Study of K_{e4} Decays*

CLAUDE KACSER
Department of Physics and Astronomy, University of Maryland, College Park, Maryland

AND

PAUL SINGER† AND TRAN N. TRUONG
Department of Physics, Columbia University, New York, New York
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It is assumed that the $K \rightarrow \pi+\pi+e+\nu$ process has a Born term dominated by K^* exchange. By assuming a "nearly conserved" axial current we obtain a Goldberger-Treiman-type relation by which the K_{e4} amplitude is related to the K^* width and $K \rightarrow \mu+\nu$ amplitude. This is used to determine the left-hand cut for a set of partial-wave dispersion relations. With the assumption of elastic unitarity for pion-pion scattering, integral equations of the Muskhelishvili-Omnès type are obtained, which are then solved using various assumed forms for the pion-pion $T=0$ interaction. The effect of the ρ resonance is included in the $T=1$ P-wave amplitude. We obtain agreement with the experimental rate for K_{e4}^+ decay when the S-wave pion-pion interaction is described by a scattering-length approximation with a scattering length of $\alpha_0=(1\pm0.3)$ pion Compton wavelengths. With this value of α_0, the two-pion invariant mass distribution is in good agreement with experiment, and the total P-wave contribution to the total rate is predicted to be 18%. If a σ meson ($T=0$, S-wave resonance at 400 MeV) is assumed to dominate the S-wave pion-pion interaction, the calculated rate becomes larger than the experimental one by two orders of magnitude. The possibility of T violation is discussed.

I. INTRODUCTION

A SIGNIFICANT number of events of the type $K\rightarrow\pi\pi e\nu$ has been reported recently,[1] thus allowing a detailed check of the theoretical models proposed for this decay.

In addition to the obvious interest related to the weak interaction responsible for this process, it is also of special interest for strong-interaction physics. The final state containing two pions, which could be in an even or odd angular-momentum state, provides an ideal testing ground for the π-π interaction, being free of any disturbance from other strongly interacting particles. Needless to say, in interpreting different measurable quantities of this process, one has to be careful as to the possibility of separating the strong- and weak-interaction effects.

In this article, we shall calculate the rate and some of the decay spectra for the K_{e4} process with the inclusion of the final strong interaction between the pions. Theoretical treatments of this process have already appeared in the literature. The early ones are concerned with rough estimates for the rates,[2–5] while in subsequent works[6–9] the π-π interaction has been included with various degrees of reliability. Generally, on including the π-π interaction the previous workers were unable to reproduce the experimental rate (usually by about one order of magnitude), unless a certain assumption is made for the weak axial current (Ref. 9), which we find somewhat implausible.

Our work comes close in some details to that of Ref. 7, although there are significant differences which will be mentioned at the appropriate points in this paper.

For the leptonic current we take a V-A form. The strangeness-changing weak current of the decay then has a vector as well as an axial-vector part. Arguments have already been given in the past[5] to show that the vector current gives a very small contribution. For the axial vector part, we employ a technique which was first used by Bég, Cornwall, and Woo[10] in dealing with the 3π decay of the intermediate vector boson. Namely, we *assume* the three-particle ($K\pi\pi$) weak axial current to be dominated by the resonant states $K^*\pi$. Then, we use the concept of a "nearly conserved" axial vector current which we express mathematically by assuming its divergence to obey an unsubtracted dispersion relation. A pole approximation is then made, through which the K_{e4} decay is related to the rate for $K_{\mu2}$ decay. This approach has proved successful in deriving the Goldberger-Treiman relation,[11] though one should re-

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† Present address: Department of Physics, Israel Institute of Technology, Haifa, Israel.

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member that its usefulness for the axial strangeness-changing current has not been yet proved or disproved experimentally. This method has also been used in Refs. 7 and 8 to relate the K_{e4} decay to the $K\bar{K}\pi\pi$-scattering amplitude. By using the $\langle K^*\pi|A_\mu|0\rangle$ current we avoid any direct reference to the unknown $K\bar{K}\pi\pi$ amplitude, and our rate will be given in terms of the K^* width, the $K\to\mu\nu$ decay rate, and the pion-pion interaction parameters.

In order to include the π-π final-state interaction we use the amplitude calculated from the $K^*\pi$ current as a Born term in an *un*subtracted one-dimensional dispersion relation, which we assume to hold for the decay amplitudes. When elastic unitarity is assumed, the integral equation obtained is of the Omnès-Mushkelishvili type. We are able to obtain an approximate solution, which can be expressed in closed form. Different forms for the S- and P-wave shifts are fed into the equations. Once a definite form for the phase shifts is assumed, there is no free parameter and the model gives definite predictions for any measurable quantities.

Before proceeding with the calculations, we would like to touch briefly on a point so far avoided, which the reader may already have noticed. We have assumed the $\langle K\pi\pi|J_\mu{}^A|0\rangle$ axial vector-current matrix element for this decay to be given by the $\langle K^*\pi|J_\mu{}^A|0\rangle$ approximation, without mentioning the other possible two-particle configurations of the $K\pi\pi$ states, i.e., $\langle K\rho|$ and possibly $\langle K\sigma|$. The reason is that we have chosen to include the effects of the ρ (and σ) in the phase shifts, in order to preserve the unity of our treatment of the final-state interactions.

In Sec. II, a general analysis of the K_{e4} amplitude and a decomposition into partial waves is given. In Sec. III we obtain the contribution of the Born term to the various independent terms of the amplitude. In Sec. IV the dispersion relations are solved and in Sec. V we give the numerical results of our model. In Sec. VI the discussion and conclusion are presented. We also briefly discuss the possibility of time-reversal violation in K_{e4}. In Appendix A we amplify on some problems related to the solution of the integral equations while in Appendix B an alternative calculation is given for the case where the 2π channel is dominated by resonances.

II. THEORETICAL PRELIMINARIES

A. General Form of the Matrix Element

We consider the decay

$$K\to\pi+\pi+e+\nu$$

and assume the validity of the $\Delta I=\frac{1}{2}$ rule, and hence of the $\Delta S=+\Delta Q$ rule. Thus we treat specifically

$$K^+\to\pi^0+\pi^0+e^++\nu \tag{2.1a}$$

$$K^+\to\pi^++\pi^-+e^++\nu \tag{2.1b}$$

$$K^0\to\pi^-+\pi^0+e^++\nu\,. \tag{2.1c}$$

Let p, q_1, q_2, e, and ν be the four momentum of the K, the two pions, the positron, and the neutrino, respectively, with

$$p=q_1+q_2+e+\nu\,. \tag{2.2}$$

We define

$$\begin{aligned} k&=(e+\nu)\,,\\ s&=(q_1+q_2)^2=(p-k)^2\,,\\ t_1&=(q_2+k)^2=(p-q_1)^2\,,\\ t_2&=(q_1+k)^2=(p-q_2)^2\,,\end{aligned} \tag{2.3}$$

also

$$\eta=\tfrac{1}{2}(t_1-t_2)=-k(q_1-q_2)=-p(q_1-q_2)\,. \tag{2.4}$$

Then, since

$$s+t_1+t_2=m_K{}^2+2\mu^2+k^2 \tag{2.5}$$

where m_K is the K mass, and μ is the pion mass, we see that a complete set of scalar invariants consists of s, η, and k^2.

We assume the V-A form of interaction for the leptons. Then

$$S=(-i)(2\pi)^4\delta^{(4)}(p-q_1-q_2-k)\times\langle 2\pi e\nu,\,\text{out}|\mathcal{L}(0)|K,\,\text{in}\rangle\frac{1}{[(2\pi)^{3/2}]^5}, \tag{2.6}$$

with

$$\langle 2\pi e\nu,\,\text{out}|\mathcal{L}(0)|K,\,\text{in}\rangle=\langle 2\pi,\,\text{out}|J_\mu|K\rangle\times\bar{u}_\nu\tfrac{1}{2}\gamma^\mu(1-i\gamma_5)v_e(m_e/e_0)^{1/2}\,. \tag{2.7}$$

Here J_μ has both an axial vector ($J_\mu{}^A$) and a proper vector part ($J_\mu{}^V$). On invariance grounds we can write the most general form as

$$\begin{aligned}(2q_1{}^0 2q_2{}^0 2p^0)^{1/2}\langle 2\pi,\,\text{out}|J_\mu{}^A|K\rangle &=A(s,\eta,k^2)(q_1+q_2)_\mu+B(s,\eta,k^2)(q_1-q_2)_\mu\\ &\quad+C(s,\eta,k^2)k_\mu\,,\end{aligned} \tag{2.8}$$

and

$$(2q_1{}^0 2q_2{}^0 2p^0)^{1/2}\langle 2\pi,\,\text{out}|J_\mu{}^V|K\rangle=D(s,\eta,k^2)\epsilon_{\mu\delta\sigma\tau}q_1{}^\delta q_2{}^\sigma k^\tau\,. \tag{2.9}$$

Here A, B, C, and D are scalars in Lorentz space, but contain isotopic indices. From kinematical considerations the contribution of the vector part (2.9) to the K_{e4} decay is expected to be small. A rough calculation using the $\langle K^*\pi|$ approximation (as described later on in detail for the axial part) gives us a contribution of less than a few percent to the rate.[12] Henceforth, we shall concentrate in our work on the axial-vector part (2.8).

B. Isotopic Spin Structure

We use the $\Delta I=\frac{1}{2}$ rule and assume that J_μ carries away $\frac{1}{2}$ unit of isotopic spin in the form of an isotopic spinor spurion. Let α, β be the charge-state indices for

[12] The characteristic radius in our problem is $(1/s_p)=1/(67m_\pi{}^2)$. The vector contribution should be negligible (see Refs. 2–5).

the pions ($\alpha, \beta = 1, 2, 3$). Also let λ and σ be the indices for the K and the spurion, and let χ_λ, χ_σ be two-component isotopic spinors. Then each of the amplitudes A, B, C must have the form

$$\chi_\sigma^+\{A^{(+)}\delta_{\beta\alpha}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}\}\chi_\lambda, \text{ etc.} \tag{2.10}$$

The $(+)$ and $(-)$ amplitudes are simply related to the amplitudes which have a definite value of isotopic spin for the two pions. In fact

$$\delta_{\beta\alpha}=\sqrt{6}\mathcal{I}_{\beta\alpha}{}^{(0)}, \quad \tfrac{1}{2}[\tau_\beta,\tau_\alpha]=2\mathcal{I}_{\beta\alpha}{}^{(1)} \tag{2.11}$$

where $\mathcal{I}^{(0)}$ and $\mathcal{I}^{(1)}$ are projection operators for the two pions in states of total isotopic spin 0 and 1, respectively.[13]

Hence

$$A^{(0)}=\sqrt{6}A^{(+)}, \quad A^{(1)}=2A^{(-)}, \text{ etc.}, \tag{2.12}$$

where $A^{(0)}$ and $A^{(1)}$ are, respectively, the amplitudes for the normalized isotopic spin states 0 and 1.

Under the interchange of the two pions $s\to s$, $\eta\to-\eta$ and $k^2\to k^2$. Hence Bose-Einstein symmetry implies

$$\begin{aligned} A^{(0)}(s,\eta,k^2)&=A^{(0)}(s,-\eta,k^2),\\ B^{(0)}(s,\eta,k^2)&=-B^{(0)}(s,-\eta,k^2),\\ C^{(0)}(s,\eta,k^2)&=C^{(0)}(s,-\eta,k^2),\end{aligned} \tag{2.13}$$

and

$$\begin{aligned} A^{(1)}(s,\eta,k^2)&=-A^{(1)}(s,-\eta,k^2),\\ B^{(1)}(s,\eta,k^2)&=+B^{(1)}(s,-\eta,k^2),\\ C^{(1)}(s,\eta,k^2)&=-C^{(1)}(s,-\eta,k^2).\end{aligned} \tag{2.14}$$

[In (2.13) and (2.14) the symmetry with respect to $\eta\to-\eta$ seems to contradict the isotopic spin symmetry for B. However, recall that we are treating a vector amplitude.]

