Covariant Description of Several Spinless Particles*

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To describe a state of n particles it is necessary to construct a matrix element or wave function from the momentum vectors of the n particles. It is usually possible to write down a simple function having the correct angular momentum and parity. It is not clear in what sense the choice made is general. This is in contrast to the two-particle states, where the spherical harmonics form a complete orthonormal set over the phase space. The spherical harmonics are homogeneous polynomials in the components of the relative momentum of the two particles. It will be shown that homogeneous polynomials in the n-1 relative momenta of n particles entering an n-particle state, with a correction term for relativistic kinematics, form a complete orthonormal set of functions over the n-body phase space and provide a basis for a systematic classification of n-body states. There are some new quantum numbers (degeneracy indices) that enter and may or may not have physical significance. The application of these notions to ω decay is briefly considered. The basis of this classification is the determination of a larger invariance group than the rotations for a system of free particles. The Lie algebra of generators of this group furnishes a complete commuting set of operators, and it is exhibited. The eigenfunctions of this set are given.

I. INTRODUCTION

IN treating several particles covariantly, the usual approach is to combine a pair and fix its mass and spin. This pair then behaves under Lorentz transformations like a particle and the process may be iterated to give a description for any number of particles.[1] There is the problem of what order the particles should be combined in, and it is possible to introduce recoupling coefficients to relate the alternatives. This process seems unnatural when the particles enter symmetrically as, for example, do the three pions from ω decay. Koba developed a method for treating three- and four-particle states more symmetrically with nonrelativistic kinematics.[2] The generalization of this method to the relativistic case is given in the following sections. More emphasis will be placed on the underlying group-theoretical notions than was done in Ref. 2.

In a relativistic situation, it is convenient to treat the particles in a fixed coordinate system (center of mass) and then construct the general state by making a pure Lorentz transformation. The essential result is that an arbitrary Lorentz transformation entails only a rotation on the center-of-mass states.[3] These rotations are called the "little group." For several noninteracting particles, the little group is a subgroup of the invariance group. It is easy to see that the nonrelativistic, noninteracting Hamiltonian is invariant under phase-space rotations, as well as the usual spatial ones. This wider invariance provides the basis for Koba's classification of the states of several mesons. It will be shown that the same group exists in the covariant problem.

The work is carried out in momentum space in contrast to Koba's work, which was done in coordinate space, for two reasons. First, practically, one wants to display the results as a density distribution on a Dalitz plot or similar diagram for more than three particles. Secondly, the wave function is much more complicated in coordinate than in momentum space. In an Appendix the coordinate-space treatment of the two-body problem is given.

The states considered in the following sections are always represented as linear combinations of plane-wave states. This again is for mathematical convenience and ease of physical interpretation. The experimentally observable quantities are the distribution of plane waves in a state. The function specifying the linear combination is called the wave function, and its determination is the object of this work. The three-particle state is treated explicitly for definitiveness, although some proofs are given for n particles. The generalization to n only requires some notation.

II. WAVE FUNCTION

A linear combination of three plane-wave states is sought that will transform according to an irreducible representation of the Lorentz Group. The plane-wave state of a particle with mass m, momentum k, and energy ω will be denoted by $|k\omega\rangle$. Under Lorentz transformations, they behave like

$$L|k,\omega\rangle=|k'\omega'\rangle\,, \qquad (1)$$

where $k_\mu'=L_{\mu\nu}k_\nu$. The two L's should not be confused; the one that occurs in Eq. (1) is an operator in Hilbert space; $L_{\mu\nu}$ is a four by four matrix.

