TABLE I. The five largest eigenvalues and projections b_i for the scattering of scalar mesons with the exchange of a mass $3\mu^2$ scalar meson. The value of s_0 was also $3\mu^2$.

Number	Eigenvalue	b_i
1	0.160	−0.334
2	0.00945	0.0283
3	0.00118	−0.00464
4	0.000235	+0.00106
5	8.19×10^{-5}	−0.00028

The fact that the Born terms used in most calculations have relatively simple structure for $s>4\mu^2$ (for example the scalar exchange Born term is positive definite and monotonically decreasing for $s>4$) means that only a few of the coefficients b_i will be important, allowing the N function to be represented over a wide range of coupling constants as a simple function of g^2. To illustrate this point the two scalar problem has been solved for a meson exchange mass of $3\mu^2$ with 50 mesh points. In Table I are given the eigenvalues for the five largest coefficients b_i. For the range $0<g^2<5000$ the N function is represented by

$$N(s)=\sum_{i=1}^{4}\frac{b_iU_i(s)}{(1/g^2)-\lambda_i}$$

and D is given by

$$D(s)=1-\sum_{i=1}^{4}\frac{b_iV_i(s)}{(1/g^2)-\lambda_i}$$

to an accuracy of a few percent. The value of s_0 used was $3\mu^2$.

V. CONCLUSION

The method which has been discussed here produces solutions to N and D which are explicit functions of the coupling constant and which, for wide ranges of the coupling constant, can be approximated by a small number of terms.

The "bootstrap" problem is particularly simple when this form of the ND^{-1} solution is used as one parameter is determined directly.

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Analytic Continuation of Partial-Wave Amplitude in the Complex Angular-Momentum Plane

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An attempt is made to continue analytically the partial-wave amplitude for the scattering of two identical spinless particles in the complex l plane, exploiting unitarity and analyticity properties in s. The Froissart-Gribov representation for the partial-wave amplitude is known to be holomorphic in the region $\mathrm{Re}l>\alpha$ of the complex l plane provided the absorptive part $A_t(s,t)$ of $A(s,t)$, the scattering amplitude in the t channel, is bounded by t^α for any fixed s. Apart from the above assumptions, two crucial hypotheses on which the present analysis is based are (i) the possibility of extending unitarity in the inelastic region to complex values of l, and (ii) the boundedness condition, viz., that both $A_t(s,t)$ and $A(s,t)$ are asymptotically bounded by the maximum of $(t^\beta/s^\gamma, s^\beta/t^\gamma)$ if s and t are both sufficiently large with $\gamma>0$ and $\beta<\min(1,\gamma)$. With the help of the N/D technique it is then possible to continue analytically the partial-wave amplitude up to the line $\mathrm{Re}l=\beta$ and show that it is meromorphic in the region $\beta<\mathrm{Re}l\leqslant\alpha$. The domain of meromorphy of the partial-wave amplitude obtained by the method of analytic completion is smaller than the preceding one. The analytically continued partial-wave amplitude is bounded by $|l|^{-1/2}$ for large values of $\mathrm{Im}l$, so that a Regge representation for $A(s,t)$ can be obtained. The N/D method of analytic continuation does not work beyond the line $\mathrm{Re}l=-1$ even if one assumes $\beta<-1$. It has also been shown that accumulation of poles at $l=-\frac{1}{2}$ near threshold, a feature which has been pointed out by several authors, is also manifested in the analytically continued partial-wave amplitude.

I. INTRODUCTION

THE purpose of the present work is to discuss the problem of analytic continuation of the relativistic partial-wave amplitude in the complex angular-momentum plane and hence to investigate the singularities which one encounters in such a procedure. In the case of nonrelativistic scattering by potentials, Schrödinger equation provides a very convenient framework within which this problem has been tackled.[1] In relativistic scattering the absence of a Schrödinger equation makes the situation very much complicated. It is, however, presumed that unitarity and the analyticity properties of the scattering amplitude as contained in the Mandelstam representation play the role of a

[1] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

Schrödinger equation in the relativistic scattering theory. Therefore, the question as to how and to what extent Regge's results in potential scattering may be justified within the framework of analyticity and unitarity of the scattering amplitude is of considerable interest.

The above problem has been recently considered by several authors.[2–5] The starting point in these investigations is the Froissart-Gribov (F-G) representation for the partial-wave amplitude. In the case of identical pseudoscalar particles of mass m, the F-G representation may be written as

$$a_+(l,s)=\frac{2}{(s-4m^2)}\int_{4m^2}^{\infty}dtQ_l\left(1+\frac{2t}{s-4m^2}\right)\times\{A_t(s,t)+A_u(s,t)\}$$

$$=\frac{4}{(s-4m^2)}\int_{4m^2}^{\infty}dtQ_l\left(1+\frac{2t}{s-4m^2}\right)A_t(s,t)\,, \qquad (1)$$

where s, t, u are the usual Mandelstam variables with s the square of the center-of-mass energy. In the above, $A_t(s,t)$ and $A_u(s,t)$ denote the absorptive parts of the scattering amplitude in the t and u channels, respectively. If there exists an α such that for any fixed value of s

$$|A_t(s,t)|/t^{\alpha+\epsilon}\to 0 \quad \text{as} \quad t\to\infty\,, \qquad (2)$$

and there is at least one value s' of s such that

$$|A_t(s',t)|/t^{\alpha-\epsilon}\to\infty \quad \text{as} \quad t\to\infty\,, \qquad (3)$$

where ϵ is an arbitrarily small positive number, then Eq. (1) defines a function holomorphic in l in the region $\text{Re}l>\alpha$. In the interesting region $\text{Re}l\leqslant\alpha$ where singularities of $a_+(l,s)$ are expected to occur, the Froissart-Gribov representation is not valid. In order to investigate the singularities of $a_+(l,s)$ in the complex l plane one has, therefore, to continue $a_+(l,s)$ analytically into the region $\text{Re}l\leqslant\alpha$ by exploiting unitarity and analyticity properties in the s plane.

The analyticity properties of $a_+(l,s)$ follow directly from the Mandelstam representation for $A_t(s,t)$ and the known singularities of $Q_l(z)$. Unitarity for $a_+(l,s)$ when written in the form[3,4]

$$\text{Im}a_+(l,s)=\mathcal{R}(l,s)((s-4m^2)/s)^{1/2}\times a_+(l,s+i\epsilon)a_+(l,s-i\epsilon)\,,\quad s\geqslant 4m^2\,, \qquad (4)$$

gives us a relation between the left-hand $(-\infty\leqslant s\leqslant 0)$ and the right-hand $(4m^2\leqslant s\leqslant\infty)$ discontinuities of $a_+(l,s)$ provided $\mathcal{R}(l,s)$ is known. If s is below the threshold for inelastic processes (i.e., $s\leqslant 16m^2$ in our case) $\mathcal{R}(l,s)=1$. In the inelastic region, however, our knowledge about $\mathcal{R}(l,s)$ is very limited. Even for physical values of l the only definite restriction one can impose on $\mathcal{R}(l,s)$ is that $\mathcal{R}(l,s)>1$ if $s>16m^2$. It is thus clear that one can hardly make any progress unless one is prepared to make some working hypothesis regarding $\mathcal{R}(l,s)$. For our purpose it is sufficient to assume that, as a function of l, $\mathcal{R}(l,s)$ is holomorphic in the region of the l plane in which we are interested. As regards the behavior of $\mathcal{R}(l,s)$ as a function of s, our assumption is less restrictive. We need only assume that it is continuous and is bounded asymptotically by $s^{\delta-\eta}$, where δ is given by Eq. (41) and η is a positive number.