One finds by standard Clebsch-Gordan techniques that the amplitudes for the processes (2.1a)–(2.1c) are

$$M^{(a)}=\frac{1}{\sqrt{6}}A^{(0)}, \quad M^{(b)}=\frac{1}{2}A^{(1)}-\frac{1}{\sqrt{6}}A^{(0)}, \quad M^{(c)}=\frac{1}{\sqrt{2}}A^{(1)}, \tag{2.15}$$

and therefore the following relation between the rates exists:

$$\omega(b)=\tfrac{1}{2}\omega(c)+2\omega(a). \tag{2.16}$$

C. Partial-Wave Decomposition

We now turn to the construction of eigenstates of angular momentum for the two pions. We treat only the A and B amplitudes, since only these will be needed in the final calculations.

For fixed k^2, we go into the center-of-mass system of the two pions, in which

$$\begin{aligned} q^2&=\tfrac{1}{4}(s-4\mu^2),\\ p^2&=[s-(m_K-\sqrt{k^2})^2][s-(m_K+\sqrt{k^2})^2]/(4s),\end{aligned} \tag{2.17}$$

where p and q are the magnitudes of the 3 momenta of the K and of the pions. Further expressions which will be used throughout this article are:

$$\eta=\tfrac{1}{2}(t_1-t_2)=-p(q_1-q_2)=+2pq\cos\theta, \tag{2.18a}$$

$$t_1=\tfrac{1}{2}(m_K{}^2+k^2+2\mu^2)-\tfrac{1}{2}s+2pq\cos\theta, \tag{2.18b}$$

$$t_2=\tfrac{1}{2}(m_K{}^2+k^2+2\mu^2)-\tfrac{1}{2}s-2pq\cos\theta, \tag{2.18c}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}_1$. Also

$$(q_1+q_2)_\mu=2\omega\delta_{\mu,0},$$

and

$$(q_1-q_2)_\mu=2q\hat{\mathbf{q}}, \tag{2.19}$$

where ω is the pion energy ($\omega^2=q^2+\mu^2$), and $\hat{\mathbf{q}}$ is a unit three-vector along $\mathbf{q}_1$. Hence $\langle q_1q_2;I|J_\mu{}^A|K\rangle$ splits up into "independent" time and space components, i.e.,

$$(8q_1{}^0q_2{}^0p^0)^{1/2}\langle q_1q_2,I|J_0{}^A|K\rangle=A^{(I)}(s,\eta,k^2)2\omega, \tag{2.20}$$

and

$$(8q_1{}^0q_2{}^0p^0)^{1/2}\langle q_1q_2;I|\mathbf{J}^A|K\rangle=B^{(I)}(s,\eta,k^2)2q\hat{}. \tag{2.21}$$

Here $\langle q_1q_2;I|$ refers to a state of the two pions with definite isotopic spin I.

From (2.18) and (2.20) it is clear that $A^{(I)}$ can be expanded in a complete set of partial waves. Further from (2.13) and (2.14) $A^{(0)}$ contains only even l, and $A^{(1)}$ only odd l. Thus

$$A^{(I)}(s,\eta,k^2)=\sum_{n=0}^{\infty}A_{2n+I}{}^{(I)}(s,k^2)P_{2n+I}(\cos\theta), \tag{2.22}$$

and $A_{2n+I}{}^{(I)}(s,k^2)$ will have the phase of the partial-wave pion-pion scattering amplitude for isotopic spin I, $l=2n+I$ (at least in the elastic region $4\mu^2<s<16\mu^2$).

It is clear from (2.13) and (2.14) that an expansion in Legendre polynomials also exists for $B^{(I)}$, that is

$$B^{(I)}(s,\eta,k^2)=\sum_{n=0}B_{2n+1-I}{}^{(I)}(s,k^2)P_{2n+1-I}(\cos\theta). \tag{2.23}$$

However, the B_l do *not* correspond to eigenamplitudes of angular momentum. This is due to the presence of $\hat{\mathbf{q}}(\theta,\phi)$ in (2.21). In fact

$$\hat{\mathbf{q}}=(4\pi/3)^{1/2}\textstyle\sum_m Y_{1,m}(\theta,\phi)\boldsymbol{\chi}_m{}^*, \tag{2.24}$$

where $\boldsymbol{\chi}_m$ are a set of unit vectors[14]

$$\boldsymbol{\chi}_m=\left(-\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}}),\ \hat{\mathbf{z}},\ +\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}})\right). \tag{2.25}$$

[13] See, e.g., G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957); W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

[14] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Appendix B.

Further,

$$P_l(\cos\theta)(4\pi/3)^{1/2}Y_{1,m}(\theta,\phi)=\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,0}(\theta,\phi)(4\pi/3)^{1/2}Y_{1,m}(\theta,\phi)$$
$$=\left(\frac{4\pi}{2l-1}\right)^{1/2}Y_{l-1,m}(\theta,\phi)\langle l1;0m|l1;l-1,m\rangle\langle l1;0,0|l1;l-1,0\rangle$$
$$+\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l+1,m}(\theta,\phi)\langle l1;0m|l1;l+1,m\rangle\langle l1;00|l1;l+1,0\rangle,\quad(2.26)$$

where $\langle l_1l_2;m_1m_2|l_1l_2;LM\rangle$ is the usual Clebsch-Gordan coefficient. [Notice that $L=l$ does not enter into (2.26).]

Collecting terms from (2.21), (2.23), (2.24) and (2.26) we are led to

$$(8q_1^0q_2^0p^0)^{1/2}\langle q_1q_2;I|\mathbf{J}^A|K\rangle=2q\sum_m\chi_m{}^*\sum_L\left(\frac{4\pi}{2L+1}\right)^{1/2}Y_{L,m}(\theta,\phi)\{B_{L-1}{}^{(I)}(s,k^2)\langle L-1,1;0,m|L-1,1;L,m\rangle$$
$$\times\langle L-1,1;00|L-1,1;L,0\rangle+B_{L+1}{}^{(I)}(s,k^2)\langle L+1,1;0,m|L+1,1;L,m\rangle\langle L+1,1;00|L+1,1;L,0\rangle\}.\quad(2.27)$$

In (2.27) $I=0$ contains only even L and $I=1$ contains only odd L. Further $B_{-1}\equiv0$. It is clear that the quantity inside the curly brackets in (2.27) corresponds to the eigenamplitude for the Lth partial wave, having the appropriate pion-pion scattering phase. In particular, the $L=0$ and $L=1$ amplitudes are related to

$$L=0,\quad I=0:\ B_1{}^{(0)}(s,k^2)\quad(2.28)$$

$$L=1,\quad I=1:\ B_0{}^{(1)}(s,k^2)\langle01;0m|01;1m\rangle\langle01;00|01;10\rangle+B_2{}^1(s,k^2)\langle21;0m|21;1m\rangle\langle21;00|21;10\rangle.\quad(2.29)$$

We remark that $A_l{}^{(I)}$ will contain the standard kinematic singularity factor $(pq)^l$, as does $B_l{}^{(I)}$.

III. BORN TERMS AND DISPERSION RELATIONS

We turn now to the calculation of the $\langle2\pi|J_\mu{}^A|K\rangle$ amplitude. It is convenient for our purpose to consider the process $K+(e\nu)\to\pi+\pi$, which is the analytic continuation in the k^2 variable of the decay under study. We make the assumption that the partial-wave amplitudes $A_m{}^{(I)}$ and $B_m{}^{(I)}$ which describe this process obey an unsubtracted dispersion relation in the s variable.[15] The discontinuity across the right-hand cut, which is given by the unitarity condition and is related to the pion-pion scattering amplitude, will be treated in the next section. We consider here the contribution to the left-hand cut, which will constitute the inhomogeneous term in the integral equation.

The left-hand cut contribution can be obtained from the appropriate partial wave-projections of the amplitude for $K+\pi\to\pi+(e\nu)$. The lowest intermediate states for this process are $K\pi$ states. We shall assume that they can be approximated by the K^* resonance and we treat the K^* as a vector particle.[16] (We do not include the contribution of the κ (725-MeV $K\pi$ resonance) whose existence is uncertain. But see also note added in proof.) Henceforth, our problem is reduced to calculating the $\langle K^*\pi|J_\mu{}^A|0\rangle$ current.

The strangeness nonchanging component of the axial vector current has been considered by Bég, Cornwall, and Woo.[10] They assume that the divergence of the current vanishes as $k^2\to\infty$, and so write an unsubtracted dispersion relation for this divergence. For small k^2 the single-particle contribution dominates, i.e., the pion or the kaon, respectively. If one sets $k^2=0$ in the resulting identity, one can obtain the full structure of the form factor at $k^2=0$. Unfortunately, the method does not allow one to keep k^2 small but finite. The method of the "partially conserved current" is admittedly somewhat *ad hoc.* However, it has had startling success in the Goldberger-Treiman relation, and we hope that it also works in our case.

We turn now to the details. For general four-momenta K and q for the K^* and pion, with $k=K+q$ (the K^* having isotopic spin index κ and the pion δ) we write

[15] The various analytic continuations and dispersion relations which we assume to hold are on a safer footing than in the somewhat analogous problem of $K\to3\pi$. This is because we keep only the first order in weak interaction and hence all Feynman graphs will contain a continuous K line which can be traced through the whole graph until the final weak vertex. Thus even higher order graphs should be free of overlapping cuts, complex singularities, etc.

[16] In principle, one would disperse over $K\pi$ intermediate states. The major contribution would then arise from the vicinity of the K^*, and so only "mass shell" form factors for the K^* will enter. This argument leads to the same conclusion as reached by R. H. Capps, Phys. Rev. **131**, 1307 (1963). See also G. T. Hoff, Phys. Rev. **131**, 1302 (1963).

the most general form

$$(2K_0 2q_0)^{1/2}\langle K^*\pi|J_\mu{}^A|0\rangle=\{(\epsilon\cdot q)(K-q)_\mu\alpha(k^2)+\epsilon_\mu\beta(k^2)+(\epsilon\cdot q)(K+q)_\mu\gamma(k^2)\}\chi_\kappa{}^+\tau_\delta\chi_\sigma, \quad (3.1)$$

where χ_σ is again the spurion spinor and ϵ is the polarization vector of the K^*, with $\epsilon^2=-1$, $\epsilon\cdot K=0$ and hence $\epsilon\cdot q=\epsilon\cdot k$. The K-meson pole contributes directly only to γ, but will be used to "induce" α and β.

We use the standard (on the mass shell) $K^*K\pi$ coupling form

$$(2K_0 2q_0)^{1/2}\langle K^*\pi|J_K|0\rangle=g\epsilon\cdot(k+q)\,\chi_K{}^+\tau_\delta\chi_\sigma, \quad (3.2)$$

where k and σ now refer to the K meson. With this definition the total width $(K^{*+}\to K^0\pi^++K^+\pi^0)$ is given by

$$\Gamma_{K^{*+}}=\frac{g^2}{4\pi}\frac{\{[M^{*2}-(m_K-\mu)^2][M^{*2}-(m_K+\mu)^2]\}^{3/2}}{4M^{*5}}=(g^2/4\pi)\times 59\text{ MeV}, \quad (3.3)$$

where M^* is the K^* mass (890 MeV). For a total decay width of 51 MeV one obtains $(g^2/4\pi)=0.86$.