The required state ψ is given by

$$\psi=\int d^4k_1d^4k_2d^4k_3F(k_1,k_2,k_3)|k_1m_1\rangle|k_2m_2\rangle|k_3m_3\rangle. \qquad (2)$$

The behavior of ψ under Lorentz transformation follows from that for $|k,\omega\rangle$:

$$L\psi=|d^4k_1d^4k_2d^4k_3F(k_1,k_2,k_3)L|k_1\omega_1\rangle L|k_2\omega_2\rangle L|k_3\omega\rangle. \qquad (3)$$

The function F depends on 12 variables, but seven of

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[1] A. J. Macfarlane, Rev. Mod. Phys. 34, 41 (1962).
[2] Z. Koba, Acta Physiol. Polon. Suppl. 22, 103 (1962); M. Grynberg and Z. Koba, Acta Physiol. Polon. 23, 501 (1963).
[3] E. P. Wigner, Ann. Math. 40, 139 (1939).

them are immediately fixed by noting that the state must have an energy-momentum four-vector K_μ, and the mass of each particle is fixed. Thus ψ may be written

$$\psi=\int d^4k_1d^4k_2d^4k_3\delta(k_1+k_2+k_3-K)\times\delta(k_1^2-m_1^2)\delta(k_2^2-m_2^2)\delta(k_3^2-m_3^2)\times f(k_1k_2k_3)|k_1\omega_1\rangle|k_2\omega_2\rangle|k_3\omega_3\rangle. \quad (4)$$

The domain of integration is just the usual three-body phase space. For n particles, the analogous integration would be carried over the n-body phase space. In general, n masses of the single-particle states are fixed, and the four variables that describe the energy and momentum of the n-body system. Since there are $4n$ variables, the remaining integral is over $3n-4$ variables.

Returning now to the three-body case, integrate out all the δ functions except the energy-conserving one. The state is described by two momentum vectors $\mathbf{p}$, $\mathbf{q}$; they may be two of the original three or some convenient linear combination. Dalitz[4] and Koba[2] have used

$$\mathbf{p}=(\mathbf{p}_1-\mathbf{p}_2)/2^{1/2},\quad \mathbf{q}=(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)/6^{1/2}, \quad (5)$$

and these will be convenient for illustration.

The argument of the energy-conserving δ function may be considered as a multiplication operator $\mathcal{R}$ in a Hilbert space. In the nonrelativistic theory, this $\mathcal{R}$ would just be equal to $H-E$, the Hamiltonian minus the energy. The wave functions are the solutions of $(H-E)\psi=0$. In the covariant theory, the solutions of $\mathcal{R}\psi=0$ will again be the wave functions, although the operator is more complicated. If the substitution of Eq. (5) is used for equal masses, the operator $\mathcal{R}$ is given in the center of mass by

$$\mathcal{R}=\left(\frac{\mathbf{p}^2}{2}+\frac{\mathbf{p}\cdot\mathbf{q}}{(12)^{1/2}}+\frac{\mathbf{q}^2}{6}+m^2\right)^{1/2}+\left(\frac{\mathbf{p}^2}{2}-\frac{\mathbf{p}\cdot\mathbf{q}}{(12)^{1/2}}+\frac{\mathbf{q}^2}{6}+m^2\right)+\left(\frac{4\mathbf{q}^2}{6}+m^2\right)^{1/2}-M \quad \text{(covariant)},$$

$$\mathcal{R}=3m+\frac{\mathbf{p}^2}{2m}+\frac{\mathbf{q}^2}{2m}-M \quad \text{(nonrelativistic)}, \quad (6)$$

where M is the mass of the three-particle state. The solution of the equation $\mathcal{R}\psi=0$ involves the energy-conserving δ function times a function of the remaining variables. The invariance properties of the operator $\mathcal{R}$ suggests ways to choose this function. For example, for a single nonrelativistic particle, the operator $\mathcal{R}$ is $\mathcal{R}=(\mathbf{p}^2/2m)-E$, and the equation is

$$\mathcal{R}\psi=[(\mathbf{p}^2/2m)-E]\psi=0.$$

The solution is $\psi_{Elm}(p,\theta,\varphi)=\delta[(\mathbf{p}^2/2m)-E]Y_{lm}(\theta,\varphi)$. The choice of Y_{lm} is made because $\mathcal{R}$ is rotationally invariant. That is, the operators L_x, L_y, L_z commute with $\mathcal{R}$ and the solution is a simultaneous eigenfunction of L and $\mathcal{R}$. The L's have the property $[L,\mathcal{R}]=0$ for this problem. In the more general case, operators $\mathcal{H}$ such that $[\mathcal{H},\mathcal{R}]=0$ are found and used to classify the solutions. Mathematically, the operators $\mathcal{H}$ are said to form a Lie alegbra. They are the infinitesimal generators of the invariance group of the function $\mathcal{R}$. That is, they generate infinitesimal coordinate transformations that transform the surface $\mathcal{R}$ equals a constant into itself.