We have already noted that unitarity determines the right-hand discontinuity of $a_+(l,s)$ provided its left-hand discontinuities are given or vice versa. It is customary to regard the left-hand discontinuity or at least some of its general features such as the asymptotic behavior in s to be known. In making this approach one is mainly guided by an analogy with the case of potential scattering. There one can show that the left-hand singularities of $a_+(l,s)$ are related to those of the Fourier transform of the potential. In the present investigation there is another reason why such an approach seems to be appealing. Under some reasonable assumptions (see Secs. II and III) regarding the asymptotic behavior of the scattering amplitude, one can show that the left-hand discontinuity has a larger domain of holomorphy in the l plane than the right-hand discontinuity. It is then our task to find the nature of the singularities of the latter consistent with unitarity.

It should be emphasized that any arbitrary asymptotic behavior of the left-hand function [i.e., the function which has only the left-hand discontinuity of $a_+(l,s)$] is not consistent with unitarity. Indeed, it can be easily verified that for real l (leaving aside the limiting case where the left-hand function behavies like $\ln s$ for large s) unitarity requires that the left-hand function must vanish for large s. The boundedness condition

$$|A_t(s,t)|<\eta(t^{1-\gamma}/s)\,,\quad (s,\,t\text{ both large}) \qquad (5)$$

assumed by Mandelstam[2] just ensures the vanishing of the left-hand function like $s^{-\gamma}$ asymptotically. It has been pointed out[5] that (5) is compatible with crossing symmetry only if $\gamma=2$. In the elastic unitarity approximation followed by Mandelstam crossing symmetry is in any case violated and, therefore, the above does not consititute a serious objection. In our investigation, however, by allowing $\mathcal{R}(l,s)$ to be different from unity we have in some sense included the contributions from inelastic channels. It is, therefore, desirable that we extend our considerations to a more general asymptotic behavior of $A_t(s,t)$ consistent with crossing symmetry. This we have done in Sec. III.

It should be noted that the F-G representation (1) has a continuous cut $-\infty\leqslant s\leqslant+\infty$ in the s plane for nonintegral values of l. Several authors[2–4,6] have pointed

[2] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).
[3] G. M. Prosperi, Nuovo Cimento **26**, 541 (1962).
[4] K. Bardakci, Phys. Rev. **127**, 1832 (1962).
[5] Haridas Banerjee, Phys. Rev. **131**, 2810 (1963).
[6] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **42**, 1260 (1962) [English transl.: Soviet Phys.—JETP **15**, 873 (1962)]; A. O. Barut and D. Zwanziger, Phys. Rev. **127**, 974 (1962).

out that the cut $0\leqslant s\leqslant 4m^2$ stems from the threshold behavior of $a_+(l,s)\sim(s-4m^2)^l$ and can be eliminated if one considers the amplitude

$$T(l,s)=(1/(s-4m^2)^l)a_+(l,s)\,. \qquad (6)$$

From the point of view of analytic continuation, however, the amplitude $T(l,s)$ is not convenient. This is because the s-asymptotic behavior of $T(l,s)$ is now l-dependent. For example, if $a_+(l,s)$ behaves asymptotically like $s^{-\gamma}$, then $T(l,s)$ should behave like $s^{-l-\gamma}$. But such a behavior is, in general, not possible to guarantee in the process of analytic continuation. In his analysis Mandelstam[2] has tried to continue $a_+(l,s)$ analytically strip by strip in the complex l plane in terms of a sequence of amplitudes defined by

$$T_n(l,s)=(s-4m^2)^{-l+n}a_+(l,s)\,, \qquad (7)$$

where n is allowed to assume successively all integral values up to $n=0$. It can be shown that in the common strip of the l plane where both $T_{n-1}(l,s)$ and $(s-4m^2)^{-1}T_n(l,s)$ are defined and where, therefore, they should be identical in order that Mandelstam's method should work, the agreement of their asymptotic behavior cannot be guaranteed. It is, therefore, not surprising that on the basis of the boundedness condition (5) Mandelstam was led to conclude that $a_+(l,s)$ is meromorphic in l if $\mathrm{Re}l>-\gamma$ and that there is a fixed pole at $l=1-\gamma$ which contradicts our result (see Sec. III). In our analysis (Sec. II) we shall analytically continue $a_+(l,s)$ with the help of an auxiliary amplitude of the form

$$\mathcal{B}(l,s)=(f(l,s)/(s-4m^2)^l)a_+(l,s)\,, \qquad (8)$$

where $f(l,s)$ (i) is an entire function of l, (ii) behaves like s^l asymptotically, and (iii) is analytic on the physical sheet of the s plane except for the cut $-\infty\leqslant s\leqslant 0$. The exact form of $f(l,s)$ will be given in Sec. II. The immediate advantage of our method is that it avoids the strip-by-strip procedure of Mandelstam. It will be shown in Sec. III that if the boundedness condition (5) of Mandelstam is strictly valid, then one can analytically continue $\mathcal{B}(l,s)$ and hence $a_+(l,s)$ only up to $\mathrm{Re}l>1-\gamma$ in the complex l plane.

In Sec. IV, following Bardakci,[4] we have also obtained a domain of meromorphy for $a_+(l,s)$ by applying the tube theorem for analytic completion of meromorphic functions. The domain thus obtained is, however, found to be smaller than that derived by the N/D method in Secs. II and III. Finally, in Sec. V, we have pointed out that if one makes stronger assumptions regarding the asymptotic behavior of the scattering amplitude $A(s,t)$ and the absorptive part in the t channel $A_t(s,t)$ it is possible to extend the domain of meromorphy of $a_+(l,s)$ only up to the line $\mathrm{Re}l=-1$ and the N/D method of analytic continuation does not work for $\mathrm{Re}l\leqslant -1$. It has also been shown that the analytically continued partial-wave amplitude exhibits the feature of clustering of poles near $l=-\frac{1}{2}$ at threshold.

II. FORMULATION OF THE N/D EQUATIONS

It has already been pointed out in Sec. I that the F-G representation has a continuous cut for nonintegral values of l from $-\infty\leqslant s\leqslant+\infty$. If, in order to eliminate the kinematical cut $0\leqslant s\leqslant 4m^2$, we choose the representation (6), we find that the s-asymptotic behavior of $T(l,s)$ is l-dependent. From the point of view of analytic continuation, such a behavior of $T(l,s)$ is very inconvenient. Hence, we need a function which (i) eliminates the cut $0\leqslant s\leqslant 4m^2$ from the F-G representation i.e., keeps a gap between the left- and the right-hand cuts, and (ii) maintains the s-asymptotic behavior of the F-G representation. In order to do this we choose an auxiliary function $\mathcal{B}(l,s)$ defined by

$$\mathcal{B}(l,s)=\left(\frac{\sqrt{s}+(4m^2)^{1/2}}{\sqrt{s}-(4m^2)^{1/2}}\right)^l a_+(l,s)\,, \qquad (9)$$

$$=[\sqrt{s}+(4m^2)^{1/2}]^{2l}(a_+(l,s)/(s-4m^2)^l)\,. \qquad (10)$$

From Eq. (9), it is clear that $\mathcal{B}(l,s)$ has the same s-asymptotic behavior as $a_+(l,s)$. Moreover, if we choose the branch cuts for $[\sqrt{s}+(4m^2)^{1/2}]^{2l}$ from $-\infty\leqslant s\leqslant 0$ and for $(s-4m^2)^{-l}$ from $-\infty\leqslant s\leqslant 4m^2$, then in the s plane $\mathcal{B}(l,s)$ will have no kinematical cut from $0\leqslant s\leqslant 4m^2$. Further, the presence of the factor $[\sqrt{s}+(4m^2)^{1/2}]^{2l}$ in (9) introduces no extraneous pole in $a_+(l,s)$ because $[\sqrt{s}+(4m^2)^{1/2}]^{2l}$ never vanishes on the physical sheet.