We also define the axial vector current responsible for K-meson decay as

$$(2k_0)^{1/2}\langle K|J_\mu{}^A|0\rangle=Gk_\mu, \quad (3.4)$$

in terms of which the $K^+\to\mu^+\nu$ rate is given by

$$\Gamma_{K\to\mu\nu}=\frac{G^2}{4\pi}m_K\mu^2[1-(\mu^2/m_K{}^2)]^2,$$

where

$$G^2m_K{}^2=2.002\times10^{-14}. \quad (3.5)$$

Returning to (3.1) we write an unsubtracted dispersion relation in k^2 for $i\partial_\mu J_\mu{}^A$, and keep only the K-meson pole contribution. One readily finds

$$(M^{*2}-\mu^2)\alpha(k^2)+\beta(k^2)+k^2\gamma(k^2)\equiv 2gG[m_K{}^2/(m_K{}^2-k^2)]. \quad (3.6)$$

We set $k^2=0$ to obtain

$$(M^{*2}-\mu^2)\alpha(0)+\beta(0)=2gG.$$

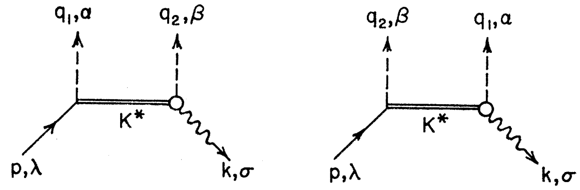

FIG. 1. Graphs contributing to the Born terms.

Further, following Bég, Cornwall, and Woo,[10] it is possible to argue that α is much smaller than β. Hence finally we have

$$\alpha(0)\simeq0, \quad \beta(0)\simeq2gG. \quad (3.7)$$

If we had written down dispersion relations directly for α, β and γ, the K-meson pole would have contributed only to γ. The lowest mass states which contribute to α and β are $K\pi\pi$. The partially conserved current technique is needed to determine the subtraction constant for α and β, given by (3.7) if we subtract at $k^2=0$. [In fact, the argument leading to $\alpha(0)\simeq0$ corresponds to showing that the α dispersion relation does not need a subtraction.] It follows that $\alpha(k^2)$ and $\beta(k^2)$ will not change greatly from their values at $k^2=0$ until k^2 reaches at least $m_K{}^2$.

In our application $k^2\leq(m_K-2\mu)^2$, hence (3.7) should be a good approximation. The third form factor γ will depend on k^2; however, it contributes only to C (Eq. 2.8), and C gives no contribution to K_{e4} decay in the limit $m_e\to0$ (see Sec. V). Hence, we will not need the actual form of γ, nor need we keep C any further in our calculations. (For $K_{\mu4}$ decay the C term, and hence γ, must be kept. However the variation of γ with k^2 will be large and hard to estimate, since the K-pole term is only one of the contributions. Thus $K_{\mu4}$ decay is a more difficult problem.)

The Born terms correspond to the two diagrams given in Fig. 1. Using the $K^*K\pi$ coupling given in (3.2) and $K^*\pi$ axial vector current from (3.1) but keeping only the β form factor we readily obtain

$$\begin{aligned}(2p_0\,2q_{10}\,2q_{20})^{1/2}&\langle q_1\alpha;q_2\beta\text{ out}|J_\mu{}^A|p,\lambda\rangle_{\text{Born}}\\
&=2g^2G\left\{(p+q_1)_\lambda\frac{[g_{\nu\mu}-(1/M^{*2})(p-q_1)_\nu(p-q_1)_\mu]}{(p-q_1)^2-M^{*2}}\chi_\sigma{}^+\tau_\beta\tau_\alpha\chi_\lambda+(p+q_2)_\lambda\frac{[g_{\nu\mu}-(1/M^{*2})(p-q_2)_\nu(p-q_2)_\mu]}{(p-q_2)^2-M^{*2}}\chi_\sigma{}^+\tau_\alpha\tau_\beta\chi_\lambda\right\}\\
&=2g^2G\left\{\chi_\sigma{}^+(\sqrt{6}\mathcal{G}_{\beta\alpha}{}^{(0)})\chi_\lambda\left[d^+\left(\frac{1}{t_1-M^{*2}}+\frac{1}{t_2-M^{*2}}\right)(q_1+q_2)_\mu+d^-\left(\frac{1}{t_1-M^{*2}}-\frac{1}{t_2-M^{*2}}\right)(q_1-q_2)_\mu\right]\right.\\
&\quad\left.+\chi_\sigma{}^+(2\mathcal{G}_{\beta\alpha}{}^{(1)})\chi_\lambda\left[d^+\left(\frac{1}{t_1-M^{*2}}-\frac{1}{t_2-M^{*2}}\right)(q_1+q_2)_\mu+d^-\left(\frac{1}{t_1-M^{*2}}+\frac{1}{t_2-M^{*2}}\right)(q_1-q_2)_\mu\right]\right\} \quad (3.8)\end{aligned}$$

where we have used (2.11), disregarded any k_μ part, and where

$$
\begin{aligned}
d^{+}&=\frac{3}{2}-\frac{1}{2}\frac{m_K{}^2-\mu^2}{M^{*2}}=1.36\,,\\
d^{-}&=\frac{1}{2}+\frac{1}{2}\frac{m_K{}^2-\mu^2}{M^{*2}}=0.64\,.
\end{aligned}
\tag{3.9}
$$

The width of the K^* is irrelevant in (3.8) and (3.9) *away* from the left-hand cut, once it has been incorporated into the coupling constant, g. The form (3.8) enables us to immediately identify $\tilde{A}^{(0)}$, $\tilde{A}^{(1)}$, $\tilde{B}^{(0)}$ and $\tilde{B}^{(1)}$, where the tilde labels the Born contribution. We find

$$
\begin{aligned}
\tilde{A}^{(0)}&=(2\sqrt{6})g^2Gd^{+}\left(\frac{1}{t_1-M^{*2}}+\frac{1}{t_2-M^{*2}}\right)\\
\tilde{A}^{(1)}&=4g^2Gd^{+}\left(\frac{1}{t_1-M^{*2}}-\frac{1}{t_2-M^{*2}}\right)
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\tilde{B}^{(0)}&=(2\sqrt{6})g^2Gd^{-}\left(\frac{1}{t_1-M^{*2}}-\frac{1}{t_2-M^{*2}}\right),\\
\tilde{B}^{(1)}&=4g^2Gd^{-}\left(\frac{1}{t_1-M^{*2}}+\frac{1}{t_2-M^{*2}}\right).
\end{aligned}
\tag{3.11}
$$

(3.10) and (3.11) show explicitly the symmetry properties (2.13) and (2.14).

We are interested in the partial-wave expansion of the $\tilde{A}$'s and $\tilde{B}$'s. To this end we use the standard definition of the Legendre polynomial of the second kind to express our Born partial-wave amplitudes:

$$
Q_l(x)=\frac{1}{2}\int_{-1}^{1}\frac{1}{x-z}P_l(z)dz\,,
\tag{3.12}
$$

and we also make use of the relation

$$
Q_l(-x)=(-1)^{l+1}Q_l(x)\,.
$$

Then making use of (2.18) we obtain

$$
\begin{aligned}
\tilde{A}^{(0)}(s,\cos\theta)&=\frac{(2\sqrt{6})g^2Gd^{+}}{pq}\sum_{l=\text{even}}(2l+1)Q_l(x(s))P_l(\cos\theta)\,,\\
\tilde{A}^{(1)}(s,\cos\theta)&=\frac{-4g^2Gd^{+}}{pq}\sum_{l=\text{odd}}(2l+1)Q_l(x(s))P_l(\cos\theta)\,,
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\tilde{B}^{(0)}(s,\cos\theta)&=\frac{-(2\sqrt{6})g^2Gd^{-}}{pq}\sum_{l=\text{odd}}(2l+1)Q_l(x(s))P_l(\cos\theta)\,,\\
\tilde{B}^{(1)}(s,\cos\theta)&=\frac{4g^2Gd^{-}}{pq}\sum_{l=\text{even}}(2l+1)Q_l(x(s))P_l(\cos\theta)\,,
\end{aligned}
\tag{3.14}
$$

with

$$
x(s)=(\bar{s}-s-2M^{*2})/4pq\,,
\tag{3.15}
$$

and

$$
\bar{s}=m_K{}^2+k^2+2\mu^2.
\tag{3.16}
$$

We notice that in the physical decay region of K_{e4} one has

$$
\frac{Q_2}{Q_0}\simeq\frac{1}{3}\frac{(4pq)^2}{(\bar{s}-s-2M^{*2})^2}\ll 1\,,
$$

and similarly

$$
\frac{Q_3}{Q_1}\ll 1\,.
$$

This is so because $2M^{*2}$ is much larger than s so that we can take only the leading terms in the expansion. This can always be checked *a posteriori* and is certainly valid here, as our calculation shows. Hence, using the definitions (2.22) and (2.23)

$$
\tilde{A}_0{}^{(0)}(s)\simeq\frac{(2\sqrt{6})g^2Gd^{+}}{pq}Q_0(x(s))\,,
\tag{3.17}
$$

$$
\tilde{A}_1{}^{(1)}(s)\simeq\frac{-12g^2Gd^{+}}{pq}Q_1(x(s))\,,
\tag{3.18}
$$

$$
\tilde{B}_1{}^{(0)}(s)\simeq\frac{-(6\sqrt{6})g^2Gd^{-}}{pq}Q_1(x(s))\,,
\tag{3.19}
$$

$$
\tilde{B}_0{}^{(1)}(s)\simeq\frac{4g^2Gd^{-}}{pq}Q_0(x(s))\,.
\tag{3.20}
$$

The full partial-wave amplitudes will have the same kinematic singularities as those of the Born amplitudes (3.17)–(3.20). We define new amplitudes from which the kinematic singularities have been removed:

$$\begin{aligned} &\mathcal{A}_0^{(0)}=A_0^{(0)}; && \mathcal{A}_1^{(1)}=A_1^{(1)}/2pq; \\ &\mathcal{B}_1^{(0)}=B_1^{(0)}/2pq; && \mathcal{B}_0^{(1)}=B_0^{(1)}. \end{aligned} \tag{3.21}$$

Recall that the upper index refers to the isotopic spin, the lower to the Legendre expansion of the scalar amplitudes A and B.

From examination of the expansions (2.22), (2.27), and (2.28) one sees that $\mathcal{A}_0^{(0)}$ and $\mathcal{B}_1^{(0)}$ have the $I=0$ S-wave phase of the π-π scattering, while $\mathcal{A}_1^{(1)}$ and $\mathcal{B}_0^{(1)}$ correspond to the P-wave phase.