III. LIE ALGEBRA

Consider a function of n variables $F(x_1,x_2,\cdots,x_n)$. The problem is find a set of operators $\mathcal{H}_a$ such that

$$[\mathcal{H}_a,F]=0 \quad \text{(commute with the Hamiltonian)},$$
$$[\mathcal{H}_a,\mathcal{H}_b]=C_{ab}{}^c\mathcal{H}_c \quad \text{(form a Lie algebra)}, \quad (7)$$
$$\mathcal{H}_a{}^\dagger=\mathcal{H}_a \quad \text{(Hermitian)}.$$

For the case when F is a sum of squares,

$$F=x_1^2+x_2^2+\cdots+x_n^2,$$

there are $n(n-1)/2$ operators

$$L_{ab}=-i(x_a\partial/\partial x_b-x_b\partial/\partial x_a)$$

with the commutation relation

$$[L_{ab},L_{cd}]=i(\delta_{ac}L_{bd}+\delta_{bd}L_{ac}+\delta_{ad}L_{cb}+\delta_{bc}L_{da}). \quad (8)$$

This is a simple generalization of the known properties of the rotation group to the orthogonal group in n variables O_n. For F equal to a sum of squares, the orthogonal transformation moves a point of the surface F=constant to another point on that surface. If F is more complicated, a rotation will move a point off the surface F=constant. This may be compensated by making a rotation followed by a radial displacement. The simplest rotations are those about axes such that only two variables change:

$$x_i'=x_i\cos\theta-x_j\sin\theta,$$
$$x_j'=x_i\sin\theta+x_j\cos\theta,$$
$$x_s'=x_s,\quad s\neq i \text{ or } j.$$

Stated in terms of differentials for a small θ,

$$x_i'=x_i-x_jd\theta,$$
$$x_j'=x_j+x_id\theta,$$
$$x_s'=x_s.$$

At the new point x', the function F has changed to

$$F'=F+dF=F+[x_i(\partial F/\partial x_j)-x_j(\partial F/\partial x_i)]d\theta.$$

[4] R. H. Dalitz, Phil. Mag. 44, 1068 (1953); Phys. Rev. 94, 1046 (1954).

To return to the surface $F=\text{constant}$, it is necessary to make a radial displacement such that

$$dF=-(x_i(\partial F/\partial x_j)-x_j(\partial F/\partial x_i))d\theta\,,$$

but for a radial displacement $dF=(\partial F/\partial r)dr$, so that

$$dr=-\frac{x_i(\partial F/\partial x_j)-x_j(\partial F/\partial x_i)}{\partial F/\partial r}d\theta\,.$$

In terms of Cartesian coordinates, $dx_a=x_a dr/r$. The combined displacements give

$$\begin{aligned} dx_i&=(-x_j-Rx_i)d\theta\,,\\ dx_j&=(x_i-Rx_j)d\theta\,,\\ dx_s&=-Rx_s d\theta \quad s\neq i,j\,, \end{aligned}$$

where $R=(x_i\partial F/\partial x_j-x_j\partial F/\partial x_i)/\sum_a (x_a\partial F/\partial x_a)$. The change in an arbitrary function G under this coordinate transformation is given by

$$\begin{aligned} dG&=\sum\frac{\partial G}{\partial x_a}dx_a=\sum\frac{\partial G}{\partial x_a}\left(\frac{dx_a}{d\theta}\right)d\theta\\ &=i\mathcal{L}_{ij}Gd\theta\,, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_{ij}&=(-i)\textstyle\sum_a (dx_a/d\theta)(\partial/\partial x_a)\\ &=(-i)\left(x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}\right)+iRx_a\frac{\partial}{\partial x_a}\\ &=L_{ij}+iRx_a(\partial/\partial x_a)\\ &=L_{ij}-\textstyle\sum Rx_a p_a\,. \end{aligned}$$

