Solutions to the N and D Equations as Explicit Functions of the Coupling Constant*

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A method for obtaining solutions to the N and D equations as explicit functions of the coupling constant is developed. The analytic properties of N and D in the coupling-constant plane are investigated. Finally this method, which is particularly useful in bootstrap calculations, is used to find the self-consistent parameters in several types of scalar and pseudoscalar meson bootstrap calculations.

I. INTRODUCTION

SINCE their introduction by Chew and Mandelstam,[1] the N and D equations have been employed extensively in dynamical calculations in high-energy physics. The general procedure has been the following: one starts with an assumed interaction whose strength is determined by a coupling constant and the ND^{-1} technique is used to generate a partial-wave scattering amplitude which satisfies unitarity on the physical cut and has the input left-hand singularity. The integral equation for N is solved in one of several ways, either by iteration or by a numerical Fredholm technique. The disadvantages of these two methods of solving the N equation is that the dependence of the coupling constant is not explicit and for each new value of the coupling constant the equation must be solved again. The deficiency becomes particularly important in the case of bootstrap calculations in which one adjusts the coupling constant to produce a resonance or a bound state at a particular energy.

The purpose of this paper is to propose another method of solving the ND^{-1} which, while requiring the use of a computer, has the advantage of making the dependence of the solution on the coupling constant explicit. This method makes use of the simple analytic structure possessed by the N and D functions in the coupling constant plane. The procedure is then applied to several types of bootstrap calculations.

The following section presents the solution to the ND^{-1} equations. Section III contains the formulation of the bootstrap problem in terms of this form of the solutions to ND^{-1}. In Sec. IV numerical results are presented.

II. SOLUTION FOR THE N AND D FUNCTIONS

To illustrate this approach, based on analyticity of the N function in the coupling constant plane, we will consider the S wave scattering of equal-mass scalar mesons. The method obtained is by no means limited to this simple example.

The ND^{-1} method is a mathematical procedure for constructing a scattering amplitude which satisfies elastic unitarity on the physical or "right-hand cut" and has a prescribed discontinuity across the "left-hand cut." The partial-wave amplitude T which is analytic in the energy variable s is written:

$$T=e^{i\delta}\sin\delta/\rho=ND^{-1}, \tag{2.1}$$

where $\rho=[(s-4\mu^2)/s]^{1/2}$ is the appropriate two-body phase-space factor. The left-hand singularities are carried by N, while D is analytic on the left (s less than the physical threshold). The equations satisfied by N and D are the following:

$$N=\frac{g^2}{\pi}\int_L ds'\frac{\sigma(s')D(s')}{s'-s} \tag{2.2}$$

and

$$D=1-\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty} ds'\frac{\rho(s')N(s')}{(s'-s_0)(s'-s)}, \tag{2.3}$$

where $\sigma(s)$ is the discontinuity of T across the left cut and g^2 is a parameter (coupling constant) which controls the strength of this singularity. Since the scattering amplitude is unchanged if both N and D are multiplied by a constant we remove this ambiguity by normalizing D to one at $s=s_0$. The linear integral equation for N is obtained by substitution of Eq. (2.2) into (2.3).

$$N=g^2B+\frac{g^2}{\pi}\int_{4\mu^2}^{\infty} ds' \times\frac{[B(s')-((s-s_0)/(s'-s_0))B(s)]\rho(s')N(s')}{s'-s}, \tag{2.4}$$

where

$$B(s)=\frac{1}{\pi}\int_L ds'\frac{\sigma(s')}{s'-s}. \tag{2.5}$$

It is this equation which is usually solved by iteration or matrix inversion techniques. The method of solution proposed here is the Hilbert-Schmidt method[2] for treating inhomogeneous linear integral equations such as Eq. (2.4). The first step of the approach is to produce a symmetric kernel in Eq. (2.4). This is done by considering the following modification of N and B:

$$n(s)=\left(\frac{\rho(s)}{s-s_0}\right)^{1/2}N(s),\quad b(s)=\left(\frac{\rho(s)}{s-s_0}\right)^{1/2}B(s). \tag{2.6}$$

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[2] See, for example, R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. II, Chap. 3.

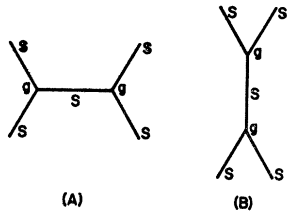

FIG. 1. Diagrams representing the scalar meson bootstrap: (a) is the interaction produced by exchange of the scalar meson which in turn produces the bound state (b) which is itself the scalar meson.