We finally write unsubtracted dispersion relations for each of these amplitudes, with the appropriate $e^{-i\delta}\sin\delta$ factor, and with the appropriate Born term. Thus, generally we have

$$\begin{aligned} \mathcal{A}_0^{(0)}(s)&=\tilde{\mathcal{A}}_0^{(0)}(s)+\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-s-i\epsilon}\mathcal{A}_0^{(0)}(s')e^{-i\delta_0(s')}\sin\delta_0(s'), \\ \mathcal{A}_1^{(1)}&=\tilde{\mathcal{A}}_1^{(1)}(s)+\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-s-i\epsilon}\mathcal{A}_1^{(1)}(s')e^{-i\delta_1(s')}\sin\delta_1(s'), \\ \mathcal{B}_1^{(0)}(s)&=\tilde{\mathcal{B}}_1^{(0)}(s)+\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-s-i\epsilon}\mathcal{B}_1^{(0)}(s')e^{-i\delta_0(s')}\sin\delta_0(s'), \\ \mathcal{B}_0^{(1)}(s)&=\tilde{\mathcal{B}}_0^{(1)}(s)+\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-s-i\epsilon}\mathcal{B}_0^{(1)}(s')e^{-i\delta_1(s')}\sin\delta_1(s'). \end{aligned} \tag{3.22}$$

The $\mathcal{A}$ and $\mathcal{B}$ are also functions of k^2, which is to be kept fixed in the dispersion integrals. The Born terms $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are likewise functions of k^2 (compare 3.16), and we have dropped the dependance on k^2 only in the form factors α and β.

At this stage we simplify the notation by writing

$$\begin{aligned} &\mathcal{A}_0^{(0)}=a_0, && \mathcal{A}_1^{(1)}=a_1, \\ &\mathcal{B}_1^{(0)}=b_0, && \mathcal{B}_0^{(1)}=b_1, \end{aligned} \tag{3.23}$$

where the single subscript now refers to I, and is a complete label. Then, our current has the form:

$I=0$:

$$\begin{aligned} &A^{(0)}(s,\eta,k^2)(q_1+q_2)_\mu+B^{(0)}(s,\eta,k^2)(q_1-q_2)_\mu \\ &\qquad=a_0(s,k^2)(q_1+q_2)_\mu+b_0(s,k^2)\eta(q_1-q_2)_\mu, \end{aligned} \tag{3.24}$$

$I=1$:

$$\begin{aligned} &A^{(1)}(s,\eta,k^2)(q_1+q_2)_\mu+B^{(1)}(s,\eta,k^2)(q_1-q_2)_\mu \\ &\qquad=a_1(s,k^2)\eta(q_1+q_2)_\mu+b_1(s,k^2)(q_1-q_2)_\mu. \end{aligned} \tag{3.25}$$

IV. SOLUTIONS OF THE INTEGRAL EQUATIONS

In the last section, we have obtained various integral equations for the a's and b's. They are of the Muskhelishvili-Omnès form.[17] We exemplify our treatment by dealing in detail with the following integral equation:

$$G(s)=CQ_0(x(s))+\frac{1}{\pi}\int_4^\infty ds'\frac{e^{-i\delta_l}\sin\delta_l(s')G(s')}{s'-s-i\epsilon}, \tag{4.1}$$

where C is a constant. (The other type of integral equation will have a Q_1 as the inhomogeneous term and can be treated similarly.) Since $Q_0(x(s))$ has only the left-hand cut and is separated from the right-hand cut it is possible to write down the solution either as the product of two cuts or as the sum of the two cuts. We choose the latter form. Provided the integral converges the solution of equation (4.1) is[17]

$$\begin{aligned} G(s)=CQ_0(x(s))&+\frac{Ce^{u_l(s)-u_l(4)}}{\pi}\int_4^\infty\frac{ds'}{s'-s-i\epsilon} \\ &\times[e^{-u_l(s')+u_l(4)}e^{i\delta_l}\sin\delta_lQ_0(x(s'))]+P_n(s)e^{u_l(s)}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &\exp(u_l(s)-u_l(4)) \\ &\quad=\exp\left(\frac{s-4}{\pi}\int_4^\infty ds'\frac{\delta_l(s')}{(s'-s-i\epsilon)(s'-4)}\right)\equiv\frac{1}{D(s)}. \end{aligned} \tag{4.3}$$

Here D is the denominator function in the N/D method for the partial-wave amplitude for pion-pion scattering, and $P_n(s)$ is a polynomial of order n. We need $G(s)$ only for $4\mu^2<s<(m_K-\mu)^2$ and hence evaluate (4.2) using approximations which are valid for this range. Notice that for these physical values of s, $x(s)$ as given by Eq. (3.16) is much larger than unity. Hence, for these values of s we shall expand $Q_0(x(s))$ as follows:

$$Q_0(x(s))=\frac{1}{x(s)}\left[1+\frac{1}{3}\frac{1}{[x(s)]^2}+\cdots\right],$$

and we also use this expansion inside the integral in (4.2), since the high-energy part of the integral is well

[17] R. Omnès, Nuovo Cimento 8, 316 (1958). N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

damped. (See discussion in Appendix A.) It is seen that this series converges very fast in the physical region. The expansion remains valid for larger (unphysical) values of s up to perhaps $s\sim\frac{1}{2}M^{*2}$. As $s\to\infty$, $x(s)$ approaches unity so that the expansion breaks down. However, this does not affect the validity of the evaluated integral. The accuracy of the approximation can be checked by computing higher order terms in the integral and, as will be shown in the Appendix A, they are indeed small. In the following we approximate $Q_0(x(s))$ and $Q_1(x(s))$ by their first leading terms.

It is well known that the solution of the Muskhelishvili-Omnès equation is defined only to within a term $P_n(s)e^{u(s)}$. In order to make the solution unique we make the following ansatz: Among the many solutions as given by Eq. (4.2) we select the solution which yields the best behavior for $G(s)$. By "best behavior" we mean those solutions which make $G(s)$ tend to zero fastest as $s\to\infty$. This eliminates the possibility of adding an arbitrary term such as $P_n(s)e^{u(s)}$.

Typically we now have to calculate expressions like:

$$^{(n)}F_l(s)=\frac{C}{(s+s_p)^n}+\frac{Ce^{u_l(s)-u_l(4)}}{\pi}\times\int_4^\infty ds'\frac{e^{i\delta_l}\sin\delta_l(s')e^{-u_l(s')+u_l(4)}}{(s'-s-i\epsilon)(s'+s_p)^n},\quad(4.4)$$

where $s_p\gg1$.

This expression for $^{(n)}F_l(s)$ can be put in a more convenient form by considering the integral equation which is satisfied by $^{(1)}F_l(s)$:

$$^{(1)}F_l(s,s_p)=\frac{C}{s+s_p}+\frac{1}{\pi}\int_4^\infty ds'\frac{e^{i\delta_l}\sin\delta_l(s')\,^{(1)}F_l(s',s_p)ds'}{s'-s}.\quad(4.5)$$

The solution of Eq. (4.5) is, of course, simply

$$^{(1)}F_l(s,s_p)=\frac{C}{s+s_p}+\frac{Ce^{u_l(s)-u_l(4)}}{\pi}\times\int_4^\infty ds'\frac{e^{i\delta_l}\sin\delta_l(s')e^{-u_l(s')+u_l(4)}}{(s'-s)(s'+s_p)},\quad(4.6)$$

which can also be put in the form

$$^{(1)}F_l(s,s_p)=\frac{Ce^{u_l(s)-u_l(-s_p)}}{s+s_p}.\quad(4.7)$$

This clearly has the correct cut, discontinuity, pole and residue, and satisfies our ansatz for uniqueness of solution. Differentiating $^{(1)}F_l(s,s_p)$ n times with respect to s_p one obtains

$$\frac{\partial^n}{\partial s_p{}^n}\,^{(1)}F_l(s,s_p)=(-1)^n n!\left[\frac{C}{(s+s_p)^{n+1}}+C\frac{e^{u_l(s)-u_l(4)}}{\pi}\int_4^\infty\frac{e^{i\delta_l}\sin\delta_l(s')e^{-u_l(s')+u_l(4)}}{(s'-s)(s'+s_p)^{n+1}}\right].$$

Therefore,

$$^{(n)}F_l(s)=\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}(^{(1)}F_l)}{\partial s_p{}^{n-1}}=C\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial s_p{}^{n-1}}\left[\frac{e^{u_l(s)-u_l(-s_p)}}{s+s_p}\right].\quad(4.8)$$

Using the formula (4.8), we obtain the following expressions for the a's and b's:

$$a_0=-8\sqrt{6}g^2Gd^+\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p},\quad(4.9a)$$

$$b_0=16\sqrt{6}g^2Gd^-\frac{\partial}{\partial s_p}\left[\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p}\right],\quad(4.9b)$$

$$a_1=32g^2Gd^+\frac{\partial}{\partial s_p}\left[\frac{e^{u_1(s)-u_1(-s_p)}}{s+s_p}\right],\quad(4.9c)$$

$$b_1=-16g^2Gd^-\frac{e^{u_1(s)-u_1(-s_p)}}{s+s_p},\quad(4.9d)$$

where $s_p=2M^{*2}-\bar{s}$. We note that b_0 and a_1 have the correct second-order pole and residue at $-s_p$ and no first-order pole here.

We shall try to characterize the S and P wave pion-pion phase shifts in a simple manner so that the e^u factor can be obtained without using numerical methods. We shall discuss first the P-wave pion-pion interaction because its behavior is well established experimentally. Since the ρ resonance is far from the physical K_{e4} region, the enhancement factor given by Frazer and Fulco[13] reduces to the simple form

$$e^{u_1(s)-u_1(-s_p)}=(s_\rho+s_p)/(s_\rho-s).\quad(4.10)$$

This of course corresponds to the Breit-Wigner form far from the resonance position, where the imaginary part is negligible.

The situation for the S-wave pion pion interaction is much more complicated because the present experimental evidence is conflicting. We shall consider the following possibilities for the S-wave pion-pion phase shift:

A. Scattering-Length Approximation

We use the form derived by Chew and Mandelstam[18] where the left-hand cut contribution to the S-partial

[18] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

wave pion-pion amplitude is neglected. One can use a better form for the S-wave pion pion phase shift where the left-hand cut ρ contribution is taken into account, but for the present purpose it is unnecessary. We approximate the phase shift by

$$[(s-4)/s]^{1/2}\cot\delta_0=\frac{1}{\alpha_0}+H(\tfrac{1}{4}s-1)\,, \tag{4.11}$$

where α_0 is the scattering length and H_0 is defined as

$$H(x)=\frac{2}{\pi}[x/(x+1)]^{1/2}\ln(\sqrt{x}+(x+1)^{1/2})\,. \tag{4.12}$$

The enhancement factor is given by

$$e^{u_0(s)-u_0(-s_p)}=\frac{(1/\alpha_0)+H(-\frac{1}{4}s_p+1)}{(1/\alpha_0)+H(\frac{1}{4}s-1)-i[(s-4)/s]^{1/2}\theta(s-4)}\,. \tag{4.13}$$

This form is obtained by considering the N/D method for the S-wave dispersion relation, and $e^{u_0(s)-u_0(-s_p)}$ is simply $D(-s_p)/D(s)$.

