introduction), Martin[1] has shown that, in some cases, all partial-wave amplitudes become uniquely determined by the spectral functions. However, even in these cases he is not able to conclude that the s- and p-wave physical amplitudes agree with the partial-wave amplitude analytically continued from high l. Our example, under the assumption of an explicit form for the high momentum-transfer has been explicitly shown to have this property.

As a final remark, we mention the fact[1] that if the asymptotic behavior of the amplitude is of polynomial type, then it would be impossible to have agreement between the interpolated and the physical partial-wave amplitude. It is clear that terms of polynomial type introduce Kronecker deltas into the amplitude which give contributions at discrete values of angular momentum, which contributions cannot be reproduced by a smooth connection with higher angular momenta. This gives the important result[1] that MASD forbids polynomial asymptotic behavior of the scattering amplitude in t.

ACKNOWLEDGMENTS

I am indebted to Professor M. L. Goldberger for a critical reading of the manuscript and to Dr. John R. Taylor for helpful discussions.

PHYSICAL REVIEW VOLUME 137, NUMBER 6B 22 MARCH 1965

Bounds for a Class of Bethe-Salpeter Amplitudes*

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(Received 12 November 1964)

For a certain wide class of kernels involving trilinear coupling of scalar particles, the absorptive part of the Bethe-Salpeter amplitude for forward scattering is bounded from above and below. The bounds are expressed in the form $B_1 s^{\alpha_1} \leqslant A(s) \leqslant B_2 S^{\alpha_2}$, where s is the squared c.m. energy and B_1 and B_2 are positive constants. Expressions for the exponents α_1 and α_2 are given as functions of the coupling constant g. For the straight ladder model, α_1 and α_2 coincide for all values of g, the common expression agreeing with an exact result of Nakanishi. For the more complicated models, α_1 and α_2 do not in general coincide. However, in the strong-coupling limit $g \to \infty$, we find that $\alpha_2/\alpha_1 \to 1$; moreover, the common asymptotic behavior $\alpha_{1,2} \to_{g\to\infty} g/4\pi m$ is the same for all the models, including the straight-ladder model.

I. INTRODUCTION

USING techniques discussed in two earlier papers,[1,2] we consider here the problem of setting upper and lower bounds on the absorptive part of the forward elastic scattering amplitude for a certain wide class of ladder-like models. We deal with theories involving scalar particles which couple trilinearly.

In general, the absorptive amplitude A satisfies a Bethe-Salpeter equation, as symbolized in Fig. 1. For an inclusive treatment, one would have to take for the kernel K a sum over all possible irreducible diagrams; and for the Born term A_B a similar sum, evaluated on the mass shell $k^2=0$. But as we shall understand the term here, a particular *model* is characterized by the choice of a particular one of the irreducible diagrams for the kernel K and corresponding Born term A_B. The class of such models which will come under discussion here is characterized by the examples shown in Fig. 2 for the irreducible kernels. The heavy lines (spinless "nucleons") correspond to particles of mass m, except for the *external* nucleons which are taken, for reasons of kinematic simplicity, to be massless. The wavy lines represent exchanged particles ("mesons"). In general terms, the class of irreducible diagrams which we consider consists of those in which each wavy line joins two solid lines, without further connections (no loops or self-energy and vertex corrections).

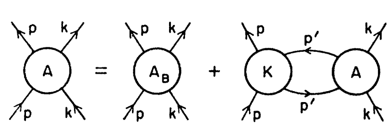

FIG. 1. Notation for the integral equation.

Insofar as the kernel K is concerned, the exchanged particles are taken to be massless. But in the Born term A_B, which is described by the same diagram as for K, we suppose that *one* of the exchanged particles has a finite mass μ. Although we could set $\mu=0$ without embarrassment insofar as the absorptive amplitude is concerned, we would encounter infrared divergence troubles for the real part of the amplitude. In order to

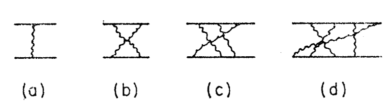

FIG. 2. Examples of irreducible diagrams.