The $\mathcal{L}_{ab}$ satisfy the same commutation rule as L_{ab} and they commute with F, the first two requirements of Eq. (7), but they are not Hermitian. The Hermitian conjugates of $\mathcal{L}_{ab}$ is given by

$$\begin{aligned} \mathcal{L}_{ab}{}^\dagger&=L_{ab}{}^\dagger-\textstyle\sum p_a{}^\dagger x_a{}^\dagger R^\dagger\\ &=L_{ab}-p_a x_a R\\ &=L_{ab}-Rx_a p_a+i(\partial/\partial x_a)(x_a R)\\ &=\mathcal{L}_{ab}+i(\partial/\partial x_a)(x_a R)\,. \end{aligned}$$

If $\mathcal{L}_{ab}$ is separated into a Hermitian part $\mathcal{H}_{ab}$ and an anti-Hermitian part A_{ab}, both parts commute with F, and they are given by

$$\begin{aligned} \mathcal{H}_{ab}&=\mathcal{L}_{ab}+(i/2)(\partial/\partial x_a)(x_a R)\,,\\ A_{ab}&=-(i/2)(\partial/\partial x_a)(x_a R)\,. \end{aligned}$$

The $\mathcal{H}_{ab}$ satisfy the same commutation rule as do the $\mathcal{L}$'s. This may be seen by taking the difference in the commutators for $\mathcal{L}$ and $\mathcal{L}^\dagger$. The commutator for $\mathcal{L}$ written in terms of $\mathcal{H}$ and A is

$$\begin{aligned} &[\mathcal{H}_{ab},\mathcal{H}_{cd}]+[\mathcal{H}_{ab},A_{cd}]+[A_{ab},\mathcal{H}_{cd}]+[A_{ab},A_{cd}]\\ &\quad=i(\delta_{ac}\mathcal{H}_{bd}+\delta_{bd}\mathcal{H}_{ac}+\delta_{ad}\mathcal{H}_{cb}+\delta_{bc}\mathcal{H}_{da})\\ &\qquad\qquad+i(\delta_{ac}A_{bd}+\delta_{bd}A_{ac}+\delta_{ad}A_{cb}+\delta_{bc}\mathcal{H}_{da})\,. \end{aligned}$$

The term $[A_{ab},A_{cd}]$ vanishes. Under Hermitian conjugation, the equation splits into two parts: odd and even. The odd part is just the required commutation relation for the $\mathcal{H}$'s.

From the set of operators L_{ij} or $\mathcal{H}_{ij}$ it is necessary to choose a complete commuting set. The larger the number of particles the more complicated this set will be, and unlike the two-particle case there is more than one choice. The number of operators required is $3n-4$, since there are that many variables that are not fixed by δ functions. In the three-body case, it has been conventional to take $J_p{}^2$, the angular momentum associated with the 1–2 pair in Eq. (5); $J_q{}^2$, the angular momentum of 3 relative to the 1–2 center of mass; and J^2 and J_3, the total angular momentum and its z component. In addition, Koba introduces an operator Λ which is the sum of the squares of the 15 possible L_{ab}. This same set will be satisfactory in the relativistic problem, since the L's and $\mathcal{H}$'s have the same commutation rules.

The eigenfunction of the set of L's are like the spherical harmonics; they depend on the angles of a point not its radius. It is easy to see that the same functions are eigenfunctions of the analogous set of $\mathcal{L}$'s, since the difference between the L's and $\mathcal{L}$'s is in the term $\sum x_a(\partial/\partial x_a)$, but this is just $r(\partial/\partial r)$, and vanishes for a function of the angles. To find eigenfunctions of the $\mathcal{H}_{ij}$, one further modification is necessary. The eigenfunctions of the sets L and $\mathcal{L}$ are not orthogonal with the integral $\int d^n x\,\delta(F)$. If a change is made to polar coordinates, this integral becomes $\int r^{n-1}drd\Omega\,\delta(F)$, where $d\Omega$ stands for the $(n-1)$-dimensional polar coordinates. If the δ function is used to do the radial integration, the integral becomes