B. Resonance Behavior

We approximate the S-wave pion-pion phase shift by

$$[(s-4)/s]^{1/2}\cot\delta_0(s)=\frac{1}{4\gamma}(s_R-s)+H\left(\frac{s}{4}-1\right). \tag{4.14}$$

This gives the following enhancement factor

$$e^{u(s)-u(-s_p)}=\frac{s_R+s_p+4\gamma H[-(s_p/4)+1]}{s_R-s+4\gamma H\left(\frac{s}{4}-1\right)-i4\gamma[(s-4)/s]^{1/2}\theta(s-4)}\,, \tag{4.15}$$

where γ is the reduced width, and in terms of the full width Γ it takes the following value:

$$4\gamma=\bar{s}_R^{1/2}(\bar{s}_R/\bar{s}_R-4)^{1/2}\Gamma\,. \tag{4.16a}$$

Here $\bar{s}_R$ is the resonance position and is related to s_R by

$$\bar{s}_R=s_R+4\gamma H((\bar{s}_R/4)-1)\,. \tag{4.16b}$$

Equations (4.14) and (4.15) are derived by making an extra subtraction in the D equation.[19]

Using Eqs. (4.10), (4.13), and (4.15) and the fact that

$$H(-x)=\frac{x}{x-1}H(x-1)\quad\text{for}\quad x>1\,,$$

after some algebra we obtain the following expressions for the a's and b's:

$$a_1=-\frac{32g^2Gd^+}{(s+s_p)^2}\,, \tag{4.17}$$

$$b_1=-\frac{16g^2Gd^-(s_\rho+s_p)}{(s+s_p)(s_\rho-s)}\,, \tag{4.18}$$

(*i*) *Scattering Length*

$$a_0=-\frac{(8\sqrt{6})g^2Gd^+}{s+s_p}\times\frac{\frac{1}{\alpha_0}+H\left(\frac{s_p}{4}\right)}{\frac{1}{\alpha_0}+H\left(\frac{s}{4}-1\right)-i[(s-4)/s]^{1/2}\theta(s-4)}\,, \tag{4.19a}$$

$$b_0=-\frac{16\sqrt{6}g^2Gd^-}{(s+s_p)^2}\times\frac{\frac{1}{\alpha_0}+H\left(\frac{s_p}{4}\right)-\frac{1}{\pi}}{\frac{1}{\alpha_0}+H\left(\frac{s}{4}-1\right)-i[(s-4)/s]^{1/2}\theta(s-4)}\,, \tag{4.19b}$$

(*ii*) *Resonance*

$$a_0=-\frac{8\sqrt{6}g^2Gd^+}{s+s_p}\times\frac{s_R+s_p+4\gamma H\left(\frac{s_p}{4}\right)}{s_R-s+4\gamma H\left(\frac{s}{4}-1\right)-i4\gamma[(s-4)/s]^{1/2}\theta(s-4)}\,, \tag{4.20a}$$

$$b_0=-\frac{16\sqrt{6}g^2Gd^-}{(s+s_p)^2}\times\frac{s_R-s+4\gamma\left[H\left(\frac{s_p}{4}\right)-\frac{1}{\pi}\right]}{s_R-s+4\gamma H\left(\frac{s}{4}-1\right)-i4\gamma[(s-4)/s]^{1/2}\theta(s-4)}\,, \tag{4.20b}$$

where we have used the fact that $s_p\gg s$ in the physical decay region of K_{e4} to simplify our final expressions.

On examining Eqs. (4.9), (4.10), (4.19), and (4.20) we find $a_0\gg b_0$ and $b_1\gg a_1$. The enhancement factor for a_0 at $s=4$ is simply $1+\alpha_0H(s_p/4)\simeq1+1.3\alpha_0$ for the scattering length formula, and is equal to $1+(s_p/s_R)+(4\gamma/s_R)H(s_p/4)$ for the resonance. It is simple to see that by switching off the pion-pion interaction the enhancement factor is unity as it should be. For the resonant pion-pion interaction we expect a very large enhancement because s_p is much larger than s_R, and as our calculation in the next section shows it is not

[19] A. V. Efremov and D. V. Shirkov, Zh. Eskperim. i Teor. Fiz. 42, 1344 (1962) [English transl.: Soviet Phys.—JETP **15**, 932 (1962)] and references given in this paper for previous works.

possible to obtain the correct rate for K_{e4} in this situation. If α_0 is negative corresponding to a repulsive pion-pion interaction, it is expected that the enhancement factor should be less than unity. This is certainly the case for our formula; however, our formula is no longer correct because the demoninator function has a zero on the left-hand cut corresponding to the existence of a "ghost" in the solution of the pion-pion equation. We are unable to obtain a simple equation for the D function for a repulsive S-wave pion-pion interaction which is free from the ghost. Should the situation require, the formula obtained here is simple but general enough to use with a better D function for pion-pion scattering.

One question which may be asked concerns the reliability of the scattering length formula [Eq. (4.11)] and the resonance formula [Eq. (4.14)] which are used to calculate the ratio $D(-s_p)/D(s)$ in order to obtain the absolute enhancement factor. It is recalled that s_p is large; therefore we have to extrapolate the D function in the pion-pion equations to a distant point on the left-hand cut. In order to see the reliability of our procedure we have also used the D function for the S-wave pion-pion interaction where the effect of the ρ exchange in the crossed channel is taken into account in an approximate manner by using the Shirkov crossing relation. Taking the ρ width as 100 MeV, we find that the enhancement factor obtained with this more complicated formula is about 10% larger than with Eq. (4.13). We have also constructed the S-wave D-functions by the pole approximation such that they reproduced the set of S-wave pion-pion phase shift given by Taylor and one of us[20] and we find that the enhancement factor is less than 10% larger than that given by Eq. (4.11). In the absence of a satisfactory set of solutions to the pion-pion equation we choose to characterize the S-wave phase shift by Eqs. (4.11) and (4.14) because of their simplicity.

V. NUMERICAL RESULTS

We write the S-matrix element for the K_{e4} decay in the form

$$S=(-i)(2\pi)^4\delta^4(p-q_1-q_2-\nu-e)\langle f|M|i\rangle \times\frac{1}{[(2\pi)^{3/2}]^5}\frac{1}{(8q_1^0q_2^0p^0)^{1/2}}(m_e/e^0)^{1/2} \tag{5.1}$$

where the positron spinor (denoted by v_e) is normalized to $\bar{v}_e v_e=-1$, and the neutrino spinor to $u_\nu^* u_\nu=1$. (Here the asterisk denotes Hermitian conjugation, and $\bar{e}=e^*\gamma_0$; in general we follow the conventions used by Schweber.[21]) Further

$$\langle f|M|i\rangle=\bar{u}_\nu\gamma^\mu\tfrac{1}{2}(1-i\gamma_5)v_e M_\mu. \tag{5.2}$$

The transition rate is then given by

$$\Gamma=(2\pi)^4\int\delta^{(4)}(p-q_1-q_2-\nu-e)\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3} \times\frac{d^3e}{(2\pi)^3}\frac{d^3\nu}{(2\pi)^3}\frac{m_e}{8q_1^0q_2^0p^0e^0}\sum_{\text{spins}}|\langle f|M|i\rangle|^2, \tag{5.3}$$

where after summing over the lepton spins

$$\sum_{\text{spins}}|\langle f|M|i\rangle|^2=M_\mu M_\nu^*\frac{1}{2\nu^0 m_e} \times[\nu^\mu e^\nu+\nu^\nu e^\mu-(\nu\cdot e)g^{\mu\nu}-i\epsilon^{\alpha\mu\beta\nu}\nu_\alpha e_\beta], \tag{5.4}$$

with ϵ being the usual antisymmetric tensor.

The general form of M_μ has been given in Eq. (2.8) with $k_\mu=(e+\nu)_\mu$. It is easy to see that C gives no contribution to (5.4) if we neglect m_e^2 compared with other typical meson masses. (This was the reason for neglecting a detailed study of the C term in the previous sections.)

In order to integrate in (5.3) it is convenient to use the variables

$$\begin{aligned} Q&=q_1+q_2, \quad F=\nu+e=K,\\ R&=q_1-q_2, \quad L=e-\nu, \end{aligned} \tag{5.5}$$

with

$$\begin{aligned} p&=Q+F, \quad s=Q^2,\\ \eta&=\tfrac{1}{2}(t_1-t_2)=F\cdot R. \end{aligned} \tag{5.6}$$

Similar integrations have already been described in the literature in detail.[3,22] We hence only give here the result after integrating over R, F, and L.

The K_{e4} decay rate is given by

$$\begin{aligned}\Gamma_{K_{e4}}=(3\times2^9\times\pi^5 m_K)^{-1}\int ds\, h\bigg(r_0^2\bigg\{|a_0|^2 m_K^2 N(s)+\tfrac{2}{3}\operatorname{Re}(a_0b_0^*)h^2m_K^2\bigg[\frac{m_Kx^5}{5}-sM(s)\bigg]\\ +|b_0|^2h^4\bigg[\frac{M(s)}{5}+\frac{2sm_K^3y}{3}N(s)-\frac{2sm_K^3x^5}{15}\bigg]\bigg\}+r_1^2\bigg\{|a_1|^2\frac{h^2}{3}M(s)+\tfrac{2}{3}\operatorname{Re}(a_1b_1^*)h^2m_K^2\bigg[\frac{m_Kx^5}{5}-sN(s)\bigg]\\ +|b_1|^2h^2\bigg[\tfrac{1}{3}m_K^2N(s)+2m_Ksy\bigg(\tfrac{1}{2}xy-\frac{s}{4}\ln\frac{m_K^2}{s}\bigg)-\tfrac{2}{3}m_Ksx^3\bigg]\bigg\}\bigg)\end{aligned} \tag{5.7}$$

[20] J. G. Taylor and Tran N. Truong, Nuovo Cimento **25**, 946 (1962).
[21] S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson and Company, New York, 1961).
[22] R. H. Dalitz, Phys. Rev. **99**, 915 (1955).

where a_0, a_1, b_0, b_1, are defined in (3.23) and

$$N(s)=\frac{1}{4}x^3y-\frac{3}{8}sxy+\frac{3}{16}s^2\ln\frac{m_K^2}{s}, \tag{5.8a}$$

$$M(s)=m_K^4\left(\frac{1}{6}x^5y-\frac{5}{24}sx^3y+\frac{5}{16}s^2xy-\frac{5}{32}s^3\ln\frac{m_K^2}{s}\right), \tag{5.8b}$$

$$x=\frac{m_K^2-s}{2m_K};\quad y=\frac{m_K^2+s}{2m_K};\quad h^2=\frac{s-4}{s}, \tag{5.8c}$$

and r_0, r_1 are isotopic spin factors which take the following values for the decays (2.1a)–(2.1c), respectively:

$$r_0=\frac{1}{\sqrt{12}};\quad r_1=0, \tag{5.9a}$$

$$r_0=-\frac{1}{\sqrt{6}};\quad r_1=\frac{1}{2}, \tag{5.9b}$$

$$r_0=0;\quad r_1=\frac{1}{\sqrt{2}}. \tag{5.9c}$$

Once a definite choice has been made for the π-π interaction, expression (5.7) gives us the rate for K_{e4} decay and the spectrum of the invariant mass of the two pions. For the two-pion P-wave interaction we use, we have only one specific form (ρ-meson dominance), as explained in the previous section. For the S-wave we have assumed either a scattering-length approximation or a resonance dominance. We present first the scattering-length case.

For different values of the scattering length, the rate for $K^+\rightarrow e^+\nu\pi^+\pi^-$ calculated from (5.7) with (4.17), (4.18) and (4.19) is given in Table I.

The experimental rate for $K^+\rightarrow e^+\nu\pi^+\pi^-$ was recently reported[1] on the basis of approximately 100 events to be

$$\Gamma(K^+\rightarrow\pi^+\pi^-e^+\nu)=(3.5\pm0.7)\times10^3\ \mathrm{sec}^{-1}. \tag{5.10}$$

We obtain this result for an attractive pion-pion interaction with a scattering length of about 1.0 Compton wavelengths.

TABLE I. Predicted rates versus scattering length.

α_0	$\omega(a)$[a]	$\omega(c)$[b]	$\omega(b)$[c]
−0.5	55	1290	750
−0.3	160	1290	970
0.0	410	1290	1500
0.5	910	1290	2500
1.0	1400	1290	3400
1.5	1800	1290	4300
2.0	2200	1290	5000
2.5	2500	1290	5600

[a] $\omega(a)$ = rate (per sec) of $K^+\rightarrow\pi^0+\pi^0+e^++\nu$.
[b] $\omega(b)$ = rate (per sec) of $K^+\rightarrow\pi^++\pi^-+e^++\nu$.
[c] $\omega(c)$ = rate (per sec) of $K^0\rightarrow\pi^-+\pi^0+e^++\nu$.