* Work supported by the U. S. Air Force Office of Research, Air Research and Development Command.
[1] G. Tiktopoulos and S. B. Treiman, Phys. Rev. **135**, B711 (1964).
[2] G. Tiktopoulos and S. B. Treiman, Phys. Rev. **136**, B1217 (1964).

deal with models free from this disease in any of their parts, we retain this one finite mass and drop all further discussion of the real part of the scattering amplitude.

The simplest of the models is that corresponding to the kernel of Fig. 2(a), which generates the straight-ladder diagrams. It has been widely discussed in the literature in various limits and approximations.[3] In particular, Nakanishi[4] has recently obtained an exact solution to the problem for the case where the exchanged particles (all but one of them, as above) are massless. For very large scattering energy $s^{1/2}$, the amplitude grows like s^α. According to Nakanishi, the exact expression for the exponent is given by

$$\alpha = -\tfrac{3}{2} + [\tfrac{1}{4} + (g/4\pi m)^2]^{1/2}, \tag{1}$$

where g is the coupling constant. An *upper* bound on α which we had earlier obtained happens to coincide exactly with this result.[1] For later reference, let us note that the first two terms in an asymptotic expansion for the strong coupling limit are given by

$$\alpha \to_{g\to\infty} (g/4\pi m) - \tfrac{3}{2} + \cdots. \tag{2}$$

In the present paper we consider the more general class of ladder models which has been characterized above. For each model we seek both an upper and a lower bound on the forward scattering absorptive amplitude, in the form of expressions which again grow like s^α for large energies. In particular, we are concerned with setting bounds on the exponent α. For this more general class of models, exact results are of course not available for comparison. However, the following remarkable result emerges from the present analysis. The absorptive amplitude $A(s)$ is bounded in the form

$$B_1 s^{\alpha_1} < A(s) < B_2 s^{\alpha_2}, \tag{3}$$

where B_1 and B_2 are positive constants and the exponents α_1 and α_2 are certain explicit functions of the coupling constant g, $\alpha_2 > \alpha_1$. In the strong coupling limit we find that $\alpha_2/\alpha_1 \to 1$; and the leading term in the asymptotic expansion is

$$\alpha_1, \alpha_2 \to_{g\to\infty} (g/4\pi m). \tag{3'}$$

This limiting behavior is common to all of the models under discussion, and since $\alpha_2/\alpha_1 \to 1$ in the strong coupling limit, we conclude that the leading term (3′) is in fact exact in this limit. It of course coincides with the exact result for the straight ladder, Eq. (2).

At the end of the paper we make a few comments on the problem of setting bounds for the amplitude generated by including the *sum of all* irreducible graphs of the described class in the kernel.

II. PRELIMINARIES

In the notation of Fig. 1, the Bethe-Salpeter equation for the off-mass-shell absorptive amplitude takes the form

$$A(p,k) = A_B(p,k) + \int \frac{d^4p'}{(2\pi)^4} K(p,p') \frac{A(p',k)}{(m^2-p'^2)^2}, \tag{4}$$

where $k^2=0$ is held fixed at its physical value. In terms of the scalar invariants p^2, p'^2, $k^2=0$, and

$$s=(p+k)^2,\quad s'=(p'+k)^2,\quad y=(p'-p)^2,$$

this can be written

$$A(s,p^2,k^2) = A_B(s,p^2,k^2) + \frac{1}{32\pi^3}\frac{1}{s-p^2}\int dp'^2 ds' \times \frac{A(s',p'^2,k^2)}{(m^2-p'^2)^2}\int K(y,p^2,p'^2)dy. \tag{5}$$

At the end, the variable p^2 is to be set at its physical value $p^2=0$.

For a given irreducible diagram involving the exchange of n massless mesons, the kernel K has the form

$$K = 2\pi g^2\left[\frac{g^2}{(2\pi)^3}\right]^{n-1}\int \frac{d^4q_1\cdots d^4q_n\,\delta(q_1^2)\cdots\delta(q_n^2)}{(m^2-p_1^2)(m^2-p_2^2)\cdots(m^2-p_{2n-2}^2)} \times\delta(p-p'-\sum_1^n q_i), \tag{6}$$

where the q_i are the momenta of the exchanged mesons and the p_i are the momenta of the intermediate nucleons. The Born term is $A_B = \frac{1}{2}K$, with $p'^2=0$; but here, as mentioned, we take *one* of the exchanged mesons to have finite mass μ.

