixed angles, where $c(\theta)$ is some function of the center-of-mass scattering angle θ and s is the square of the center-of-mass energy. We show that if the scattering amplitude is $O(\exp\{-c(\theta)s^{1/2}\})$ as $s\to\infty$ where $c(\theta)>0$ for some fixed θ, then the entire scattering amplitude for this θ is uniquely determined by the function associated with the left-hand cut in the fixed-angle dispersion relation. The spectral function for the right-hand cut is exhibited as the Fourier transform of an analytic function which is defined by a power series in a certain domain of its regularity. The power series involves only quantities which are determined by the left-hand cut. Unitarity is not explicitly used in the proof, so that this asymptotic requirement plus the nature of the left-hand cut have somehow to imply the unitarity condition and hence the mass spectrum in the s channel at all energies.

If the scattering amplitude is $O[\exp\{-c(\theta)s^{1/2}\}]$ where $c(\theta)>0$ for a nonzero interval in θ, then the amplitude is uniquely determined by its left-hand cut for all θ by analytic continuation. In this case, the solutions of the partial-wave dispersion relations are also unique.

I. INTRODUCTION

RECENTLY, the p-p elastic scattering cross section was measured by the Cornell-Brookhaven collaboration group at energies ranging from 10 to 30 BeV and various scattering angles.[1] Orear[2] has analyzed these data and finds that outside the diffraction peak, they can all be fit within two standard deviations by the simple exponential form

$$\frac{d\sigma}{d\Omega}=A\exp[-as^{1/2}\,|\sin\theta|], \tag{I.1}$$

where A and a are constants, s is the square of the center-of-mass energy, and θ is the center-of-mass scattering angle.

According to Cerulus and Martin,[3] if the total cross section behaves like a constant at high energies, and if the usual analyticity assumptions are correct, the scattering amplitude $F(s,z)$ cannot decrease faster than $\exp[-c(\theta)s^{1/2}\ln s]$ as $s\to\infty$ for $\theta\neq 0$ or π:

$$|F(s,z)|\geq A\exp[-c(\theta)s^{1/2}\ln s]\quad\text{as}\quad s\to\infty. \tag{I.2}$$

Here, A is again a suitable constant, $z=\cos\theta$ and $c(\theta)$ is constant times a known function of θ which vanishes for $\theta=0$ or π, and is well approximated by $|\sin\theta|$ for a wide range of θ[4].

As the form (I.1) is not much larger than the form (I.2), Kinoshita[4] has suggested that the scattering amplitude may in fact attain the lower bound in Eq. (I.2) as $s\to\infty$. As it also seems to grow as fast as it can in the forward direction consistent with analyticity and unitarity,[5] this would then be an indication that for some ill-understood reason, the scattering amplitude has many extremal properties at high energies.

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