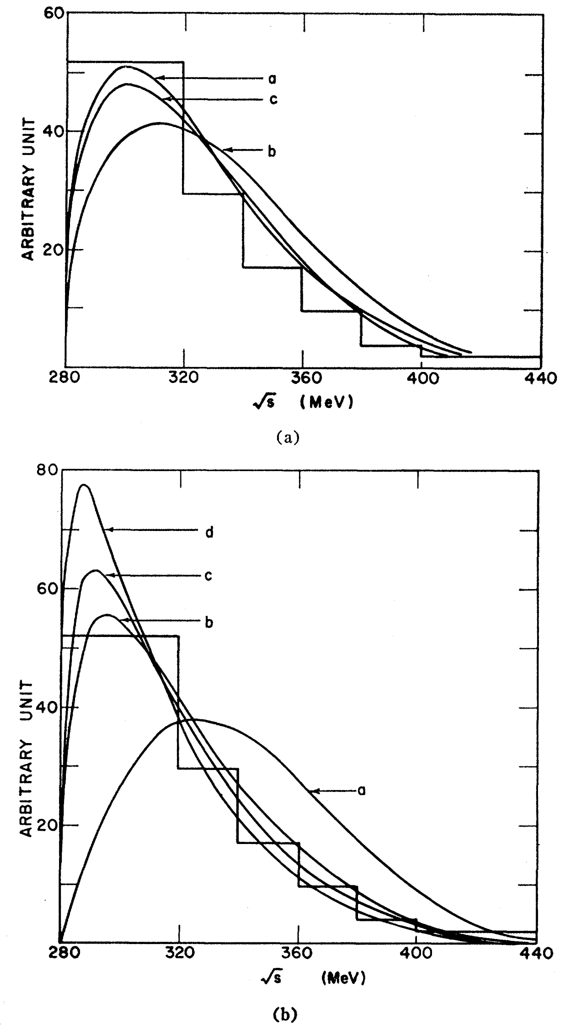

FIG. 2(a) Two-pion spectra for various values of the S-wave scattering length, α_0: a: S-wave phase space (no P-wave included); b: $\alpha_0=0$; c: $\alpha_0=0.5$. (b) Two-pion spectra for various values of the S-wave scattering length, α_0: a: $\alpha_0=-0.5$; b: $\alpha_0=1.0$; c: $\alpha_0=1.5$; d: $\alpha_0=2.5$. In each case the histogram is the experimental data of Ref. 1.

In Fig. 2 we give the distribution of the invariant mass of the two pions for different values of the scattering length (α_0). The tendency of an attractive S-wave pion-pion interaction is to enhance the low end of the two-pion spectrum while the opposite is true for the effect of the P wave. In Fig. 3 we plot separately, for illustration, the contribution of the S wave to the two-pion spectrum for $\alpha_0=1$ (which is our favored value), the separate P-wave distribution, and their appropriate sum.

The terms a_1 and b_0 turn out to be numerically much smaller than b_1 and a_0, respectively [remember that a_0, b_0 (a_1,b_1) refer to $S(P)$ waves]. The contribution of the term "Re($a_0b_0^*$)" in (5.7) is about 0.5% of the contribution to the rate of the "$|a_0|^2$" term. The term "Re($a_1b_1^*$)" contributes about 5% of the total P-wave

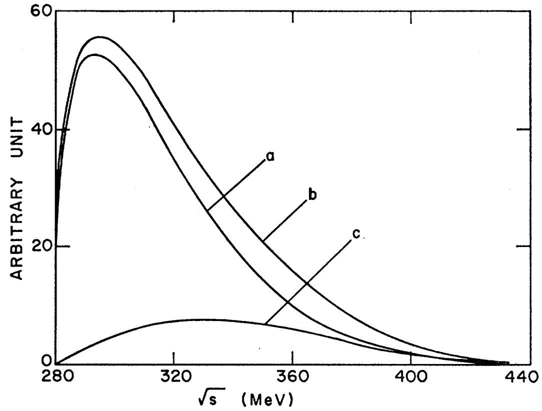

FIG. 3. Two-pion spectra for $\alpha_0=1.0$: a: S-wave contribution; c: P-wave contribution; b: Sum of S and P-waves.

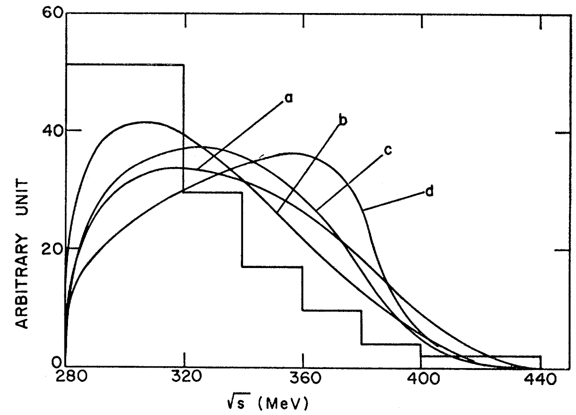

FIG. 4. Two-pion spectra for a σ resonance of mass, m_σ and total width Γ_σ (in units of MeV): a: $m_\sigma=400$, $\Gamma_\sigma=70$; b: $m_\sigma=400$, $\Gamma_\sigma=130$; c: $m_\sigma=380$, $\Gamma_\sigma=70$; d: $m_\sigma=380$, $\Gamma_\sigma=50$. The histogram is taken from Ref. 1.

term. Hence, we verify in our model that the amplitudes A and B of Eq. (2.8) can be taken to be pure S and P waves, respectively, to a very good accuracy.

The other alternative we have tried for the behavior of the S-wave phase shifts is of a resonant type [Eq. (4.20)]. Brown and Singer[23] have suggested the existence of an S-wave π-π resonance ("σ" particle) with a mass of about 400 MeV and a width of about 70–100 MeV, in order to explain the $\eta\to 3\pi$ rate and spectrum, and have also interpreted the $K\to 3\pi$ decays in terms of this resonance. We have varied in our calculation the mass of the σ between 370 and 410 MeV and the width between 50 and 120 MeV. In Fig. 4 we give the π-π invariant mass distribution for several choices of these parameters.[24] It turns out that in this case the P wave is negligible and the decay amplitude is given by $K\to\sigma+(e\nu)$, as assumed by Brown and Faier.[9] However, the rate for K_{e4} calculated from (5.7) with a σ, comes out to be larger than the experimental one by at least two orders of magnitude. For example, for $m_\sigma=390$ MeV, $\Gamma_\sigma=100$ MeV we get for the decay rate $K^+\to\pi^+\pi^-e^+\nu$, $1/\tau=9.06\times10^5$ sec^{-1}, which is about 260 times larger than the experimental rate. For larger σ mass and width one obtains smaller rates, but even for $m_\sigma=410$ MeV and $\Gamma_\sigma=120$ MeV we get $1/\tau=6.04\times10^5$ sec^{-1}, which is still about 170 times too large. For $m_\sigma=380$ MeV and $\Gamma_\sigma=50$ MeV one obtains $1/\tau=1.68\times10^6$ sec^{-1}. In Appendix B we discuss an alternative treatment incorporating the σ resonance. It leads to the same rate as obtained here.

Another possible test of the pion-pion interaction would be the measurement of the forward-to-backward asymmetry of one pion in the c.m. system of the pions, due to the interference of S and P waves. Before proceeding to the complete expression, we shall briefly explain this distribution in a simplified way. Let us write approximately our weak current of the $K\pi\pi$ states as

$$|a(s)|(q_1+q_2)_\mu e^{i\delta_0(s)}+|b(s)|(q_1-q_2)_\mu e^{i\delta_1(s)}$$

in an obvious notation. Then the differential rate for K_{e4} decay contains a term proportional to

$$dsd(\cos\theta)|a(s)||b(s)|P(s)\cos(\delta_0-\delta_1)\cos\theta\,,$$

where $P(s)$ comes from the phase space. Obviously, such a term does not contribute to the total rate. Nevertheless, it might be useful in detecting the presence and type of the pion-pion interaction. By integrating this term over $\cos\theta$, so as to get the forward hemisphere events minus the backward hemisphere pion events, one obtains an expression which is a function of s and proportional to $\cos(\delta_0-\delta_1)$.[25] For an S-wave resonance in the physical region the asymmetry dis-

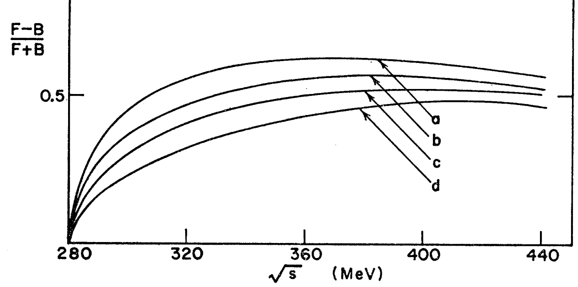

FIG. 5. Forward-backward asymmetry of the two pions for various values of α_0: a: $\alpha_0=0.5$; b: $\alpha_0=1.0$; c: $\alpha_0=1.5$: d: $\alpha_0=2.5$.

[23] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); Phys. Rev. **133**, B812 (1964).

[24] N. P. Samios, A. H. Bachman, R. M. Lea, T. E. Kalogeropoulos, and W. D. Shephard, Phys. Rev. Letters **9**, 139 (1962); J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **130**, 2481 (1963); F. S. Crawford, R. A. Grossman, L. J. Lloyd, L. R. Price, E. C. Fowler, Phys. Rev. Letters **11**, 564 (1963) also Erratum, *ibid.* **14**, 421 (1964); R. D. Fabbro, M. DePretis, R. Jones, G. Marini, A. Odian, G. Stoppini, L. Tau, *ibid.* **12**, 674 (1964). The two-pion spectra in $\pi+N\to 2\pi+N$ and $\gamma+p\to 2\pi+p$ can be understood without invoking two pion resonances. V. V. Anisovich and L. G. Dakhno, Phys. Letters **10**, 221 (1964).

[25] The usefulness of investigating this distribution was in fact first suggested by E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **44**, 765 (1963) [English transl: Soviet Phys.—JETP **17**, 517 (1963)].

tribution would go through 0 at the position of the resonance.

We shall present the asymmetry term as a function of the π-π invariant mass. By using our expressions (3.24) and (3.25) we obtain for the asymmetry distribution of the positive pion in the center of mass of the pions with respect to the line of flight of the two-pions in the K-meson rest system

$$dA=\int_{\cos\theta>0} d\Gamma-\int_{\cos\theta<0} d\Gamma=r_0r_1(3\times 2^8\times\pi^5 m_K)^{-1}ds$$
$$\times\left\{\mathrm{Re}(a_0a_1^*)\frac{m_K{}^3h^2}{30}[y(x^2-2s)^2+2(x^4+2s^2)y-8s^{5/2}]+\mathrm{Re}(a_0b_1^*)\frac{m_Kh^2}{2}\left[\frac{m_Kx^4}{4}-\frac{s}{3}[(x^2-2s)y+2s^{3/2}]\right]\right.$$
$$+\mathrm{Re}(a_1b_0^*)\frac{m_K{}^3h^4}{4}\left[\frac{m_Kx^6}{6}-\frac{s}{15}[y(x^2-2s)^2+2y(x^4+2s^2)-8s^{5/2}]\right]$$
$$\left.+\mathrm{Re}(b_0b_1^*)m_K{}^2h^2\left[\frac{m_Kh^2}{60}[y(x^2-2s)^2+2y(x^4+2s^2)-8s^{5/2}]+\frac{y(s-4)}{3}[(x^2-2s)y+2s^{3/2}]-\frac{(s-4)x^4}{4}\right]\right\}. \tag{5.11}$$

In Fig. 5 the asymmetry distribution is plotted for various choices of the scattering length. (The resonance case is discussed in the next section.) As the terms a_1, b_0 are much smaller in our model than a_0, b_1, respectively, the main contribution in (5.11) comes from the term "$\mathrm{Re}(a_0b_1^*)$." The term "$\mathrm{Re}(a_0a_1)$," which is the biggest among the small terms, does not exceed about 5% of the main term.