Introducing the new variables

$$x=-p^2,\quad u=s-p^2,\qquad x'=-p'^2,\quad u'=s'-p'^2, \tag{7}$$

we find that Eq. (6) can be rewritten

$$A(u,x) = A_B(u,x) + \frac{1}{32\pi^3}\int_{\mu^2}^{u}\frac{du'}{u'}\int_{xu'/u}^{u'} dx' \times\frac{A(u',x')}{(m^2+x')^2}\int_0^{y_0} K(y,x,x')dy, \tag{8}$$

where

$$y_0 = (u-u')\left(\frac{x'}{u'}-\frac{x}{u}\right). \tag{9}$$

The fixed quantity $k^2=0$ is not displayed in the above notation.

It is out of the question to compute exactly the appropriate kernel function for each model, and to solve the corresponding integral Eq. (8). Instead, we seek bounds on the absorptive amplitude. We proceed from the observation that the kernel K is positive definite and that Eq. (8) is of the Volterra type. Thus, as discussed in the earlier papers, the amplitude A is

[3] The literature can be traced from Ref. 1.

[4] N. Nakanishi, Phys. Rev. **135**, B1430 (1964).

minorized (majorized) by any trial function $\bar{A}$ which satisfies an equation analogous to (8) in which there appears a minorized (majorized) kernel $\bar{K}$ and minorized (majorized) Born term $\bar{A}_B$. Notice that A_B is just equal to $\frac{1}{2}K'$; the prime reminds us that for the Born term the kernel is to be computed for the situation where one of the exchanged particles has finite mass μ. In effect, therefore, we minorize A with a function A_1 for which

$$A_1(u,x)\leqslant \tfrac{1}{2}K_1'(u,x,0)+\frac{1}{32\pi^3}\int_{\mu^2}^{u}\frac{du'}{u}\int_{xu'/u}^{u'}dx' \times\frac{A_1(u',x')}{(m^2+x')^2}\int_0^{y_0}K_1(y,x,x')dy, \quad (10)$$

where $K_1\leqslant K$, $K_1'\leqslant K'$; and we majorize A with a function A_2 which satisfies a similar equation with the inequality sign reversed and with majorized kernels $K_2\geqslant K$, $K_2'\geqslant K'$.

Our first task will be to seek tractable bounds on the kernel K for the general models under discussion. In Eq. (6) the denominators $D_i=m^2-p_i^2$ vary in a correlated way in the course of integration, the correlation depending on the details of the diagram in question. However, we can minorize $K(y,x,x')$ by replacing each denominator by the maximum value which it reaches for fixed y, x, x'. Similarly we majorize K by replacing the denominators by their smallest values. Now a typical momentum p_i in Eq. (6) can be written

$$p_i=p-Q_1=p'+Q_2,$$

where

$$Q_1=\sum_{j=1}^{i}q_j,\quad Q_2=\sum_{j=i+1}^{n}q_i.$$

With $Q^2=\mu_1^2$, $Q'^2=\mu_2^2$, we then find

$$\begin{aligned}D_i&=m^2-(p-Q_1)^2\\&=m^2+x-\mu_1^2+(1/2y)(y-x+x')(y+\mu_1^2-\mu_2^2)\\&\quad-(\cos\theta/2y)[(y+x'-x)^2+4yx]^{1/2}\\&\qquad\times[(y+\mu_1^2-\mu_2^2)^2-4y\mu_1^2]^{1/2},\end{aligned}$$

where θ is the angle between the vectors p and Q_1 in the center-of-mass frame $\mathbf{p}-\mathbf{p}'=0$. For given values of y, x, and x', the quantity D_i is largest when $\cos\theta=-1$ and when μ_1 and μ_2 are at their smallest possible values, $\mu_1=\mu_2=0$. Similarly, D_i is smallest when $\cos\theta=1$ and again $\mu_1=\mu_2=0$. Thus,