$$\int d\Omega\; r^{n-1}(F)(dF/dr)\,. \tag{9}$$

Since the eigenfunctions of L are orthonormal with the integral $d\Omega$, it is suggested that the eigenfunctions of the set $\mathcal{H}$ will be those of L divided by $[r^{n-1}(dF/dr)]^{1/2}$. If φ_A are the eigenfunctions of the set of operators L, then

$$\int d\Omega\;\varphi_A\varphi_B=\delta_{AB}\,,$$

where A and B stand for a set of $n-1$ indices. The functions

$$\psi_A=\{[x_a(\partial F/\partial x_a]/r^n\}\,\varphi_A{}^{1/2} \tag{10}$$

are orthogonal with the integral (9). It is further true that the functions ψ_A are the eigenfunctions of $\mathcal{H}$. This follows from the equation

$$\mathcal{H}_{ab}\left(\frac{x_a(\partial F/\partial x_a)}{r^n}\right)^{1/2}\varphi=\left(\frac{x_a(\partial F/\partial x_a)}{r^n}\right)^{1/2}L_{ab}\varphi\,. \tag{11}$$

To summarize the results of this section, if $F(x_1\cdots x_n)=\text{constant}$ is the energy-conserving surface, then the operators

$$\mathcal{H}_{ab}=L_{ab}-\frac{L_{ab}F}{(\mathbf{r}\cdot\mathbf{p}F)}\mathbf{r}\cdot\mathbf{p}+\frac{i}{2}\frac{\partial}{\partial x_a}\left(x_a\frac{LF}{\mathbf{r}\cdot\mathbf{p}F}\right)$$

commute with F and are the Hermitian generators of the Lie group that commutes with F. The function ψ_A

$$\psi_A=\left(\frac{x_a(\partial F/\partial x_a)}{r^n}\right)^{1/2}\varphi_A$$

are the simultaneous eigenfunctions of a complete set of operators, if φ_A are the eigenfunctions of the analogous set of operators for the orthogonal group.

IV. APPLICATION TO ω DECAY

One of the great advantages of the Koba wave functions is that they can readily and naturally be classified according to permutation symmetry. This is true for the same reason that the spherical harmonics have this property for two-body problems. The Koba functions with index Λ are homogeneous polynomials of degree Λ in the components of the vectors $\mathbf{p}$ and $\mathbf{q}$, divided by the radius $(\mathbf{p}^2+\mathbf{q}^2)^{\Lambda/2}$ to make them dimensionless. The spherical harmonics of order l is a homogeneous polynomial of degree l in the components of the single vector describing the two-body system. The Koba wave functions resemble the spherical harmonics in another respect that makes them useful for treating collision phenomena. A partial-wave expansion is useful, because at moderate energies only the lower partial waves are expected to contribute significantly. This is because the wave function behaves like $(kr)^l$ at the origin. In a similar way, the exponent index Λ measures behavior at the origin of the wave function. For the three-body problem, the origin means the vanishing of the quantity

$$R=[(\mathbf{r}_1-\mathbf{r}_2)^2+(\mathbf{r}_2-\mathbf{r}_3)^2+(\mathbf{r}_3-\mathbf{r}_1)^2]^{1/2};$$

at the origin the Koba wave function behaves like $(KR)^\Lambda$. A generalized partial-wave expansion that uses Λ as the principal index may be justified in problems with a low Q value.

In the case of ω decay, an antisymmetric spatial wave function is required since the isospin is 0. In Koba's table the first antisymmetric functions occur for $\Lambda=2$ and has $J=1$. It is easy to construct in terms of the three momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$.

$$\begin{aligned}\psi&=(\mathbf{p}_1\times\mathbf{p}_2+\mathbf{p}_2\times\mathbf{p}_3+\mathbf{p}_3\times\mathbf{p}_2)/N\\&=(\mathbf{p}\times\mathbf{q})/(\mathbf{p}^2+\mathbf{q}^2).\end{aligned}$$

The equality is always up to constant factors. The constant N is given by

$$N=(\mathbf{p}_1-\mathbf{p}_2)^2+(\mathbf{p}_2-\mathbf{p}_3)^2+(\mathbf{p}_3-\mathbf{p}_1)^2$$

in terms of the three momenta. This is the same as the matrix element used in the original discussion of ω decay.[5]

[5] M. L. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, Phys. Rev. **125**, 687 (1962).