VI. SUMMARY AND DISCUSSION

We have given in this paper a detailed treatment of the weak and strong interaction aspects of the K_{e4} decays. By *assuming* a model for the weak interaction, namely a "nearly conserved" axial strangeness changing current and the dominance in the $\langle K\pi\pi|$ states of the $\langle K^*\pi|$, we obtain a Goldberger-Treiman-like relation Eq. (3.7) which relates the K_{e4} decay strength to the $K\rightarrow\mu+\nu$ rate and the strong $K^*K\pi$ vertex. The rate, as well as the different decay spectra discussed in the previous section, are changed by significant amounts when the final-state pion-pion interaction is included. For this reason, we have developed in Secs. III and IV a detailed treatment of this effect. Before comparing the results of our calculation with the experimental data, we would like to stress that even if our treatment of the weak interaction is inadequate, the method of handling the strong final-state interaction effect is quite general and reliable. See also note added in proof.

Let us now summarize our main results: Without any pion-pion interaction in the S and P waves, the decay rate we obtain is about 3.5 times smaller than the experimental rate. By including only the effect of the ρ resonance, the calculated rate is still smaller than the experimental rate by a factor of 2.5. Assuming that our treatment of the weak part is adequate, the question one would like to ask is what type of S-wave pion-pion interaction would make the calculated rate equal to the experimental one. We find that when the ρ meson and the S-wave pion-pion interaction in the form of the scattering length are introduced, we obtain the experimental rate Eq. (5.10) for a scattering length of $\alpha_0=(1.0\pm0.3)\hbar/m_\pi c$. This is consistent with the value obtained by the analysis of the pion-nucleon phase shift and other phenomenological analyses of the π-π interaction.[26] This is the only unknown parameter in the calculation. Once it is fixed we can make the predictions on the two-pion spectrum, the S to P wave ratio, and the forward-backward asymmetry. For this value of the scattering length ($\alpha_0=1.0$) the total contribution of the P wave is 18% of the total decay rate for $K^+\rightarrow\pi^++\pi^-+e^++\nu$. The two-pion spectrum is in good agreement with the experimental one (see Fig. 3). With the existing experimental data, our curves seem already to rule out a too large scattering length [i.e., $2.5(\hbar/m_\pi c)$]. It is seen that while the S-wave distorts the two pion invariant mass distribution towards smaller values of $\sqrt{s}$, the P-wave effect tends to bring it back to the "pure phase space" distribution. This accounts for the observed spectrum which does not differ very much from the S-wave phase space. If a σ resonance dominates the π-π $T=0$ channel [Eq. (4.14)] then *our* calculation predicts a rate which is larger than the experimental one by about two orders of magnitude. Because of the unusually large enhancement factor involved, it would be interesting to find out whether the number obtained does not depend on our model. For this purpose, we have written the amplitude for the K_{e4} decay in the form suggested by Brown and Faier,[9] i.e., $\langle\sigma|A_\mu|K\rangle J_\mu$, where J_μ is the lepton current. This form is also consistent with our result that if σ is included, the P wave is negligible. In Appendix B we calculate $\langle\sigma|A_\mu|K\rangle$ by using again the Bég, Cornwall and Woo technique[10] and we obtain again that the rate is about 200 times larger than the observed one.

As far as the two-pion invariant mass distribution is concerned a σ with mass of $\sim$400 MeV and width of $\sim$130 MeV is not ruled out by the existing experimental results.[1] However, a σ with parameters 350–380 MeV

[26] For a recent review of the experimental and theoretical status of pion-pion interactions, see J. Hamilton in *Strong Interactions and High Energy Physics*, Scottish Universities' Summer School 1963, edited by R. G. Moorehouse (Oliver and Boyd, Edinburgh 1964).

and 50–70 MeV gives a spectrum (see Fig. 4) which disagrees with the experimental data.

For repulsive S-wave pion-pion interactions, our calculation shows that the rate is smaller than that which is obtained by switching off the S-wave pion-pion interactions effect. This is certainly what we expected for a repulsive interaction, therefore our calculation for K_{e4} is not consistent with the repulsive S-wave pion-pion interaction.

A scattering length of approximately 1 is favored by both the rate and the π-π mass distribution *in our model*. It is very desirable to compare also the data on the backward-forward asymmetry to our prediction (Fig. 5). For the resonance case, this distribution goes through zero at the resonance mass; however, it does not rise in the other direction about 2–3%. This is due to the very small P/S ratio predicted by our model.

Additional tests have been suggested recently by Maksymowicz and Cabibbo.[27] Our model can be easily used to calculate the proposed distributions.

We would like now to comment on the difference between our method and others in the literature. In Ref. 8, there is no serious attempt to treat the strong effects in the manner of our paper. The effect of the strong interaction is treated by the unitary symmetry scheme and no other prediction than the rate can be made. In this respect our calculation is similar to that given in Ref. 7, that is, the strong interaction is handled by the use of dispersion relation techniques. However, there is a basic difference between our paper and Ref. 7. In Ref. 7 the K_{e4} rate is related to the $K+\bar{K}\to 2\pi$ amplitude while in this paper it is related to the process $(e\nu)+K\to 2\pi$. The numerical value for the K_{e4} rate given by Ref. 7 is 3.5 times smaller than the experimental rate even when as large a scattering length as $\alpha_0=2.5(\hbar/m_\pi c)$ is used. Furthermore, the Born term for the integral equation $K+\bar{K}\to 2\pi$ behaves at infinity as $\ln s$ (to be contrasted with $(\ln s)/s$ in our case); therefore a cutoff has to be introduced in their numerical procedure. It is not clear how the results given in Ref. 7 depend on the cutoff.

Finally, we would like to discuss the possibility of time-reversal invariance violation as suggested by the experiment of Christenson, Cronin, Fitch, and Turlay[28] (CPT invariance is assumed). We shall assume that T violation, if it exists, is connected with the weak current involving strongly interacting particles. Instead of using Eq. (2.8) we now can write:

$$(2q_1{}^0 2q_2{}^0 2p^0)^{1/2}\langle 2\pi,\text{out}|J_\mu{}^A|K\rangle \simeq a_0(s)e^{i\varphi_0}(q_1-q_2)_\mu+b_1(s)e^{i\varphi_1}(q_1+q_2)_\mu,$$

where we have made the S- and P-wave approximation and neglected the s dependence of the Born cut. The angles φ_0 and φ_1 are due to T violation. The strong pion-pion interaction is assumed to be T invariant; therefore $a_0(s)=f/D_0(s)$ and $b_1(s)=g/D_1(s)$ where f and g are real. (It is always possible to make this separation because the Born cut is separated from the final-state pion-pion scattering cut and hence the amplitude can be written as the product of the two cuts.) Furthermore, since φ_0 and φ_1 are associated with the Born amplitude, they depend weakly on s and do not have the threshold behavior of δ's. In our calculation, the presence of T violation would make $\beta(0)$ as given by Eq. (3.7) complex. Since our S and P-wave amplitudes are both generated by $\beta(0)$, we have $\varphi_0=\varphi_1$; therefore our calculation would not be affected.

Let us now entertain the possibility that $\varphi_0\neq\varphi_1$ and $\varphi_0-\varphi_1$ is large[29] (our model is unable to allow for this possibility). It has been pointed out that it is possible to measure the difference in the phase of a_0 and b_1, namely δ_0 and δ_1, by measuring the up-down asymmetry of the positron and the backward-forward asymmetry of the pions.[27] If there were a T violation, the measured phase angle would be $\phi=\delta_0-\delta_1+\varphi_0-\varphi_1$. In principle, it is possible to detect the presence of φ_0 and φ_1 by selecting events which are sufficiently close to the two-pion threshold in order to make $\delta_0=\delta_1=0$. However, phase space is unfavorable for this choice. With a very large number of events an extrapolation procedure to threshold may be made, but this is out of the question at this moment. At best we can hope for some measurement of ϕ averaged over the two-pion energy. If T invariance holds, we do *not* expect ϕ to be near $\frac{1}{2}\pi$. This is so because most of the pions have small relative energy (experimentally the two-pion spectrum peaks at a total energy of 300 MeV) and it is not possible for the S-wave phase shift δ_0 to grow from zero at threshold to a value near $\frac{1}{2}\pi$ in a small energy interval without introducing a strong variation in the form factor $1/D_0(s)$. (It is a good approximation to put $\delta_1=0$ in the physical region in K_{e4} decay.) A large averaged value of δ_0 can come about either if there is a two-pion resonance of large width at an energy near 300 MeV, or if α_0 is much larger than 1. The former is ruled out by the experimental data, while the latter would distort the two-pion spectra by a factor of

$$|1/D_0(s)|^2\simeq[1+(\tfrac{1}{4}s-1)\alpha_0{}^2]^{-1}$$

and is also in contradiction with the experimental data of K_{e4}.

Note added in proof. The existence of the κ resonance in the $K\pi$, $T=\frac{1}{2}$, S state channel has been confirmed by

[27] N. Cabibbo and A. Maksymowicz, University of California Radiation Laboratory Report UCRL-11437 1964) (to be published).

[28] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters 13, 138 (1964).

[29] The effect of CP violation may not be large in weak interactions: Tran N. Truong, Phys. Rev. Letters 13, 358a (1964). For other discussions of the experimental results of Ref. 28, see J. S. Bell and J. K. Perring, Phys. Rev. Letters 13, 348 (1964); T. T. Wu and C. N. Yang, *ibid.* 13, 380 (1964); R. G. Sachs, *ibid.* 13, 286 (1964); N. Cabibbo Phys. Letters 12, 137 (1964); J. Bernstein, N. Cabibbo, and T. D. Lee, *ibid.* 12, 146 (1964); M. Lévy and M. Nauenberg, *ibid.* 12, 155 (1964).

M. Ferro-Luzzi *et al.*[30] They find a mass of 725 ± 5 MeV, and a full width $\Gamma<30$ MeV.

We hence have calculated the possible κ contribution to our Born terms. We again make use of a Goldberger-Treiman type relation to determine the $\langle\kappa|J_\mu{}^A|\pi\rangle$ vertex. The position of the equivalent left hand pole $-s_p$ for κ is somewhat nearer the physical region than for K^* ($s_p=2M_\kappa{}^2-\bar{s}$, compared with $2M^{*2}-\bar{s}$ for K^*). This implies that the enhancement factors are slightly smaller for the κ contributions than for the K^* contributions. Disregarding this small variation in the Born term denominators, we find that the Born terms arising from κ are related to those arising from K^* exchange by

$$(\tilde{A}_\kappa{}^{(0)}/\tilde{A}_{K^*}{}^{(0)})=(\tilde{A}_\kappa{}^{(1)}/\tilde{A}_{K^*}{}^{(1)})\simeq0.12(\Gamma_\kappa/\Gamma_{K^*})$$
$$(\tilde{B}_\kappa{}^{(0)}/\tilde{B}_{K^*}{}^{(0)})=(\tilde{B}_\kappa{}^{(1)}/\tilde{B}_{K^*}{}^{(1)})\simeq0.25(\Gamma_\kappa/\Gamma_{K^*})\,.$$

Recalling that in all cases the largest contribution to the rate arises from $A^{(0)}$, we see that including the κ makes essentially no change to our best value of α_0. Even as regards the various spectra and asymmetries, which do depend on $B^{(1)}$, the κ contribution is small.