$$D_i\leqslant D_{\max}=m^2+\tfrac{1}{2}(y+x+x')+\tfrac{1}{2}[(y+x+x')^2-4xx']^{1/2}, \quad (11)$$

and

$$D_i\geqslant D_{\min}=m^2+\tfrac{1}{2}(y+x+x')-\tfrac{1}{2}[(y+x+x')^2-4xx']^{1/2}. \quad (11')$$

Bounds on the kernel K of Eq. (6) can now be obtained by the use of these alternative limits on the denominator functions. The integration which remains to be carried out corresponds to that required for the determination of the phase volume of a system of n massless particles at squared barycentric energy y. With $K_1\leqslant K\leqslant K_2$, we then find

$$K_1(y,x,x')=\left(\frac{g}{4\pi}\right)^{2n}\frac{32\pi^3}{(n-1)!(n-2)!}\frac{y^{n-2}}{[D_{\max}(y,x,x')]^{2n-2}}, \quad (12)$$

$$K_2(y,x,x')=\left(\frac{g}{4\pi}\right)^{2n}\frac{32\pi^3}{(n-1)!(n-2)!}\frac{y^{n-2}}{[D_{\min}(y,x,x')]^{2n-2}}. \quad (12')$$

In connection with the integration over the variable y in Eq. (10), and the corresponding integration for the majorizing trial function A_2, we observe that $D_{\max}$ is biggest when y is at its maximum value y_0; and $D_{\min}$ is smallest when $y=y_0$. Therefore,

$$\int_0^{y_0}dy\frac{y^{n-2}}{[D_{\max}(y,x,x')]^{2n-2}}\geqslant\frac{y_0^{n-1}}{n-1}\frac{1}{[D_{\max}(y_0,x,x')]^{2n-2}}, \quad (13)$$

and

$$\int_0^{y_0}dy\frac{y^{n-2}}{[D_{\min}(y,x,x')^{2n-2}]}\leqslant\frac{y_0^{n-1}}{n-1}\frac{1}{[D_{\min}(y_0,x,x')]^{2n-2}}, \quad (13')$$

where

$$D_{\max}(y_0,x,x')=m^2+x'(u/u'), \quad (14)$$

$$D_{\min}(y_0,x,x')=m^2+x(u'/u). \quad (14')$$

We shall exploit these inequalities in order to simplify the arithmetic of the bounding equations for A_1 and A_2.

Concerning the Born term $A_B=\frac{1}{2}K'$, we deal with a similar kernel and similar approximations, except that here one of the exchanged particles is taken to have finite mass μ. We can still exploit Eqs. (11) and (11′) in order to set lower and upper bounds, since the neglect of this finite mass acts in a direction which conforms to the inequalities. The Born term is evaluated at $x'=0$, and we note that

$$D_{\max}(y,x,\ x'=0)=m^2+y+x, \quad (15)$$

$$D_{\min}(y,x,\ x'=0)=m^2. \quad (15')$$

Furthermore, since the phase volume is made larger when the mass μ is neglected, we can in fact simply take Eq. (12′) as our upper bound for the Born term K'. To obtain a lower bound, however, we must replace Eq. (12) by one in which the phase volume reflects the fact that one of the particles has finite mass μ. Minorizing the resulting expression further, for later convenience, we then adopt the bounds $K_1'\leqslant K'\leqslant K_2'$, with

$$K_1'=K_1[1-(\mu^2/y)]^{n-1}, \quad (16)$$

$$K_2'=K_2. \quad (16')$$

III. LOWER BOUND

The minorizing inequality is given by Eq. (10), with kernel K_1 given by Eq. (12) and minorized Born term taken from Eq. (16). We minorize further by exploiting the inequality (13). For trial function A_1 we shall adopt the expression