It is interesting to note that the l-asymptotic behavior of $a_+(l,s)$, i.e.,

$$a_+(l,s)\sim O\left(\frac{e^{-\xi l}}{\sqrt{l}}\right)=O\left(\frac{1}{\sqrt{l}}\left(\frac{\sqrt{s}-(4m^2)^{1/2}}{\sqrt{s}+(4m^2)^{1/2}}\right)^l\right), \qquad (11)$$

where

$$\xi=\cosh^{-1}\left(1+\frac{8m^2}{s-4m^2}\right), \qquad (12)$$

also indicates the need of a factor of the form $\{[\sqrt{s}+(4m^2)^{1/2}]/[\sqrt{s}-(4m^2)^{1/2}]\}^l$ in the representation of $a_+(l,s)$. Representation (9) has been used by several authors, in connection with the various properties of the partial-wave amplitude. In his modification of the Regge-pole formula, Khuri[7] has used this representation in order to exhibit the correct t-plane cuts of the total scattering amplitude. More recently, representation (9) has also been used by Kreps *et al.*[8] in connection with the problem of π-π scattering.

We now discuss the analytic properties of $\mathcal{B}(l,s)$ in the s plane. It is evident from Eq. (9) that the analytic properties of $\mathcal{B}(l,s)$ depend upon the Q_l function and

[7] N. N. Khuri, Phys. Rev. **130**, 429 (1963).
[8] R. E. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964).

$A_t(s,t)$, where $A_t(s,t)$ is given by the following subtracted dispersion-relation:

$$A_t(s,t)=\left(\frac{1}{\pi}\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{(u-s)}\frac{s^N}{u^N}+\sum_{1}^{N}s^{n-1}f_n(t)\right)$$
$$+\left(\frac{1}{\pi}\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{(u+s+t-4m^2)}\frac{(4m^2-s-t)^N}{u^N}+\sum_{1}^{N}(4m^2-s-t)^{n-1}f_n(t)\right), \quad (13)$$
$$=\phi(s,t)+\phi(4m^2-s-t,\,t). \quad (14)$$

In the above, $\phi(s,t)$ and $\phi(4m^2-s-t,\,t)$ are the contributions coming from the direct and the crossed channels, respectively. For the pseudoscalar equal-mass kinematics, as is the case here, the spectral function $\rho(u,t)$ represents a real symmetric function of its argument and is given by

$$\rho(u,t)=\sigma(u,t)\theta(u-4m^2)\theta(t-16m^2u/(u-4m^2))+\sigma(t,u)\theta(t-4m^2)\theta(u-16m^2t/(t-4m^2)), \quad (15)$$

where $\theta(x)$ denotes the step function and $\sigma(u,t)$ is a real function of its arguments. For the analytic properties of the $Q_l(z)$ function we use the following formulas[9]:

$$Q_l(-z\mp i\epsilon)=e^{\pm i\pi(l+1)}Q_l(z\pm i\epsilon) \quad \text{for} \quad |z|>1, \quad (16)$$

and for $|z|<1$,

$$\sin\pi l\,Q_l(z\pm i\epsilon)=\tfrac{1}{2}\pi[e^{\mp i\pi l}P_l(z)-P_l(-z)]. \quad (17)$$

We now proceed to calculate the discontinuity of $\mathcal{B}(l,s)$ across the left-hand cut. We can write Eq. (9) using Eq. (6) in the following form; i.e.,

$$\mathcal{B}(l,s)=(\sqrt{s}+(4m^2)^{1/2})^{2l}T(l,s). \quad (18)$$

It follows from Eq. (18) that

$$\mathcal{B}(l,\,s+i\epsilon)-\mathcal{B}(l,\,s-i\epsilon)$$
$$=[(s+i\epsilon)^{1/2}+(4m^2)]^{1/2}]^{2l}[T(l,\,s+i\epsilon)-T(l,\,s-i\epsilon)]$$
$$+\{[(s+i\epsilon)^{1/2}+(4m^2)^{1/2}]^{2l}-[(s-i\epsilon)^{1/2}+(4m^2)^{1/2}]^{2l}\}\times T(l,\,s-i\epsilon). \quad (19)$$

Using Eqs. (16) to (18), we obtain that

$$\mathcal{B}(l,\,s+i\epsilon)-\mathcal{B}(l,\,s-i\epsilon)=2i\sum_{k=1,2,3}f_k(l,s), \quad s<0, \quad (20)$$

where

$$f_1(l,s)=2\frac{(|s|+4m^2)^l}{|s-4m^2|^{l+1}}\left[\pi\frac{\sin 2l\xi}{\sin\pi l}\int_{4m^2}^{4m^2-s}P_l\left(1+\frac{2t}{s-4m^2}\right)dt-\pi\sin\frac{(2l\xi-\pi l)}{\sin\pi l}\int_{4m^2}^{4m^2-s}P_l\left(-1-\frac{2t}{s-4m^2}\right)dt\right.$$
$$\left.-2\sin 2l\xi\int_{4m^2-s}^{\infty}Q_l\left(-1-\frac{2t}{s-4m^2}\right)dt\right]\phi(s,t)\theta(-s), \quad (21)$$

$$f_2(l,s)=2\frac{(|s|+4m^2)^l}{|s-4m^2|^{l+1}}\left[\pi\frac{\sin 2l\xi}{\sin\pi l}\int_{4m^2}^{4m^2-s}P_l\left(1+\frac{2t}{s-4m^2}\right)dt-\pi\sin\frac{(2l\xi-\pi l)}{\sin\pi l}\int_{4m^2}^{4m^2-s}P_l\left(-1-\frac{2t}{s-4m^2}\right)dt\right.$$
$$\left.-2\sin 2l\xi\int_{4m^2-s}^{\infty}Q_l\left(-1-\frac{2t}{s-4m^2}\right)dt\right]\bar{\phi}(4m^2-s-t,\,t)\theta(-s), \quad (22)$$

$$f_3(l,s)=4\frac{(|s|+4m^2)^l}{|s-4m^2|^{l+1}}\cos^2 l\xi\int_{4m^2}^{\infty}Q_l\left(-1-\frac{2t}{s-4m^2}\right)\rho(4m^2-s-t,\,t)\theta(-s)dt, \quad (23)$$

$$\xi=\tan^{-1}(|s|/4m^2)^{1/2}, \quad (24)$$

and $\bar{\phi}(4m^2-s-t,\,t)$ denotes the real part of $\phi(4m^2-s-t,\,t)$ defined by Eqs. (13) and (14). The limits in the case of $f_3(l,s)$ are in fact determined by the support properties of the spectral function $\rho(u,t)$ (see Eq. 15). Having obtained the left-hand discontinuities, we define the left-hand function as

$$F(l,s)=\sum_{k=1,2,3}F_k(l,s), \quad (25)$$

where $F_k(l,s)$ is defined by the Cauchy integral formula

$$F_k(l,s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{f_k(l,s')}{s'-s}ds'. \quad (26)$$

In Eq. (26), subtractions are implied if necessary.

[9] Bateman Project Staff, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1964), Vol. I, p. 140.