The integral equation for n is now

$$n=g^2b+g^2\int_{4\mu^2}^{\infty}ds'k(s',s)n(s'), \tag{2.7}$$

where

$$k(s',s)=\frac{1}{\pi}\left(\frac{\rho(s)}{s-s_0}\right)^{1/2}\left\{\frac{(s'-s_0)B(s')-(s-s_0)B(s)}{s'-s}\right\}\times\left(\frac{\rho(s')}{s'-s_0}\right)^{1/2}.$$

We will now assume that k is a nondegenerate kernel and that the following integrals are bounded:

$$\int\int dsds'[k(s,s')]^2, \quad \int ds[k(s,s')]^2.$$

In this case the kernel k has a discrete set of real eigenvalues and any function which can be represented in the form

$$g(s)=\int_{4\mu^2}^{\infty}ds'k(s,s')f(s'),$$

where f is piece-wise continuous can be expanded in terms of the eigenfunctions of k. The orthonormal eigenfunctions, which will be denoted by $u_n(s)$, satisfy the following equation:

$$\lambda_nu_n(s)=\int_{4\mu^2}^{\infty}ds'k(s,s')u_n(s'). \tag{2.8}$$

We now expand n and b in terms of the u's:

$$n(s)=\sum a_nu_n(s) \quad b(s)=\sum b_nu_n(s), \tag{2.9}$$

where

$$b_n=\int_{4\mu^2}^{\infty}ds'b(s')u_n(s').$$

Substituting these expansions into Eq. (2.7) and projecting out the ith component, we obtain

$$a_i=g^2b_i+g^2\lambda_ia_i. \tag{2.10}$$

Finally

$$n(s)=\sum\frac{g^2b_iu_i(s)}{1-g^2\lambda_i}. \tag{2.11}$$

Thus we see that n is a meromorphic function of the coupling constant with poles only on the real axis, and furthermore the positions of these poles are fixed independent of energy. We may now transform back to the usual N:

$$N(s)=\left(\frac{s-s_0}{\rho(s)}\right)^{1/2}n(s)=\sum\frac{g^2b_nU_n(s)}{1-g^2\lambda_n}, \tag{2.12}$$

where

$$U_n(s)=\left(\frac{s-s_0}{\rho(s)}\right)^{1/2}u_n(s).$$

If we define

$$V_n(s)=\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\rho(s')U_n(s')}{(s'-s_0)(s'-s)}, \tag{2.13}$$

the D function may be expressed as an explicit function of g^2.

$$D(s)=1-\sum\frac{g^2b_nV_n(s)}{1-g^2\lambda_n}.$$

Both N and D have fixed poles in the coupling constant plane whose location depends on the subtraction point s_0 as well as the function B. On the other hand, T has poles in the coupling constant plane which move as a function of the energy.

Let us examine the physical significance of these poles in the g^2 plane. First, consider g^2 small but approaching the pole due to the largest positive eigenvalue λ_1. As $g^2\to 1/\lambda_1$, one term dominates both N and D:

$$N\to g^2b_1U_1(s)/(1-\lambda_1g^2),$$
$$D\to -g^2b_1V_1(s)/(1-\lambda_1g^2).$$

Since the V's have a zero at $s=s_0$ the scattering amplitude T has a bound-state pole at s_0 for $g^2=1/\lambda_1$. Increasing g^2 to the next largest eigenvalue corresponds to having a second bound state reach the point s_0. From this form of the ND^{-1} solutions it is clear that in a bootstrap calculation one must be more precise than simply stating that a bound state is to be produced at a given energy as it is clear that there are infinitely many solutions $g^2=1/\lambda_1, \cdots 1/\lambda_n$ that have a bound state at s_0.

Finally the dependence of the positions of the poles on s_0 can be understood as follows. Each pole is at a value of g^2 at which a bound state has reached s_0; clearly if s_0 is decreased, a larger coupling constant will be required to produce a bound state at a lower energy (more binding energy). The poles which lie on the negative real axis correspond to ghost poles in the ND^{-1}

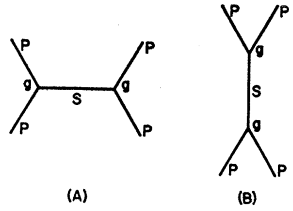

FIG. 2. Diagrams for two pseudoscalar mesons (P) producing a scalar meson (S) from the force provided by the exchange of the scalar meson.