$$A_1=c''u^\alpha(m^2+x)^{-\beta}\theta(ur-x)\theta(u-u_0)\,, \tag{17}$$

where $(1-r)u_0>\mu^2$, $1>r>0$. We are interested in finding the largest value of the parameter α consistent with our minorizing equation. Owing to the presence of the theta functions in Eq. (17), it is clear that we need only consider the minorization problem for $x<ur$, $u>u_0$. The minorizing equation in this range can then be written

$$\frac{u^\alpha}{(m^2+x)^\beta}<c'\frac{(u-x)^{n-2}}{(m^2+u)^{2n-2}}\left(1-\frac{\mu^2}{u-x}\right)^{n-1} +\frac{\lambda}{u}\int_{u_0}^{u}du'u'^\alpha\int_{xu'/u}^{u'r}dx'\frac{(u-u')^{n-1}[(x'/u')-(x/u)]^{n-1}}{(m^2+x')^{\beta+2}[m^2+x'(u/u')]^{2n-2}}, \tag{18}$$

where

$$c'=(16\pi^3/c'')(g/4\pi)^{2n}[1/(n-1)!(n-2)!]\,, \tag{19}$$

$$\lambda=(g/4\pi)^{2n}1/(n-1)!^2. \tag{19$'$}$$

Since our theta functions ensure that

$$u-x\geqslant u(1-r)\geqslant u_0(1-r)>\mu^2\,,$$

it is clear that we can get an absolute lower bound on the Born term [first term on the right side of Eq. (18)] in the form

$$\frac{c}{u^n}\leqslant c'\frac{(u-x)^{n-2}}{(m^2+u)^{2n-2}}\left(1-\frac{\mu^2}{u-x}\right)^{n-1}, \tag{20}$$

where the parameter c depends on u_0, r, and c' [hence on c'' of Eq. (17)]. Since these latter parameters are at our disposal, we can for later purposes choose c as large as needed. Introducing a new variable of integration $z=u'/u$, we now rewrite the minorizing equation in the following form:

$$\begin{aligned}(m^2+x)^{-\beta}-\lambda\int_0^1 dz\, z^{\alpha+n-1}(1-z)^{n-1}\int_{xz}^{\infty}dx'(x'-xz)^{n-1}(m^2+x')^{-\beta-2}(m^2z+x')^{-2n+2}\\ \leqslant cu^{-\alpha-n}-\lambda\int_0^1 dz\, z^{\alpha+n-1}(1-z)^{n-1}\int_{urz}^{\infty}dx'(x'-xz)^{n-1}(m^2+x')^{-\beta-2}(m^2z+x')^{-2n+2}\\ -\lambda\int_0^{u_0/u}dz\, z^{\alpha+n-1}(1-z)^{n-1}\int_{xz}^{urz}dx'(x'-xz)^{n-1}(m^2+x')^{-\beta-2}(m^2z+x')^{-2n+2}.\end{aligned} \tag{21}$$

The left-hand side of this inequality depends only on the variable x and must therefore by itself be nonpositive for all values of $x\geqslant0$. It is easily shown that this requirement sets the condition $\beta\geqslant\alpha+n\geqslant0$. However, it is also easily shown that α will be largest if we choose the equality

$$\alpha=\beta-n. \tag{22}$$

With this choice, the integral on the left side of Eq. (21) has the value

$$1/f(\alpha)[(n-1)!^2/m^{2n}(m^2+x)^\beta]\,, \tag{23}$$

with

$$f(\alpha)=(\alpha+2)(\alpha+3)\cdots(\alpha+n+1)(\alpha+n)(\alpha+n+1)\cdots(\alpha+2n-1). \tag{24}$$