Following the procedure given by one of the authors,[5] we finally obtain the explicit form of $F(l,s)$, i.e.,

$$F(l,s)=F_1(l,s)+\frac{4}{\pi}\frac{(\sqrt{s}+(4m^2)^{1/2})^{2l}}{(s-4m^2)^{l+1}}\int_{4m^2}^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4m^2}\right)\phi(4m^2-s-t,\,t)\,, \tag{27}$$

where

$$F_1(l,s)=\frac{4}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{(u-s)}\frac{s^N}{u^N}\left[\frac{(\sqrt{s}+(4m^2)^{1/2})^{2l}}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{(\sqrt{u}+(4m^2)^{1/2})^{2l}}{(u-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{u-4m^2}\right)\right]$$
$$+4\sum_{1}^{N}s^{n-1}\int_{4m^2}^{\infty}f_n(t)\frac{(\sqrt{s}+(4m^2)^{1/2})^{2l}}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right). \tag{28}$$

The first and the second term on the right-hand side in Eq. (27) will be referred to as the direct-channel and the crossed-channel left-hand functions, respectively. In order to check that $F(l,s)$, as given by Eqs. (27) and (28), satisfies all the requirements of a left-hand function, it is interesting to note that our $F(l,s)$ has no right-hand cut. The discontinuity of $F(l,s)$ across the left-hand cut is the same as that of the auxiliary partial-wave amplitude $\mathcal{B}(l,s)$. This will be clear if we write Eq. (27) in the form

$$F(l,s)=4\frac{(\sqrt{s}+(4m^2)^{1/2})^{2l}}{(s-4m^2)^{l+1}}\int_{4m^2}^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4m^2}\right)A_t(s,t)$$
$$-\frac{4}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{(u-s)}\frac{s^N}{u^N}\frac{(\sqrt{u}+(4m^2)^{1/2})^{2l}}{(u-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{u-4m^2}\right). \tag{29}$$

The most significant feature of our representation for $F(l,s)$ is that its l-asymptotic behavior can be easily examined, owing to the presence of Q_l functions with argument positive and greater than unity. Indeed, for $|z|>1$, we can use the following formula[9];

$$Q_l(z)\sim\left(\frac{1}{2\pi l}\right)^{1/2}\times\frac{\exp\{(l+\frac{1}{2})\ln[z-(z^2-1)^{1/2}]\}}{(z^2-1)^{1/4}},\quad |l|\to\infty\,, \tag{30}$$

and obtain that

$$|F(s,l)|<\text{constant}\,|l|^{-1/2},\quad |l|\to\infty\,. \tag{31}$$

This asymptotic behavior will be true throughout the domain of validity of the representation (27).

We now start the formulation of the N/D integral equation for $\mathcal{B}(l,s)$. The unitarity relation for $\mathcal{B}(l,s)$ (for $s\geqslant 4m^2$) is given by

$$\mathcal{B}(l,\,s+i\epsilon)-\mathcal{B}(l,\,s-i\epsilon)=\phi(l,s)\mathcal{B}(l,\,s+i\epsilon)\mathcal{B}(l,\,s-i\epsilon)\,, \tag{32}$$

where

$$\phi(l,s)=\left(\frac{s-4m^2}{s}\right)^{1/2}\frac{(s-4m^2)^l}{(\sqrt{s}+(4m^2)^{1/2})^{2l}}\mathcal{R}(l,s)\,. \tag{33}$$

In the elastic unitarity approximation $\mathcal{R}(l,s)=1$. We make the usual N/D ansatz for $\mathcal{B}(l,s)$, i.e.,

$$\mathcal{B}(l,s)=N(l,s)/D(l,s)\,, \tag{34}$$

where $N(l,s)$ has the left-hand cut $(-\infty\leqslant s\leqslant 0)$ and $D(l,s)$ has the physical cut $(4m^2\leqslant s\leqslant\infty)$. $N(l,s)$ and $D(l,s)$ satisfy the following integral equations:

$$N(l,s)=\frac{1}{\pi}\int_{-\infty}^{0}ds'\frac{\sum_{k=1,2,3}f_k(l,s')}{s'-s}D(l,s')\,, \tag{35}$$

and

$$Dl,s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{\phi(l,s')N(l,s')}{s'-s}\,. \tag{36}$$

Eliminating $D(l,s)$ from Eqs. (35) and (36), we obtain

$$N(l,s)=F(l,s)+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{F(l,s')-F(l,s)}{s'-s}\times\phi(l,s')N(l,s')ds'\,. \tag{37}$$

Our problem now is to investigate whether the integral equation (37) is nonsingular. Equation (37) would be nonsingular provided

$$\int_{4m^2}^{\infty}|F(l,s)|^2ds<\infty\,, \tag{38}$$

and

$$\int_{4m^2}^{\infty}\int_{4m^2}^{\infty}dsds'\left|\phi(l,s')\cdot\frac{F(l,s')-F(l,s)}{s'-s}\right|^2<\infty\,. \tag{39}$$

In the following section, we show that conditons (38) and (39) are indeed satisfied provided one makes certain assumptions regarding the asymptotic behavior of $A(s,t)$, the total scattering amplitude, and $A_t(s,t)$, the absorptive part in the t channel. Once it is proved that the integral equation (37) is nonsingular, we can conclude[10] that if the resolvent exists for at least one value of l then the solution of the integral equation is a meromorphic function of l. The poles of $N(l,s)$ in the complex l plane are independent of s and cancel in $\mathcal{B}(l,s)$. Thus, in the l plane, the poles of $\mathcal{B}(l,s)$ and, therefore, of $a_+(l,s)$ are to be identified with the zeros of $D(l,s)$ which are s-dependent. Since $F(l,s)$ is bounded by $|l|^{-1/2}$ asymptotically, it is quite clear from Eq. (37) that the same bound holds for the kernel and therefore for $N(l,s)$. It follows that the analytically continued partial-wave amplitude is bounded by $|l|^{-1/2}$ asymptotically so that, by making use of the Sommerfeld-Watson transformation[1] and the fact that $a_+(l,s)$ is meromorphic, one can obtain the Regge representation for the total scattering amplitude $A(s,t)$.

III. DOMAIN OF MEROMORPHY OF THE PARTIAL-WAVE AMPLITUDE OBTAINED BY THE N/D METHOD

Our subsequent discussions are based on the following ansatz regarding the asymptotic behavior of $A(s,t)$ and $A_t(s,t)$:

$$A(s,t)\,,\;A_t(s,t)<\eta\max\left(\frac{t^\beta}{s^\gamma},\frac{s^\beta}{t^\gamma}\right),\quad s,\,t>R,\qquad(40)$$

where $\gamma>0$ and η, R are sufficiently large positive numbers. It is now possible to show (see Appendices A and B) that if $\beta<\min(1,\gamma)$ there exists a left-hand function $F(l,s)$ such that (i) $F(l,s)$ is holomorphic for $\mathrm{Re}\,l>\max(\beta,-\gamma)$, and (ii) asymptotically $F(l,s)$ vanishes, i.e., as $s\to\infty$

$$|F(l,s)|<\text{constant}\times s^{-\delta},\quad \delta>0.\qquad(41)$$

The above properties of $F(l,s)$ are sufficient to guarantee that the integral equation for $N(l,s)$ is nonsingular. We first remark that if $\delta<\frac{1}{2}$, $F(l,s)$ would not be square-integrable so that condition (38) would not hold. In order to get around this difficulty we divide the integral Eq. (37) for $N(l,s)$ by $s^{1/2-\delta+\epsilon}$ and obtain

$$M(l,s)=\frac{F(l,s)}{s^{1/2-\delta+\epsilon}}+\frac{1}{\pi}\int_{4m^2}^{\infty}K(s,s')\left(\frac{s'}{s}\right)^{1/2-\delta+\epsilon}M(l,s')ds',\qquad(42)$$

where

$$M(l,s)=N(l,s)/s^{1/2-\delta+\epsilon}\qquad(43)$$

and

$$K(s,s')=\frac{F(l,s')-F(l,s)}{s'-s}\times\frac{(s'-4m^2)^{l+1/2}}{s'^{1/2}(\sqrt{s'}+(4m^2)^{1/2})^{2l}}R(l,s').\qquad(44)$$

[10] J. D. Tamarkin, Ann. Math. 28, 127 (1927).