On the basis of the preceding discussion, it seems reasonable to include the relativistic correction. The function $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R}&=\omega_1+\omega_2+\omega_3-M\\&=(\mathbf{p}^2/2+\mathbf{p}\cdot\mathbf{q}/3^{1/2}+\mathbf{q}^2/6+m^2)^{1/2}\\&\quad+(\mathbf{p}^2/2-\mathbf{p}\cdot\mathbf{q}/3^{1/2}+\mathbf{q}^2/6+m^2)^{1/2}\\&\quad+(2\mathbf{q}^2/3+m^2)^{1/2}-M.\end{aligned}$$

The correction factor C is $(p\partial R/\partial p+q\partial R/\partial q)^{1/2}/(\mathbf{p}^2+\mathbf{q}^2)^{3/2}$. In terms of the ω's, this is

$$C=[M-m^2(1/\omega_1+1/\omega_2+1/\omega_3)]^{1/2}/(\omega_1^2+\omega_2^2+\omega_3^2-3m^2)^{3/2},$$

where a numerical constant has been dropped. This factor is almost constant over the Dalitz plot and should make no change in the earlier analysis. The ratio C at the center to C at the edge of the Dalitz plot is

$$\left(\frac{3}{2}\right)^{3/2}\left(\frac{M-m}{M}\right)^{1/2}\left(\frac{M+m}{M+3m}\right).$$

For a nonrelativistic case, $M=3m$, and this factor is 1; for the extreme relativistic case, M/m infinite, the ratio is $(\frac{3}{2})^{3/2}$.

The second antisymmetric state is a $\Lambda=J=3$ state. It is easy to write down in terms of the three momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ as a third-rank tensor. To avoid subscripts, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are used for $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

$$\begin{aligned}\psi_{ijk}=\,&a_ia_jb_k+b_ib_jc_k+c_ic_ja_k-a_ia_jc_k-b_ib_ja_k-c_ic_jb_k\\&+a_ib_ja_k+b_ic_jb_k+c_ia_jc_k-a_ic_ja_k-b_ia_jb_k-c_ib_jc_k\\&+b_ia_ja_k+c_ib_jb_k+a_ic_jc_k-c_ia_ja_k-a_ib_jb_k-b_ic_jc_k\\&-\tfrac{1}{5}(\mathbf{c}^2-\mathbf{b}^2+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{c})(\delta_{ij}a_k+\delta_{ik}a_j+\delta_{jk}a_i)\\&-\tfrac{1}{5}(\mathbf{a}^2-\mathbf{c}^2+2\mathbf{b}\cdot\mathbf{c}-2\mathbf{a}\cdot\mathbf{b})(\delta_{ij}b_k+\delta_{ik}b_j+\delta_{jk}b_i)\\&-\tfrac{1}{5}(\mathbf{b}^2-\mathbf{a}^2+2\mathbf{a}\cdot\mathbf{c}-2\mathbf{b}\cdot\mathbf{c})(\delta_{ij}c_k+\delta_{ik}c_j+\delta_{jk}c_i).\end{aligned}$$

The identification of the tensor indices with the z component of angular momentum is difficult but unnecessary. Since the ω sample is unpolarized, the required quantity is the average over the magnetic quantum number which is $\psi_{ijk}\psi_{ijk}$. After some calculation, this is, except for numerical factors,

$$\begin{aligned}&27(\mathbf{p}_1^6+\mathbf{p}_2^6+\mathbf{p}_3^6)\\&\quad-109(\mathbf{p}_1^4\mathbf{p}_2^2+\mathbf{p}_1^4\mathbf{p}_3^2+\mathbf{p}_1^2\mathbf{p}_2^4+\mathbf{p}_1^2\mathbf{p}_3^4+\mathbf{p}_2^4\mathbf{p}_3^2+\mathbf{p}_2^2\mathbf{p}_3^4)\\&\quad-42\mathbf{p}_1^2\mathbf{p}_2^2\mathbf{p}_3^2,\end{aligned}$$

which may be compared with the experimental distribution. The successive states may be tested in the same manner until one is confident that all reasonable possibilities are excluded.