The left-hand side of Eq. (21) will then be nonpositive for all $\alpha\leqslant\bar\alpha$, where $\bar\alpha$ is the largest root of

$$f(\alpha)=\lambda[(n-1)!^2/m^{2n}];$$

or, from Eq. (19′),

$$f(\alpha)=(g/4\pi m)^{2n}. \tag{25}$$

Concerning the right side of Eq. (21), one can show that the second term is bounded above, for all values of x and u, by an expression of the form (constant) $\times u^{-(\alpha+n)}$, the proportionality constant being a definite but uninteresting function of λ, α, and n. In the Born term, we can always choose the coefficient c large enough so that the Born term dominates the second term on the right side. Thus, with $\bar c$ some new constant, still at our disposal, we can rewrite Eq. (21) in the form

$$\left\{1-\frac{1}{f(\alpha)}\left(\frac{g}{4\pi m}\right)^{2n}\right\}(m^2+x)^{-\alpha-n}\leqslant\bar c n^{-\alpha-n} -\lambda\int_0^{u_0/u}dz\, z^{\alpha+n-1}(1-z)^{n-1}\int_{xz}^{urz}dx'(x'-xz)^{n-1}\times(m^2+x')^{-\beta-2}(m^2z+x')^{-2n+2}.$$

We replace this by the more stringent requirement

$$-\epsilon(m^2+x)^{-\alpha-n}\leqslant\bar c u^{-\alpha-n}-\lambda(m^2)^{-\alpha-n-2}\int_0^{u_0/u}dz\, z^{\alpha+n-1}\int_{xz}^{\infty}dx'(x'-xz)^{n-1}(m^2z+x')^{-2n+2} \leqslant\bar c u^{-\alpha-n}-\lambda(m^2)^{-\alpha-n-2}\frac{(n-1)!}{(\alpha+2)(2n-1)\cdots(n-2)}(u_0/u)^{\alpha+2}(m^2+x)^{-n+2}, \tag{26}$$

where $\epsilon=[1/f(\alpha)](g/4\pi m)^{2n}-1$. This last inequality is of the form $\lambda' R^{n-2}-\epsilon R^{\alpha+n}\leqslant \bar{c}$, where $R=u/(u+m^2)<1$. Thus, for any $\epsilon>0$, no matter how small, one can choose $\bar{c}(\epsilon)$ sufficiently large so that the inequality (26) is satisfied for all x and u in the range under discussion. We then conclude that the trial function (20) provides a lower bound on the absorptive amplitude for any α arbitrarily smaller than the largest root $\bar{\alpha}$ of Eq. (25). Finally, we remark that the physical amplitude, on the mass shell, is obtained by setting $x=0$. Then $u=s+x\rightarrow s$, and we conclude that the physical amplitude is bounded according to

$$A(s)\geqslant A_1(s)=B_1 s^{\alpha_1}, \tag{27}$$

where α_1 is arbitrarily close to $\bar{\alpha}$ and B_1 is a constant.

For small values of g (notice that $\alpha\rightarrow -2$ if $n>1$), we have no reason to think that this lower bound on the exponent is particularly realistic in the general case.[5] However, for the special case $n=1$, which corresponds to the straight ladder model, the solution of Eq. (25) coincides precisely with the exact solution, Eq. (1), obtained by Nakanishi.

Moreover, for any of the models (n arbitrary) we observe that in the strong coupling limit

$$f(\alpha)\underset{g\rightarrow\infty}{\longrightarrow}\alpha^{2n}\left[1+\frac{n(2n+1)}{\alpha}+\cdots\right];$$

hence

$$\alpha_1\underset{g\rightarrow\infty}{\longrightarrow}\frac{g}{4\pi m}-(n+\tfrac{1}{2})+\cdots. \tag{28}$$

Thus, for *all* of the models, the leading term in the strong coupling limit is the same quantity, $g/4\pi m$; and this coincides with the exact leading term for the straight ladder model.

IV. UPPER BOUND

To obtain an upper bound A_2 on the absorptive amplitude, we reverse the inequality direction in Eq. (10) and employ the majorizing inequalities of Eqs. (12′), (13′), and (16′). We shall majorize further by extending to zero the lower limit on the u' integration. The trial function is taken to be

$$A_2(u,x)=c'' u^{\alpha}(m^2+x)^{-\beta}, \tag{29}$$

where we are now interested in finding the smallest value of α for which the majorizing equation is satisfied, with c'' and β suitably chosen. In terms of a variable of integration $z=u'/u$, the majorizing equation then reads