The inhomogeneous term in the integral equation for $M(l,s)$ is clearly square-integrable. Thus, in order to show that Eq. (42) is nonsingular, we have only to prove that the integral I, defined by

$$I=\int_{4m^2}^{\infty}\int_{4m^2}^{\infty}|K(s,s')|\left(\frac{s'}{s}\right)^{1-2\delta+2\epsilon}dsds'$$
$$\approx\int_{4m^2}^{\infty}ds\int_{4m^2}^{\infty}ds'\left|\frac{F(l,s')-F(l,s)}{s'-s}\times\frac{(s'-4m^2)^{l+1/2}}{s'^{1/2}(\sqrt{s'}+(4m^2)^{1/2})^2}\right|^2\left(\frac{s'}{s}\right)^{1-2\delta+\epsilon}|\mathcal{R}(l,s')|^2,\qquad(45)$$

should be convergent. If we now assume that

$$|\mathcal{R}(l,s)|<\text{constant}\times s^{\delta-\eta},\qquad(46)$$

where η is a positive number, and substitute $s'=\lambda s$ we obtain

$$I\approx\text{constant}\times\int_{4m^2}^{\infty}s^{-1-2\eta}ds\int_{4m^2/s}^{\infty}\left|\frac{\lambda^{-\delta}-1}{\lambda-1}\right|\lambda^{1-2\eta+2\epsilon}d\lambda<\infty,$$

since we can always choose $\epsilon\ll\eta$. It follows that $M(l,s)$ and, therefore, $N(l,s)$ is a meromorphic function of l. We have thus shown that if $A(s,t)$ and $A_t(s,t)$ obey the boundedness condition (40), then $a_+(l,s)$ is a meromorphic function of l in the domain $\mathrm{Re}\,l>\max(\beta,-\gamma)$. The restriction $\mathrm{Re}\,l>\max(\beta,-\gamma)$ comes through the fact that our representation [Eq. (27)] for $F(l,s)$ breaks down if $\mathrm{Re}\,l<\max(\beta,-\gamma)$. If $\delta>\frac{1}{2}$, instead of Eq. (42), we can consider the integral equation (37) for $N(l,s)$ directly and show that it is nonsingular. Our conclusions would be the same as in the case already considered.

It may be noted that in contradistinction with the boundedness condition assumed by Mandelstam,[2] viz.,

$$|A_t(s,t)|<\eta t^{1-\gamma}/s,\quad |s|,\;|t|>R,\qquad(5)$$

where $\gamma>0$, condition (40) is consistent with crossing symmetry and the former is only a particular case of the latter. On the basis of the boundedness condition (5) Mandelstam has claimed that $a_+(l,s)$ is meromorphic in the domain $\mathrm{Re}\,l>-\delta$, where $\delta=\min(\gamma,\frac{3}{2})$. Condition (5), however, implies the existence of a singularity of $a_+(l,s)$ on the line $\mathrm{Re}\,l=1-\gamma$ and since this lies within the domain of meromorphy the singularity must be a pole. But this is not possible. In order to see this we consider the equation for $D(l,s)$,

$$D(l,s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{k(l,s')}{s'-s}N(l,s'),$$

where $k(l,s)$ is bounded for large s. Let us first assume that $N(l,s)$ is bounded on the line $\mathrm{Re}\,l=1-\gamma$. Since

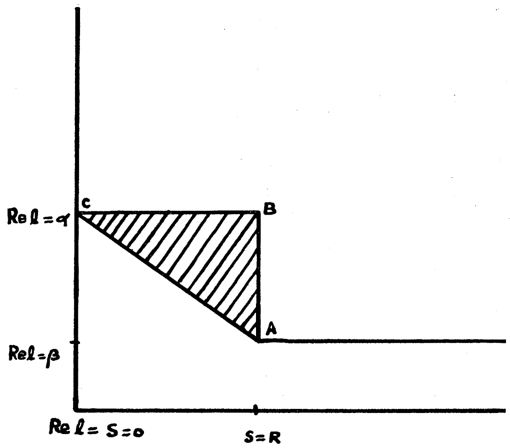

FIG. 1. The shaded triangle ABC is the convex hull of the base. The partial-wave amplitude is meromorphic in the tube with base ABC.

$N(l,s)$ is the solution of a nonsingular integral equation it must vanish asymptotically. It follows that on line $\text{Re}l=1-\gamma$, $D(l,s)$ asymptotically tends to unity and, therefore, $a_+(l,s)$ cannot have any pole on that line for large s. If, however, $N(l,s)$ has apole at, say, $l=1-\gamma+i\xi$ then in its neighborhood $a_+(l,s)$ would behave as

$$a_+(l,s)\approx f(s)/[(l-1+\gamma-i\xi)-\phi(s)],$$

where $f(s)$ and $\phi(s)$ both vanish asymptotically. Again we conclude that $a_+(l,s)$ cannot have a pole at $l=1-\gamma+i\xi$ for large s. According to our results, however, $a_+(l,s)$ can be analytically continued by the N/D method only into the region $\text{Re}l>1-\gamma$ and it would be meromorphic there. Thus, the above difficulty does not arise.

IV. DOMAIN OF MEROMORPHY OBTAINED BY THE METHOD OF ANALYTIC COMPLETION

Apart from the N/D technique, the only other method which has been employed so far in connection with the analytic continuation of partial-wave amplitude is analytic completion with the help of the tube theorem.[4] We shall now show that the domain of meromorphy of $a_+(l,s)$ obtained by this latter method on the basis of the assumptions already made in the present investigation is smaller than that obtained in the preceding section.

We begin by quoting the definition of a tube.[11,12] A tube T in the space of two complex variables $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ is the set of all points which can be represented as $(x_1,x_2)\subset S$, $-\infty<y_1, y_2<+\infty$, where S is any set in the two dimensional space of (x_1,x_2). S is called the base of the tube T. The tube T is connected if S is connected and the convex hull of a tube is the tube over the convex hull of the base. The tube theorem[12] now states that the holomorphy envelope of a connected open tube is its convex hull. In other words, if $\omega_1\{x_1',x_2'\}$ and $\omega_2(x_1'',x_2''\}$ are two points of S, then any point lying on the line $\lambda\omega_1+(1-\lambda)\omega_2$, $0\leq\lambda\leq1$ is also in S.

In order to apply the tube theorem we first construct the function

$$H(l,s)=\frac{1}{\mathcal{B}(l,s)}+\frac{s}{2\pi}\int_{4m^2}^{\infty}ds'\left(\frac{s'-4m^2}{s'^3}\right)^{1/2}\times\left(\frac{\sqrt{s'}-(4m^2)^{1/2}}{\sqrt{s'}+(4m^2)^{1/2}}\right)^{l}\frac{\mathcal{R}(l,s')}{s'-s}, \qquad (47)$$

which is free from the unitarity cut. In the above we have assumed that $\mathcal{R}(l,s')/s'$ vanishes asymptotically. If $\mathcal{B}(l,s)$ is holomorphic in a certain domain, $H(l,s)$ will also be holomorphic in the same domain except for the poles due to the zeros of $\mathcal{B}(l,s)$. From our assumptions [Eq. (40)] about the asymptotic behavior of $A_t(s,t)$ it follows that $H(l,s)$ is meromorphic in a tube domain given by the connected set of points $\{\text{Re}l>\alpha, \text{Re}s>0\}$ and $\{\text{Re}l>\beta, \text{Re}s>R\}$. If, following Bardakci,[4] one assumes that the tube theorem can also be used for analytic completion of domains of meromorphy, one can conclude that the domain of meromorphy of $H(l,s)$ and, therefore, of $\mathcal{B}(l,s)$ includes the set of points $\{\text{Re}l=\lambda(\alpha-\beta)+\beta; \text{Re}s>(1-\lambda)R\}$ with $0\leq\lambda\leq1$ and $\text{Im}l$, $\text{Im}s$ arbtirary. Diagrammatically these points define an open tube with the triangular base ABC in Fig. 1. Clearly, the domain of meromorphy thus obtained is smaller than the domain $\{\text{Re}l>\beta, \text{Re}s>0\}$ obtained by the N/D method in the preceding section. It should, however, be pointed out that we have not fully utilized in the analytic completion procedure all the assumptions needed in analytic continuation by the N/D method. Moreover, the latter procedure does not work at all unless $\beta<\min(1,\gamma)$, whereas it is always possible to obtain a domain of meromorphy larger than that of holomorphy for the partial-wave amplitude provided only $\beta<\alpha$.