V. CONCLUSIONS

The states of nonrelativistic particles may, for some purposes, be classified by using the invariance of the Hamiltonian under phase-space rotations. The added invariance property can be extended to the relativistic problem. This leads to a complete orthonormal set of wave functions. The larger symmetry group is not implied in any dynamical problem. Thus the quantum numbers associated with the higher symmetry are at best approximate. The utility of such quantum numbers may nonetheless be high, as, for example, the spin and orbital angular momentum in atomic physics. The group that arises here is the n-dimensional orthogonal group; in problems with spin, the group SU_n will arise in a natural way. It seems reasonable to look for the origins of unitary symmetries in the approximate quantum numbers found here.

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APPENDIX: POSITION-SPACE TREATMENT OF THE TWO-BODY PROBLEM

The equations governing the two-body case are

$$p_1^2=m_1^2\,,$$
$$p_2^2=m_2^2\,,$$
$$(p_1+p_2)^2=M\,,$$

where p_i and m_i are the four-momenta and mass of the ith particle and M is the mass of the two-body state. The p's are now to be interpreted as differential rather than multiplication operators. If the last equation is expanded and the first two used to simplify, the result is

$$\tilde{\omega}_1\tilde{\omega}_2-\mathbf{p}_1\cdot\mathbf{p}_2=\tfrac{1}{2}(M^2-m_1^2-m_2^2)\,,$$

where $\tilde{\omega}_i=(\mathbf{p}_i^2+m_i^2)^{1/2}$. This equation is squared again to eliminate the radicals; the result is a fourth-order equation for the two-body problem.

$$\mathbf{p}_1^2\mathbf{p}_2^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2+m_2^2\mathbf{p}_1^2+m_1^2\mathbf{p}_2^2+(m_1^2+m_2^2-M^2)\mathbf{p}_1\cdot\mathbf{p}_2=M^2p_0^2$$
$$p_0^2=(M^4+m_1^4+m_2^4-2M^2m_1^2-2M^2m_2^2-2m_1^2m_2^2)/4M^2\,.$$

The transformation to relative and center-of-mass coordinates is made with the change of variables

$$\mathbf{p}_1=(\omega_1/M)\mathbf{P}+\mathbf{p}\,,\quad \mathbf{P}=\mathbf{p}_1+\mathbf{p}_2\,,$$
$$\mathbf{p}_2=(\omega_2/M)\mathbf{P}-\mathbf{p}\,,\quad \mathbf{p}=(\omega_2/M)\mathbf{p}_1-(\omega_1/M)\mathbf{p}_2\,,$$

where

$$\omega_1=(M^2+m_1^2-m_2^2)/2M\quad \omega_2=(M^2-m_1^2+m_2^2)/2M;$$

and these are the center-of-mass energies of particles one and two. The change in spatial coordinates is given by

$$\mathbf{r}_1=\mathbf{R}+(\omega_2/M)\mathbf{r}\,,\quad \mathbf{R}=(\omega_1/M)\mathbf{r}_1+(\omega_2/M)\mathbf{r}_2\,,$$
$$\mathbf{r}_2=\mathbf{R}-(\omega_1/M)\mathbf{r}\,,\quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2\,.$$

The center-of-mass coordinates are $\mathbf{P}$ and $\mathbf{R}$, and the internal coordinates are $\mathbf{p}$ and $\mathbf{r}$. Both of these pairs of variables are canonical, if $\mathbf{p}_1\mathbf{r}_1$ and $\mathbf{p}_2\mathbf{r}_2$ are canonical pairs. In these coordinates, the equation becomes

$$\mathbf{P}^2\mathbf{p}^2-(\mathbf{p}\cdot\mathbf{P})^2+M^2\mathbf{p}^2-\mathbf{P}^2p_0^2=p_0^2M^2\,.$$