$$(m^2+x)^{-\beta}-\lambda\int_0^1 dz\, z^{\alpha-n+1}(1-z)^{n-1}(m^2+xz)^{-2n+2}\int_{xz}^{\infty}dx'(x'-xz)^{n-1}(m^2+x')^{-\beta-2}$$
$$\geqslant\frac{c'}{m^{4n-4}}(u-x)^{n-2}u^{-\alpha}-\lambda\int_0^1 dz\, z^{\alpha-n+1}(1-z)^{n-1}(m^2+xz)^{-2n+2}\int_{uz}^{\infty}dx'(x'-xz)^{n-1}(m^2+x')^{-\beta-2}. \tag{30}$$

We are concerned with x and u in the range: $0\leqslant x\leqslant u$, $u\geqslant\mu^2$. The quantities c' and λ are again given by Eqs. (19) and (19′).

One now finds that the right side of Eq. (30) can be made negative for all x and u in the range of interest, provided that $\beta\leqslant\alpha-n+2$ and provided that c' is chosen small enough (the largest permissible value of c' depends on λ,α,β,n but is of no interest here). To obtain the smallest value of α consistent with the inequality (30) we shall in fact take

$$\alpha-n+2=\beta\geqslant 0. \tag{31}$$

Choosing c' small enough, we require now only that the left side of Eq. (30) shall be nonnegative for all $x\geqslant 0$. With the parameters α and β related according to Eq. (31), we find

$$\int_0^1 dz\, z^{\alpha-n+1}(1-z)^{n-1}(m^2+xz)^{-2n+2}\int_{xz}^{\infty}dx'(x'-xz)^{n-1}\times(m^2+x')^{-\beta-2}=\frac{1}{h(\alpha)}\frac{(n-1)!^2}{m^{2n}(m^2+x)^{\beta}}, \tag{32}$$

where

$$h(\alpha)=(\alpha+1)\alpha\cdots(\alpha-n+2)(\alpha-n+3)\times(\alpha-n+2)\cdots(\alpha-2n+4). \tag{33}$$

The majorizing equation is therefore satisfied for all $\alpha\geqslant\alpha_2$, where α_2 is the smallest root, consistent with $\beta=\alpha-n+2\geqslant 0$, of

$$h(\alpha)-(g/4\pi m)^{2n}. \tag{34}$$

For the physical amplitude, on the mass shell, we have

$$A(s)\leqslant B_2 s^{\alpha_2}, \tag{35}$$

where B_2 is a constant and α_2 is given by Eq. (34).

For the special case $n=1$, which corresponds to the straight ladder model, the solution of Eq. (34) gives an

[5] We know, in fact, that the amplitude behaves *at least* like s^{-1} (hence $\alpha\geqslant -1$), independent of g. This follows from the study of the large-s behavior of the type of nonplanar graphs associated with iterations of our irreducible kernels ($n>1$). For the asymptotics of nonplanar Feynman graphs see G. Tiktopoulos, Phys. Rev. **131**, 2373 (1963).

upper bound α_2 which coincides exactly with the lower bound α_1 obtained earlier, both agreeing with the correct value of the exponent obtained by Nakanishi. For the general model, we observe that in the strong coupling limit[6]

$$\alpha_2 \underset{g\to\infty}{\longrightarrow} \frac{g}{4\pi m}+(n-\tfrac{5}{2})+\cdots. \tag{36}$$

The leading term in α_2 is identical to the leading term for α_1; i.e., the ratio of upper to lower bound on the exponent approaches unity in the strong coupling limit. So the correct exponent is fully determined as regards the leading behavior in the strong coupling limit. The result, in this limit, is common to *all* the models under discussion.