We would like to remark that the use of the Froissart bound for $A_t(s,t)$ in the procedure of analytic completion with the help of the tube theorem, as done by Bardakci,[4] cannot be justified rigorously. A Froissart bound, or an extension thereof,[3,4] for the absorptive part in the t channel is valid only if $\cos\theta$, where θ is the corresponding angle of scattering, is restricted to lie within the Lehmann ellipse. This means that $\text{Im}s$ cannot be arbitrary; in particular, if we take $0<\text{Re}s<\epsilon$ then $|\text{Im}s|<8m^2$. It follows that the region where $\mathcal{B}(l,s)$ is holomorphic for $\text{Re}l>1$ cannot form a tube domain, and hence the tube theorem cannot be applied unless some extra assumptions regarding the validity of a Froissart bound outside the Lehmann ellipse are made.

[11] Wightman's lectures on analytic functions of several complex variables, in *Relations de dispersion et particules élementaires* (Hermann et Cie, Paris, 1960).

[12] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, New Jersey, 1948), Chap. V.

V. CONCLUSIONS

It may be argued that if in our ansatz (40) for the asymptotic behavior of $A_t(s,t)$, β is assumed to be negative it may be possible to extend the domain of meromorphy of $a_+(l,s)$ to the left half of the complex l plane. Indeed, in our investigations the only restriction imposed on β was $\beta<\min(1,\gamma)$. Thus our conclusion that $a_+(l,s)$ would be meromorphic if $\mathrm{Re}l>\beta$ will remain true even if β is negative, unless the integral equation for $N(l,s)$ ceases to be nonsingular for reasons which were not relevant for positive values of $\mathrm{Re}l$. It may be easily verified that so long as $\mathrm{Re}l>-1$, $F(l,s)$ is holomorphic in $0<s<\infty$ and behaves as $s^{-\delta}$ (see Appendix B) asymptotically. Thus, the kernel of the integral equation for $N(l,s)$ remains square integrable. But at $\mathrm{Re}l=-1$, (i) $F(l,s)$ and, therefore, the kernel develops a fixed pole due to the poles of the Q_l functions at the negative integral values of l, (ii) the kernel ceases to be square-integrable due to the presence of the factor $(s'-4m^2)^{l+1/2}$. Thus, even if $\beta<-1$, our method of analytic continuation does not work beyond the line $\mathrm{Re}l=-1$ and there will, in general, be a singularity of $a_+(l,s)$ other than a pole at $l=-1$. Since there is a unique correspondence between singularities of $a_+(l,s)$ in the l plane and the asymptotic behavior of $A_t(s,t)$, consistency demands that, in general, β cannot be less than -1. This is in agreement with the observations of Gribov and Pomeranchuck.[13]

Apart from singularities at negative integral values of l, several authors[14] have discussed the accumulation of poles of the partial-wave amplitude near $l=-\frac{1}{2}$ at threshold which is essentially of kinematic origin.[15] From the above discussion it is, however, clear that if $\beta<-\frac{1}{2}$, nothing becomes wrong with $N(l,s)$ at $l=-\frac{1}{2}$ and $s=4m^2$ except that it may have a fixed pole at $l=-\frac{1}{2}$. In that case $D(l,s)$ will also have a fixed pole at $l=-\frac{1}{2}$ and this will cancel in N/D. In any case, in the neighborhood of the threshold $D(l,s)$ may be represented by

$$D(l,s)=1-(s-4m^2)^{l+1/2}\psi(l),$$

where $\psi(l)$ is at most meromorphic in l near $l=-\frac{1}{2}$. If $\psi(l)$ is bounded and nonzero at $l=-\frac{1}{2}$, the zeros of $D(l,s)$ are given by

$$l=-\tfrac{1}{2}+a/\ln(s-4m^2)+2im\pi/\ln(s-4m^2), \quad (48)$$

where $a=-\ln\psi(-\frac{1}{2})$ and m is any integer. Equation (48) clearly shows the accumulation of zeros of $D(l,s)$, i.e., poles of $a_+(l,s)$ near $l=-\frac{1}{2}$ at threshold. It is also easy to convince oneself that clustering of poles around $l=-\frac{1}{2}$ at threshold remains true even if $\psi(l)$ has a zero or a pole at $l=-\frac{1}{2}$.

In conclusion, let us summarize our results. From the validity of the Mandelstam representation with a finite number of subtractions for the total scattering amplitude one concludes that there exists a domain of holomorphy of the form $\mathrm{Re}l>\alpha$ of the relativistic partial wave amplitude $a_+(l,s)$ defined by Eq. (1). Using the tube theorem for analytic completion it is possible to prove that $a_+(l,s)$ is meromorphic in a larger domain if in addition to condition (2) one also assumes the boundedness condition (40) with $\beta<\alpha$ for the asymptotic behavior of $A_t(s,t)$. For the N/D method of analytic continuation to work, one has to make use of the boundedness condition (40) with the further restriction $\beta<\min(1,\gamma)$. The domain of meromorphy $\mathrm{Re}l>\beta$ obtained by this latter method is larger than the corresponding domain obtained by using the tube theorem. It should be emphasized that the mere existence of a domain of meromorphy $\mathrm{Re}l>\beta$ in the complex l-plane is not of much physical interest unless one can at the same time show that the analytically-continued partial-wave amplitude is bounded asymptotically in l in this domain. The above requirement for the partial wave amplitude is guaranteed in the N/D method of analytic continuation.

[13] V. N. Gribov and I. Ya. Pomeranchuck, Zh. Eksperim. i Teor. Fiz. 43, 1556 (1962) [English transl.: Soviet Phys.—JETP 16, 1098 (1963)]; P. G. O. Freund and R. Oehme, Phys. Rev. 129, 2361 (1963).

[14] B. R. Desai and R. G. Newton, Phys. Rev. 130, 2109 (1963); V. N. Gribov and I. Ya. Pomeranchuck, Phys. Rev. Letters 9, 238 (1962); B. R. Desai and B. Sakita, Phys. Rev. 136, B226 (1964).

[15] M. Froissart, Proceedings of Seminar on Theoretical Physics, Trieste, 1962.

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APPENDIX A

In this appendix we shall prove some results which will be used in Appendix B.

From

$$A_t(s,t)=\mathrm{Re}A_t(s,t)+i\pi\rho(s,t), \quad (A1)$$

it follows that if, for $s, t>R$, $A_t(s,t)$ is bounded by $\max(s^\beta/t^\gamma,t^\beta/s^\gamma)$ then $\rho(s,t)$, being its imaginary part, cannot behave worse than $\max(s^\beta/t^\gamma,t^\beta/s^\gamma)$ for $s, t>R$. Thus, we obtain

$$\rho(s,t)\leq\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right), \quad s,t>R. \quad (A2)$$

According to Eq. (14),

$$A_t(s,t)=\phi(s,t)+\phi(4m^2-s-t,t), \quad (A3)$$

where, for $t>R$, $\phi(s,t)$ is given by

$$\phi(s,t)=\frac{1}{\pi}\int_{4m^2}^{t}\frac{\rho(u,t)}{u-s}du+\frac{1}{\pi}\int_{t}^{\infty}\frac{\rho(u,t)}{u-s}\frac{s^p}{u^p}du+\sum_{1}^{N'}s^{n-1}f_n(t), \quad (A4)$$

with
$$\beta<p\leq\beta+1.$$

Substituting (A2) in (A4) we get, after some lengthy but straightforward calculations,

$$\phi(s,t)\approx\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du+\max\left(\frac{t^\beta}{s^\gamma},\frac{s^\beta}{t^\gamma}\right)+\sum_1^{N'}s^{n-1}f_n(t)\,,\quad\text{for}\quad s,t>R,\quad\text{(A5)}$$

and

$$\phi(4m^2-s-t,\,t)\approx\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u+s+t-4m^2}du+\max\left(\frac{t^\beta}{(4m^2-s-t)^\gamma},\frac{(4m^2-s-t)^\beta}{t^\gamma}\right)+\sum_1^{N'}(4m^2-s-t)^{n-1}f_n(t)\,,\quad\text{for}\quad s,t>R.\quad\text{(A6)}$$