This has the further attractive feature that our model does not depend crucially on the Goldberger-Treiman type relation for $\langle\kappa|J_\mu{}^A|\pi\rangle$, provided it gives a result correct to within a factor of 2.

Finally we should emphasize that our use of the Goldberger-Treiman type relation for $\langle K^*|J_\mu{}^A|\pi\rangle$, even if it ultimately proves to be inapplicable, only affects our calculation of the over-all rates, and hence our best choice for α_0. The spectra presented for various choices of α_0 are *independent* of the particular value of $\langle K^*|J_\mu{}^A|\pi\rangle$.

ACKNOWLEDGMENTS

We would like to thank Dr. G. Barton, Dr. M. A. B. Bég, Dr. L. M. Brown, and Dr. B. Sakita for interesting conversations. One of us (Paul Singer) would like to thank Professor R. G. Sachs and Professor R. F. Sawyer for their hospitality at the Summer Institute for Theoretical Physics, Madison, Wisc., where part of this work was done. We should like to thank Nancy Shen for carrying out the necessary programming and computations.

APPENDIX A

In this Appendix we wish to discuss the ambiguity of the Muskhelishvili-Omnès equation and the validity of our method of obtaining the solution to the integral equation.

1. The Ambiguity of The Muskhelishvili-Omnès Equation

We define

$$N(\nu)=a \tag{A1a}$$
$$D(\nu)=1+ah(\nu)+b\nu-ia[\nu/(\nu+1)]^{1/2}\theta(\nu) \tag{A1b}$$
$$A(\nu)=[\nu/(\nu+1)]^{1/2}e^{i\delta}\sin\delta=N(\nu)/D(\nu) \tag{A1c}$$

[30] M. Ferro-Luzzi, R. George, Y. Goldschmidt-Clermont, V. P. Henri, *et al.*, Phys. Letters **12**, 255 (1964); which also contains earlier references.

where $s=4(\nu+1)$. The scattering length approximation is obtained by putting $b=0$. Equation (4.6) can be rewritten in terms of the N and D functions

$$F(s,s_p)=\frac{c}{s+s_p}+\frac{c}{\pi D(s)}\times\int_4^\infty ds'\frac{[(s'-4)/s']^{1/2}N(s')D(s')}{(s'-s)(s'+s_p)}\,. \tag{A2}$$

Using (A1a), (A1b), and (A1c) in Eq. (A2), after some algebra we obtain the following expression for the amplitude in the scattering length case,

$$F(s,s_p)=\frac{c}{s+s_p}\frac{D(-s_p)}{D(s)} \tag{A3}$$

which is identical to Eq. (4.7).

For the resonant pion-pion interaction, the situation is different however. Instead of arriving at Eq. (A3) we obtain

$$F(s,s_p)=\frac{c}{s+s_p}\frac{D(-s_p)}{D(s)}+\frac{cbD(0)}{D(s)}, \tag{A4}$$

which corresponds to adding a polynomial of zero degree multiplying $1/D$ to the solution of (A3). This comes from the well-known result that the solution (A2) is defined within a term $P_n(s)[1/D(s)]$, where $P_n(s)$ is a polynomial of nth degree. The extra term in Eq. (A4) has its origin in the approximation that the resonance is of the kinematical type, that is, the resonance is not driven by the left-hand cut but by a subtraction. The pion-pion amplitude in this case behaves as $1/\nu$ as $\nu\to\infty$ and thus $\delta(\infty)=\pi$. Instead of choosing the phase shift as given by Eq. (A1), we can make the pole approximation for N, and the $\pi\pi$ amplitude then behaves as $1/\ln\nu$. In this case it is simple to show that formula (A3) is reproduced. This problem does not arise in the nonrelativistic case because there the phase shift is well defined, that is, the function $D(s)$ which is the inverse of the Jost function can be written in the unsubtracted form and $D(s)\to1$ as $s\to\infty$. Our *ansatz* as stated in the text is inspired by the situation in the nonrelativistic theory where the solution is unique. The application of our *ansatz* eliminates the possibility of choosing the expression (A4) as the solution to the integral equation. In place of this expression we must choose solution (A3).

2. Validity of Our Approximate Solution of the Integral Equation

We note that the inhomogeneous term in the integral equations behave at worst as $\ln s/s$; therefore the integral equations converge and there is no need to make a cutoff. Because of this good behavior, the approximation made in Sec. IV is an excellent one. We wish now to calculate the higher order terms in the power series expansion of $Q_0(x(s))$ in the integral equation. Typically we must show that the higher order terms in the expansion for $B(s)$ contribute very little to the integral equation, where

$$B(s)=\frac{cQ_0(x(s))}{4pq}=\frac{c}{s+s_p}\left[1+\frac{16p^2q^2}{(s+s_p)^2}+\cdots\right]. \quad (A5)$$

Here $x(s)=(\bar{s}-s-2M^{*2})/4pq$. We must show that $G^{(1)}(s)\gg G^{(2)}(s)$ for physical values of s in K_{e4} decay,

$$G^{(1)}(s)=\frac{1}{s+s_p}+\frac{e^{u_0(s)-u_0(4)}}{\pi}\int_4^\infty ds'\frac{1}{(s'-s-i\epsilon)(s'+s_p)}\times e^{-u_0(s')+u_0(4)}e^{i\delta_0(s')}\sin\delta_0(s)\,, \quad (A6)$$

and

$$G^{(2)}(s)=\frac{1}{3}\frac{(s-m_K^2)^2(s-4)}{s(s+s_p)^3}+\frac{1}{3}\frac{e^{u_0(s)-u_0(4)}}{\pi}\int_4^\infty ds'\frac{(s'-m_K^2)^2(s'-4)}{(s'-s-i\epsilon)s'(s'+s_p)^3}\times e^{-u_0(s')+u_0(4)}e^{i\delta_0(s')}\sin\delta_0(s')\,. \quad (A7)$$

We find

$$G^{(1)}(s)=\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p}. \quad (A8)$$

To evaluate (A7) it is convenient to decompose into partial fractions the inhomogeneous term

$$\frac{1}{3}\frac{(s-m_K^2)^2(s-4)}{s(s+s_p)^3}=\frac{\alpha}{s+s_p}+\frac{\beta}{s}+\frac{\gamma}{(s+s_p)^2}+\frac{\delta}{(s+s_p)^3},$$

where

$$\begin{aligned}\alpha&=\frac{1}{3}+\frac{4}{3}\left(\frac{m_K^4}{s_p^3}\right)\simeq\frac{1}{3},\\ \beta&=-\frac{4}{3}\left(\frac{m_K^4}{s_p^3}\right)\sim 0,\\ \gamma&=-\tfrac{1}{3}(4+2m_K^2)-(\tfrac{2}{3}+\beta)s_p\approx -55\,,\\ \delta&=\frac{1}{3s_p}(s_p-m_K^2)^2(s_p-4)\approx 2130\,.\end{aligned} \quad (A9)$$

Using Eqs. (4.4) and (4.8), we arrive at

$$G^{(2)}(s)=\alpha\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p}+\beta\frac{e^{u_0(s)-u_0(0)}}{s}-\gamma\frac{\partial}{\partial s_p}\left(\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p}\right)+\frac{\delta}{2}\frac{\partial^2}{\delta s_p^2}\left(\frac{e^{u_0(s)-u_0(-s_p)}}{s+s_p}\right). \quad (A10)$$

By using the various forms for the $\exp u(s)$ factor it is found that $G^{(2)}(s)\ll G^{(1)}(s)$ for all physical values of s in K_{e4} decay.

APPENDIX B

In this Appendix we present a different approach for calculating the K_{e4} decay when σ dominates the pion-pion S-wave interaction. The decay is due then to the $\langle\sigma|A_\mu|K\rangle J_\mu$ interaction, where J_μ is the V-A leptonic current. Brown and Faier[9] have remarked that for equal strengths of the $\langle\sigma|A_\mu|K\rangle$ and $\langle 0|A_\mu|K\rangle$ currents one obtains the correct rate for K_{e4}. We proceed to calculate $\langle\sigma|A_\mu|K\rangle$ by using again the method of Ref. 10.

One has generally

$$2(E_\sigma E_K)^{1/2}\langle\sigma|A_\mu|K\rangle=(p_K+p_\sigma)f_1(s)+(p_K-p_\sigma)f_2(s)\,, \quad (B1)$$

where

$$s=(p_K-p_\sigma)^2=(p_e+p_\nu)^2=k^2\,.$$

If we take the divergence of (B1), we obtain

$$i\partial^\mu 2(E_\sigma E_K)^{1/2}\langle\sigma|A_\mu|K\rangle=\lambda(s)=(m_K^2-m_\sigma^2)f_1(s)+sf_2(s)\,. \quad (B2)$$

Assuming for the region of $s\approx 0$ that the second term can be neglected and writing an unsubtracted dispersion relation for $\lambda(s)$, one has, if one retains only the K-pole term

$$\lambda(0)=(m_K^2-m_\sigma^2)f_1(0)=g_{\sigma K\bar{K}}G(m_K^2)\,, \quad (B3)$$

where $G(m_K^2)$ is defined by the matrix element for K decay

$$\sqrt{2}E_K^{1/2}\langle 0|A_\mu|K\rangle=(p_K)_\mu G(m_K^2)\,. \quad (B4)$$

Hence

$$f_1(0)=(g_{\sigma K\bar{K}}G)/(m_K^2-m_\sigma^2)\,. \quad (B5)$$

If we use a unitary symmetric coupling to relate $\sigma K\bar{K}$ coupling to $\sigma\pi\pi$, i.e.,

$$g\sigma(\boldsymbol{\pi}\cdot\boldsymbol{\pi}+2\bar{K}K+\eta\eta)\,,$$

one has

$$f_1(0)=\frac{g_{\sigma\pi\pi}G}{m_K^2-m_\sigma^2}\simeq\frac{(2\sqrt{\pi})m_\sigma G}{m_K^2-m_\sigma^2} \quad (B6)$$

by using[23] $(g_{\sigma\pi\pi})^2/4\pi=0.9$ for $m_\sigma=400$ MeV and $\Gamma_\sigma=100$ MeV. If one compares (B6) with the requirement of Brown and Faier[9] one finds that $f_1(0)$ is too large by a factor of 14.5 to reproduce the experimental rate. Hence, we obtain again that a σ would give a rate 200 times too large, in somewhat surprising agreement with our calculation in Sec. V.[31]

A similar procedure can be used to estimate the effect of ρ in the P wave. However, as our knowledge of the $\rho K\bar{K}$ coupling is nil, we feel that the treatment we have chosen is more reliable. Nevertheless, if one uses again unitary symmetry to relate $g_{\rho K\bar{K}}$ to $g_{K^*K\pi}$ one obtains a result comparable to the one obtained in Sec. V.

[31] If, instead of assuming σ was an SU_3 singlet, we had made some other choice, the rate would not be affected by the necessary 2 orders of magnitude.