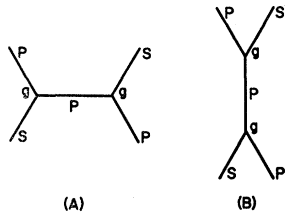

FIG. 3. Diagrams for a pseudoscalar meson (P) and a scalar meson (S) forming a pseudoscalar bound state. The force is provided by the pseudoscalar meson exchange in one crossed channel.

solution which appear for a sufficiently repulsive interaction.

From the analytic structure of N we see that the iterative solution to Eq. (2.7) (Neuman series) can converge only if $g^2<|1/\lambda_m|$ where λ_m is the eigenvalue with the largest absolute magnitude, as this solution is a power-series expansion in the coupling constant about $g^2=0$. This explains the dependence of the convergence of this expansion on the value of s_0.

III. FORMULATION OF THE BOOTSTRAP CALCULATION

In discussing bootstrap calculation let us consider as an example the simplest bootstrap, namely, the scalar meson. In this case the exchange of a scalar meson of mass μ [Fig. 1(a)] is required to produce a bound state of mass μ in the S-wave scattering amplitude for two scalar mesons [Fig. 1(b)]. Since all the scalar mesons are identical the coupling constant is the only free parameter and is adjusted to produce the bound state at the correct energy.

The interaction term produced by this particle exchange is

$$B(s)=[2/(s-4)]\ln(s-3), \tag{3.1}$$

where μ has been set equal to 1. If we now set $s_0=1$, the solution with only one bound state located at s_0 is obtained for $g^2=1/\lambda_1$. The self-consistency equation is then

$$g^2\equiv 1/\lambda_1=U_1(1)/[(\partial/\partial s)V_1(s)]_{s=1}. \tag{3.2}$$

This approach clearly has the advantage of giving the input value of g^2 directly rather than solving the N and D equations a number of times varying g^2 to produce the desired bound state.

The numerical results for this case are that the input $g^2=17.1$ while the output g^2 from Eq. (3.2) is 105 in close agreement with the values obtained by Collins[3] in a more detailed treatment of the scalar meson bootstrap.

Another bootstrap possibility is binding two identical pseudoscalar mesons in an S state to produce a scalar meson (Fig. 2). In this case the Born term is

$$B(s)=[2/(s-4)]\ln(1+(s-4)/\mu^2), \tag{3.3}$$

where μ is the mass of the scalar meson and the pseudoscalar mass is set to 1. In this case μ can be varied to produce a self-consistent value of g^2. The numerical results for this case are that the self-consistent values are $\mu^2=3.5$ and $g^2=5.3$.

The third type of bootstrap investigated here is a scalar meson of mass m and a pseudoscalar meson of mass μ which are bound to produce the pseudoscalar meson. The force is produced by the pseudoscalar meson being exchanged in one of the crossed channels (as shown in Fig. 2).

The Born term in this case is

$$B(s)=(1/4q^2)\ln(1+4q^2/\mu^2), \tag{3.4}$$

where

$$q^2=[s-(m+\mu)^2][s-(m-\mu)^2]/4s.$$

The ratio of m/μ can be varied to produce a self-consistent g^2. The self-consistent values are $g^2=4.8$ and $(m/\mu)^2=0.18$.

The simplification provided by the eigenfunction solution to N and D makes possible the investigation of two-particle bootstrap problems in which one includes the exchange of two mesons and requires the force to produce two bound states at the appropriate energies. In this calculation the ratios of the coupling constants and the ratio of the masses are the free parameters to be adjusted for self-consistency.

It should be noted that in the bootstrap problems which have been discussed here the two-particle continuum in the crossed channel and inelastic effects in the s channel have been ignored. The additional interaction produced by the continuum does not have the form which has been assumed for the exchange interaction in Eq. (2.5) in that its strength is not controlled by a coupling constant and hence cannot be easily included in the calculation scheme being discussed.

IV. NUMERICAL CONSIDERATIONS

To illustrate the advantages of the eigenfunction expansion solution from a numerical point of view we will compare this method with the most commonly employed method of matrix inversion. In either approach the interval $4\mu^2$ to ∞ is mapped into a finite interval which is then represented by finite (n) mesh. In this form Eq. (2.4) is simply a vector equation

$$\mathbf{N}=g^2\mathbf{B}+g^2K\mathbf{N}, \tag{4.1}$$

where K is an $n\times n$ matrix. The solution for N is

$$\mathbf{N}=g^2[I-g^2K]^{-1}\mathbf{B}. \tag{4.2}$$

Most of the computer time required for this calculation is required for the inversion of the matrix $[I-g^2K]$ which must be done for each value of g^2 of interest. The eigenfunction method requires the computation of the eigenvalues and eigenvectors of K which requires computer time comparable to the time required by matrix inversion. As a result producing the solution for any additional values of g^2 results in a minor increase in computer time when the eigenfunction method is used.