Substituting (A5) and (A6) in (A3), we obtain

$$A_t(s,t)\approx\left[\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du+\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u+s+t-4m^2}du\right]+\max\left(\frac{t^\beta}{s^\gamma},\frac{s^\beta}{t^\gamma}\right)+\max\left(\frac{t^\beta}{(4m^2-s-t)^\gamma},\frac{(4m^2-s-t)^\beta}{t^\gamma}\right)+\left[\sum_1^{N'}s^{n-1}f_n(t)+\sum_1^{N'}(4m^2-s-t)^{n-1}f_n(t)\right],\quad\text{for}\quad s,t>R.\quad\text{(A7)}$$

In view of the bound on $A_t'(s,t)$, the right-hand side of Eq. (A7) must be bounded by $\max(t^\beta/s^\gamma,s^\beta/t^\gamma)$ for s, $t>R$. As two terms in (A7) are already bounded by $\max(s^\beta/t^\gamma,t^\beta/s^\gamma)$ for s, $t>R$, let us consider the first and the last square brackets in (A7). The first bracket in (A7) will be written as $\mathfrak{B}_1$ and the last as $\mathfrak{B}_2$, The terms in $\mathfrak{B}_1$ cannot cancel with the terms in $\mathfrak{B}_2$ because they have different s-asymptotic behavior. We shall now prove that the terms in the same square bracket cannot cancel with each other if $s>R$. So far as $\mathfrak{B}_1$ is concerned, this will be clear if we write

$$\mathfrak{B}_1=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{s^{n+1}}\int_{4m^2}^{R}\rho(u,t)u^n du+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{(4m^2-s-t)^{n+1}}\int_{4m^2}^{R}\rho(u,t)u^n du.\quad\text{(A8)}$$

With similar reasoning one can show that the terms in $\mathfrak{B}_2$ will never cancel with each other. Thus, we get

$$\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du\leq\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right),\quad s,t>R,\quad\text{(A9)}$$

and,

$$\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u+s+t-4m^2}du\leq\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right),\quad s,t>R.\quad\text{(A10)}$$

Let us now consider the terms in $\mathfrak{B}_2$. The t-asymptotic behavior of $f_n(t)$ is independent of s. Therefore, in order that $\mathfrak{B}_2$ be bounded by $\max(s^\beta/t^\gamma,t^\beta/s^\gamma)$ for s, $t>R$, $f_n(t)$ must behave either as t^β or $t^{-\gamma}$. The possibility that $f_n(t)$ behaves as t^β can be ruled out, because if we consider the case then $s>t>R$ in (A7) the left-hand side will be bounded by s^β/t^γ, whereas the right-hand side will be bounded by $\max(s^{N'}/t^\gamma,s^{N'}t^\beta)$, which is clearly inconsistent. Hence, it follows that $f_n(t)$ must be bounded by $t^{-\gamma}$ and also that $N'=p$.

Substituting (A9) and the asymptotic behavior of $f_n(t)$ in (A5), we obtain

$$\phi(s,t)\leq\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right),\quad s,t>R.\quad\text{(A11)}$$

If we write

$$\phi(s,t)=\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du+\phi'(s,t)\,,\quad\text{(A12)}$$

where

$$\phi'(s,t)=\frac{1}{\pi}\int_{R}^{t}\frac{\rho(u,t)}{u-s}du+\frac{1}{\pi}\int_{t}^{\infty}\frac{\rho(u,t)}{(u-s)}\frac{s^p}{u^p}du+\sum_1^{p}s^{n-1}f_n(t)\,,\quad t>R,\quad\text{(A13)}$$

it follows from the known asymptotic behavior of $f_n(t)$ and $\rho(u,t)$ that

$$\phi'(s,t)\sim\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right),\quad s,t>R,\quad\text{(A14)}$$

and

$$\phi'(s,t)\sim\max(t^\beta,t^{-\gamma}),\quad t>R\quad\text{and}\quad s>0.\quad\text{(A15)}$$

Let us now consider (A9), which for $t>s>R$, reduces to

$$\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du\sim\frac{t^\beta}{s^\gamma},\quad t>s>R.\quad\text{(A16)}$$

This means[5] that the contribution from the region where $\rho(u,t)$ behaves as t^α is completely cancelled out owing to the rapid oscillations in $\rho(u,t)$. But for $s>R$ we have

$$\frac{1}{\pi}\int_{4m^2}^{R}\frac{\rho(u,t)}{u-s}du$$

$$=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{(s-\gamma)^{n+1}}\int_{-\xi}^{+\xi}\rho(\vartheta+\gamma,\,t)\vartheta^n d\vartheta\,, \quad \text{(A17)}$$

where

$$u=\vartheta+\gamma\,,\quad \xi=\tfrac{1}{2}(R-4m^2)\,,\quad \text{and}\quad \gamma=\tfrac{1}{2}(4m^2+R)\,. \quad \text{(A18)}$$

Thus, we get from (A16) that

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{(s-\gamma)^{n+1}}\int_{-\xi}^{+\xi}\rho(\vartheta+\gamma,\,t)\vartheta^n d\vartheta\sim\frac{t^\beta}{s^\gamma}\,,\; t>s>R\,. \quad \text{(A19)}$$

As each term in the left-hand side of (A19) has a different s-asymptotic behavior, we must have

$$\int_{-\xi}^{+\xi}\rho(\vartheta+\gamma,\,t)\vartheta^n d\vartheta\sim t^\beta\,,\quad t>R\,. \quad \text{(A20)}$$

We now prove some results about the s-asymptotic behavior of $A_t(s,t)$. The total scattering amplitude $A(s,t)$ is given by

$$A(s,t)=\psi(s,t)+\psi(4m^2-s-t,\,t)\,, \quad \text{(A21)}$$

where

$$\psi(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{A_t(s',t)}{(t'-t)}\cdot\frac{t^N}{t'^N}dt'+\sum_{1}^{N}t^{n-1}L_n(s)\,. \quad \text{(A22)}$$

If we now assume that both $A(s,t)$ and $A_t(s,t)$ are separately bounded by $\max(t^\beta/s^\gamma, s^\beta/t^\gamma)$ for s, $t>R$, we can set up an equation similar to (A7), with the difference that $A_t(s,t)$ is replaced by $A(s,t)$ and $\rho(s,t)$ is replaced by $A_t(s,t)$. Therefore, following the above procedure, we get

$$\frac{1}{\pi}\int_{4m^2}^{R}A_t(s,t)dt\sim s^\beta\,,\quad s>R\,. \quad \text{(A23)}$$