Finally, let us briefly consider including more than one irreducible graph in the kernel. For instance, we can include all such graphs of the same order $2n$. It is easily seen that there are at least $(n-1)!$ irreducible graphs of order $2n$ (within the class considered in this paper). We can, therefore, employ the procedure used in Sec. III for the $2n$th order kernel, the only difference being a factor of $(n-1)!$. The lower bound on the corresponding amplitude will then be of the form s^α with

$$f(\alpha)=(n-1)!(g/4\pi m)^{2n},$$

so that $\alpha\to n^{1/2}(g/4\pi m)$ in the strong coupling limit. This shows that for the amplitude $A_{\rm tot}$ generated by a kernel including all irreducible graphs of all orders the exponent α must grow faster than linearly in g. Since, clearly, a lower bound for this kernel is

$$K>\sum_{n=1}^{\infty}(n-1)!\times 32\pi^3\left(\frac{g}{4\pi}\right)^{2n}\frac{1}{(n-1)!}\frac{{y_0}^{n-1}}{{D_{\max}}^{2n-2}} =2\pi g^2\exp[(g/4\pi)^2 y_0 D_{\max}{}^{-2}],$$

a minorizing integral inequality for $A_{\rm tot}$ will then be

$$A_{\rm tot}(u,x)<\pi g^2\delta(s-\mu^2) +16\pi^3\left(\frac{g}{4\pi}\right)^4\frac{1}{(m^2+u)^2}\exp\left[\left(\frac{g}{4\pi}\right)^2\frac{(u-x)}{(m^2+u)^2}\right] +\left(\frac{g}{4\pi}\right)^2\int_{\mu^2}^{u}\frac{du'}{u}\int_{x(u'/u)}^{u'}dx'\frac{A_{\rm tot}(u',x')}{(m^2+x')^2} \times\exp\left[\left(\frac{g}{4\pi}\right)^2\frac{(u-u')(x'/u'-x/u)}{(m^2+x'u/u')^2}\right].$$

One can obtain a lower bound on $A_{\rm tot}$ of the form $u^\alpha(x+m^2)^{-\beta}$ in which α grows *quadratically* with g in the strong coupling limit. However, we have no reason to think that this bound cannot be considerably improved. The point is, of course, that the exponential kernel is too complicated to permit one to carry out the integrations even for the simplest kinds of trial functions.

[6] For $n>1$, we do not expect α_2 to be a good approximation to the true exponent for small values of g. In fact in the weak coupling limit we find $\alpha_2\to n-2$, whereas one expects that $\alpha\to -1$. More generally, it is plausible that $\alpha\leqslant\alpha_L$ where $\alpha_L=-\frac{3}{2}+[\frac{1}{4}+(g/4\pi m)^2]^{1/2}$ is the known exponent for the straight ladder model. At least for a subclass of our models, one can in fact show rigorously that the forward absorptive amplitude is everywhere bounded from above by that of the straight ladder.

PHYSICAL REVIEW VOLUME 137, NUMBER 6B 22 MARCH 1965

Corrections to the Octuplet Spurion in the Nonleptonic Decays of the Hyperons

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(Received 9 November 1964)

Corrections due to the 27-plet spurion and the second-order electromagnetic effects are calculated to the octuplet spurion in the nonleptonic hyperon decays. The 27-plet spurion predicts a relation among small deviations from the $\Delta I=\frac{1}{2}$ rule, namely $(\langle\Lambda|p\pi^-\rangle+\sqrt{2}\langle\Lambda|n\pi^0\rangle)=-(\langle\Xi^-|\Lambda\pi^-\rangle+\sqrt{2}\langle\Xi^0|\Lambda\pi^0\rangle)$ for the parity-violating amplitudes. This holds as it stands if the second-order electromagnetic effects are introduced on the assumption of the octuplet tadpole mechanism. A test of this relation, although still not possible with present experimental data, has a deep significance for the structure of the weak interactions.

1. INTRODUCTION

UNITARY symmetry[1–3] predicts a relation

$$2\langle\Xi^-|\Lambda\pi^-\rangle-\sqrt{3}\langle\Sigma^+|p\pi^0\rangle+\langle\Lambda|p\pi^-\rangle=0 \tag{1}$$

among the parity-violating amplitudes of the nonleptonic hyperon decays on the following assumptions[4]:

(a) The strong interactions are fully invariant under SU_3.

(b) The weak interactions are of the current × current type[5,6] and are CP-invariant.

(c) Among the spurions arising from the current

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