[3] P. D. B. Collins, University of California Lawrence Radiation Laboratory Report, UCRL 11463, 1964 (unpublished).

TABLE I. The five largest eigenvalues and projections b_i for the scattering of scalar mesons with the exchange of a mass $3\mu^2$ scalar meson. The value of s_0 was also $3\mu^2$.

Number	Eigenvalue	b_i
1	0.160	−0.334
2	0.00945	0.0283
3	0.00118	−0.00464
4	0.000235	+0.00106
5	8.19×10^{-5}	−0.00028

The fact that the Born terms used in most calculations have relatively simple structure for $s>4\mu^2$ (for example the scalar exchange Born term is positive definite and monotonically decreasing for $s>4$) means that only a few of the coefficients b_i will be important, allowing the N function to be represented over a wide range of coupling constants as a simple function of g^2. To illustrate this point the two scalar problem has been solved for a meson exchange mass of $3\mu^2$ with 50 mesh points. In Table I are given the eigenvalues for the five largest coefficients b_i. For the range $0<g^2<5000$ the N function is represented by

$$N(s)=\sum_{i=1}^{4}\frac{b_iU_i(s)}{(1/g^2)-\lambda_i}$$

and D is given by

$$D(s)=1-\sum_{i=1}^{4}\frac{b_iV_i(s)}{(1/g^2)-\lambda_i}$$

to an accuracy of a few percent. The value of s_0 used was $3\mu^2$.

V. CONCLUSION

The method which has been discussed here produces solutions to N and D which are explicit functions of the coupling constant and which, for wide ranges of the coupling constant, can be approximated by a small number of terms.

The "bootstrap" problem is particularly simple when this form of the ND^{-1} solution is used as one parameter is determined directly.

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Analytic Continuation of Partial-Wave Amplitude in the Complex Angular-Momentum Plane

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An attempt is made to continue analytically the partial-wave amplitude for the scattering of two identical spinless particles in the complex l plane, exploiting unitarity and analyticity properties in s. The Froissart-Gribov representation for the partial-wave amplitude is known to be holomorphic in the region $\mathrm{Re}l>\alpha$ of the complex l plane provided the absorptive part $A_t(s,t)$ of $A(s,t)$, the scattering amplitude in the t channel, is bounded by t^α for any fixed s. Apart from the above assumptions, two crucial hypotheses on which the present analysis is based are (i) the possibility of extending unitarity in the inelastic region to complex values of l, and (ii) the boundedness condition, viz., that both $A_t(s,t)$ and $A(s,t)$ are asymptotically bounded by the maximum of $(t^\beta/s^\gamma, s^\beta/t^\gamma)$ if s and t are both sufficiently large with $\gamma>0$ and $\beta<\min(1,\gamma)$. With the help of the N/D technique it is then possible to continue analytically the partial-wave amplitude up to the line $\mathrm{Re}l=\beta$ and show that it is meromorphic in the region $\beta<\mathrm{Re}l\leqslant\alpha$. The domain of meromorphy of the partial-wave amplitude obtained by the method of analytic completion is smaller than the preceding one. The analytically continued partial-wave amplitude is bounded by $|l|^{-1/2}$ for large values of $\mathrm{Im}l$, so that a Regge representation for $A(s,t)$ can be obtained. The N/D method of analytic continuation does not work beyond the line $\mathrm{Re}l=-1$ even if one assumes $\beta<-1$. It has also been shown that accumulation of poles at $l=-\frac{1}{2}$ near threshold, a feature which has been pointed out by several authors, is also manifested in the analytically continued partial-wave amplitude.

I. INTRODUCTION

THE purpose of the present work is to discuss the problem of analytic continuation of the relativistic partial-wave amplitude in the complex angular-momentum plane and hence to investigate the singularities which one encounters in such a procedure. In the case of nonrelativistic scattering by potentials, Schrödinger equation provides a very convenient framework within which this problem has been tackled.[1] In relativistic scattering the absence of a Schrödinger equation makes the situation very much complicated. It is, however, presumed that unitarity and the analyticity properties of the scattering amplitude as contained in the Mandelstam representation play the role of a

[1] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento 23, 954 (1962).