The wave function should contain $e^{i\mathbf{P}\cdot\mathbf{R}}$ from general considerations,[3] so that $\mathbf{P}$ may be considered a number, not an operator. The quantity $\mathbf{P}^2+M^2$ will be called Ω^2. With these changes, the equation is

$$\mathbf{p}^2-(\mathbf{p}\cdot\mathbf{P})^2/\Omega^2=p_0^2\,.$$

A final substitution,

$$\mathbf{p}=\mathbf{p}'+[(\Omega-M)/\mathbf{P}^2M](\mathbf{P}\cdot\mathbf{p}')\mathbf{P}\,,$$
$$\mathbf{r}=\mathbf{r}'-[(\Omega-M)/\mathbf{P}^2r](\mathbf{P}\cdot\mathbf{r}')\mathbf{P}\,,$$
$$\mathbf{p}'=\mathbf{p}+[(M-\Omega)/\mathbf{P}^2\Omega](\mathbf{P}\cdot\mathbf{p})\mathbf{P}\,,$$
$$\mathbf{r}'=\mathbf{r}+[(\Omega-M)/\mathbf{P}^2M](\mathbf{P}\cdot\mathbf{r})\mathbf{P}\,,$$

reduces the equation to

$$\mathbf{p}'^2=p_0^2\,,$$

the nonrelativistic result. The solution to this equation is

$$\psi=e^{i\mathbf{P}\cdot\mathbf{R}}j_l(p'r')Y_{lm}(\theta',\varphi')\,.$$

The substitutions used can be motivated most easily by studying the two-body system in its center of mass and then making Lorentz transformations. The following alternative approach is also interesting. The vector P is determined by general arguments about Lorentz invariance. Thus the forms of p_1 and p_2 are

$$\mathbf{p}_1=(\tfrac{1}{2}+A)\mathbf{P}+\mathbf{q}\,,$$
$$\mathbf{p}_2=(\tfrac{1}{2}-A)\mathbf{P}-\mathbf{q}\,,$$

since $\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}$. The difference in energy squared is easily calculated and is needed for the ensuing argument:

$$\omega_1^2-\omega_2^2=\Omega(\omega_1-\omega_2)=2A\mathbf{P}^2+2\mathbf{P}\cdot\mathbf{q}+m_1^2-m_2^2\,,$$

where $\omega_i=(\mathbf{p}_i^2+m_i^2)^{1/2}$. The choice of A is made so that $\omega_1+\omega_2$ is constant:

$$\begin{aligned}0=d(\omega_1+\omega_2)&=\mathbf{p}_1d\mathbf{p}_1/\omega_1+\mathbf{p}_2d\mathbf{p}_2/\omega_2\\&=(2A\mathbf{P}^2dA+2A\mathbf{P}\cdot d\mathbf{q}+\mathbf{P}\cdot\mathbf{q}dA+2\mathbf{q}\cdot d\mathbf{q})\\&\quad\times(1/\omega_1+1/\omega_2)+(\mathbf{P}^2dA+\mathbf{P}\cdot d\mathbf{q})(1/\omega_1-1/\omega_2)\,.\end{aligned}$$

After substituting the above result for $\omega_1-\omega_2$, this becomes

$$[(2\Omega^2\mathbf{P}^2-2\mathbf{P}^4)A+(2\Omega^2-2\mathbf{P}^2)\mathbf{P}\cdot\mathbf{q}+(m_2^2-m_1^2)\mathbf{P}^2]dA+[(2\Omega^2-2\mathbf{P}^2)A-2\mathbf{P}\cdot\mathbf{q}-m_1^2+m_2^2]\mathbf{P}\cdot d\mathbf{q}=0\,.$$

The equation is exact and may be integrated; with an appropriate choice of the constant of integration, A given by

$$A=[\mathbf{P}^2(m_1^2-m_2^2)-2M^2(\mathbf{P}\cdot\mathbf{q})\pm2M\Omega(\mathbf{P}\cdot\mathbf{q})]/2M^2\mathbf{P}^2\,,$$

which is the preceding substitution.