APPENDIX B

Let us consider the following representation for $F(l,s)$:

$$F(l,s)=\sum_{k=1,2,3,4}M_k(l,s)\,, \quad \text{(B1)}$$

where

$$M_1(l,s)=4\int_R^\infty dt\,\phi'(s,t)\chi(s)Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{4}{\pi}\int_R^\infty du$$

$$\times\int_R^\infty dt\frac{\rho(u,t)}{u-s}\chi(u)Q_l\left(1+\frac{2t}{u-4m^2}\right)\,, \quad \text{(B2)}$$

$$M_2(l,s)=\frac{4}{\pi}\int_R^\infty dt\int_{4m^2}^{R}du\frac{\rho(u,t)}{u-s}\chi(s)Q_l\left(1+\frac{2t}{s-4m^2}\right)$$

$$-\chi(u)Q_l\left(1+\frac{2t}{u-4m^2}\right)\,, \quad \text{(B3)}$$

$$M_3(l,s)=4\chi(s)\int_{4m^2}^{R}dt\,A_t(s,t)Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{4}{\pi}\int_{4m^2}^{\infty}du$$

$$\times\int_{4m^2}^{R}dt\frac{\rho(u,t)}{u-s}\chi(u)Q_l\left(1+\frac{2t}{u-4m^2}\right)\,, \quad \text{(B4)}$$

$$M_4(l,s)=4\chi(s)\int_R^\infty dt$$

$$\times\phi(4m^2-s-t,\,t)Q_l\left(1+\frac{2t}{s-4m^2}\right)\,, \quad \text{(B5)}$$

$$\chi(s)=\left(\frac{\sqrt{s}+(4m^2)^{1/2}}{\sqrt{s}-(4m^2)^{1/2}}\right)^l\frac{1}{s-4m^2}\,, \quad \text{(B6)}$$

with $\phi(s,t)$ and $\phi'(s,t)$ defined by Eqs. (A4) and (A13), respectively. It may easily be verified that $F(l,s)$ as given by Eq. (B1) has only the left-hand cut $-\infty\leq s\leq 0$ in the s-plane and its discontinuity across this cut is the same as that of $B(l,s)$. Thus, the above representation for $F(l,s)$ is a candidate for the left-hand function to be used in the integral equation for $N(l,s)$. In order that our discussion in Sec. III should go through, we also require that (1) $F(l,s)$ should be holomorphic in l for $\mathrm{Re}l>\max(\beta,-\gamma)$; (2) $F(l,s)$ should vanish asymptotically as

$$F(l,s)\sim s^{-\delta}\,,\quad \delta>0\,. \quad \text{(41)}$$

We shall now show that our representation (B1) for $F(l,s)$ possesses the above properties provided $\beta<\min(1,\gamma)$ with $\delta<\min(\gamma-\beta,1-\beta,1)$. This we shall do by showing that the above properties are true for each $M_k(l,s)$ ($k=1$, 2, 3, 4).

(i) $M_1(l,s)$. From Eqs. (A14) and (A15) it immediately follows that the first integral on the right-hand side of Eq. (B3) is convergent if $\mathrm{Re}l>\max(\beta,-\gamma)$ and behaves asymptotically as $s^{-\delta}$ where $0<\delta<(\gamma-\beta)$. In the second integral, which we denote by $M_1'(l,s)$, we put $t=\lambda s$ and $u=\nu s$ and obtain

$$M_1'(l,s)\equiv -\frac{4}{\pi}\int_R^\infty du\int_R^\infty dt\frac{\rho(u,t)}{u-s}\chi(u)Q_l\left(1+\frac{2t}{u-4m^2}\right)$$

$$\approx \text{const}\times s^{\beta-\gamma}\int_{R/s}^\infty d\nu\frac{\nu^\epsilon}{\nu(\nu-1)}\times\int_{R/s}^\infty d\lambda\frac{\lambda^{\beta-\gamma-\epsilon}}{(1+2\lambda/\nu)^{l+1}}, \quad \text{(B7)}$$

where

$$\epsilon=\beta \quad \text{if} \quad \nu>\lambda, \qquad \epsilon=-\gamma \quad \text{if} \quad \nu<\lambda. \quad \text{(B8)}$$

In the above, we have used the bound (A2) on the asymptotic growth of $\rho(u,t)$. It is not difficult to verify now that the integral for $M_1'(l,s)$ exists if $\text{Re}l>\max(\beta,-\gamma)$ and $\beta<\min(1,\gamma)$. Further, the asymptotic behavior of $M_1'(l,s)$ is consistent with Eq. (41). Thus, $M_1'(l,s)$ has the required properties of $F(l,s)$.

(ii) $M_2(l,s)$. The integral representation (B3) for $M_2(l,s)$ may be written as

$$M_2(l,s)=\frac{4}{\pi}\int_R^\infty \frac{dt}{t^{l+1}}\int_{4m^2}^R du\, F(s,u,t)\rho(u,t), \quad \text{(B9)}$$

where $F(s,u,t)$ is bounded in t and analytic in u in interval $4m^2\leq u\leq R$. Therefore, $F(s,u,t)$ admits of a Taylor expansion of the form

$$F(s,u,t)=\sum_{n=0}^\infty C_n(s,t)\vartheta^n, \quad \text{(B10)}$$

where

$$\vartheta=u-\tfrac{1}{2}(R+4m^2), \quad \text{(B11)}$$

and $C_n(s,t)$ is bounded in t. Substituting (B10) in (B9), we obtain

$$M_2(l,s)=\frac{4}{\pi}\int_R^\infty \frac{dt}{t^{l+1}}\sum_n C_n(s,t)\times\int_{-(R-4m^2)/2}^{(R-4m)^2/2} d\vartheta\, \vartheta^n\rho\left(\vartheta+\frac{R+4m^2}{2},t\right). \quad \text{(B12)}$$

From (A20) it now follows that the integral representation for $M_2(l,s)$ exists if $\text{Re}l>\max(\beta,-\gamma)$.

If s is large (i.e., $s>R$), one can show by expanding $(u-s)^{-1}$ in power series and following exactly the same steps as before that both integrals in the representation (B3) for $M_2(l,s)$ exist separately if $\text{Re}l>\max(\beta,-\gamma)$. Further, the contribution from the second integral is easily seen to be bounded by s^{-1}. Thus we obtain

$$M_2(l,s)\approx\frac{4}{\pi s}\sum_{n=0}^\infty \frac{1}{\left[s-\left(\frac{4m^2+R}{2}\right)\right]^{n+1}}\times\int_R^\infty dt\, Q_l\left(1+\frac{2t}{s-4m^2}\right)\int_{-(R-4m^2)/2}^{(R+4m^2)/2} d\vartheta$$

$$\times\rho\left(\vartheta+\frac{4m^2+R}{2},t\right)\cdot\vartheta^n+O(s^{-1}), \quad s>R$$

$$\approx O(s^{\beta-1})+O(s^{-1}). \quad \text{(B13)}$$

(iii) $M_3(l,s)$. Let us denote the first and the second integrals on the right-hand side of Eq. (B4) by $M_3'(l,s)$ and $M_3''(l,s)$, respectively. It is clear that $M_3'(l,s)$ exists for all l such that $\text{Re}l>-1$ and asymptotically it is majorized by the integral

$$\frac{\ln s}{s}\int_{4m^2}^R A_t(s,t)dt.$$

It now follows from (A23) that

$$M_3'(l,s)=O(s^{-\delta}), \quad \text{(B14)}$$

where $\delta<(\beta-1)$. As regards the integral representing $M_3''(l,s)$, we observe that its existence and asymptotic behavior is determined by that of the integral I defined by

$$I=\int_R^\infty \frac{du}{u(u-s)}\int_{4m^2}^R \rho(u,t)dt.$$

From (A20) it follows that

$$I\approx\text{const}\times\int_R^\infty \frac{u^{\beta-1}}{u-s}du.$$

Thus, I and, therefore, $M_3''(l,s)$ exists if $\beta<1$ and asymptotically $M_3''(l,s)$ is bounded by $s^{-\delta}$ where $\delta<(1-\beta)$.

(iv) $M_4(l,s)$. From the boundedness condition (40) and Eqs. (A3) and (A11), it follows that

$$\phi(4m^2-s-t,\, t)\leq\text{const}\times\max\left(\frac{s^\beta}{t^\gamma},\frac{t^\beta}{s^\gamma}\right),$$

for t large and $s>0$. Therefore, the integral representing $M_4(l,s)$ exists if $\text{Re}l>\max(\beta,-\gamma)$ and is bounded asymptotically by $s^{-\delta}$ where $\delta<(\gamma-\beta)$.