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$\pi^+\pi^+p$ Resonance—A Static-Model Calculation*

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The $\pi^+\pi^+p$ scattering amplitude in the $J=\frac{5}{2}$ (even-parity) state is calculated, in the static-nucleon approximation. The isobar model is *not* used, and the symmetry required by Bose statistics is maintained throughout. As a consequence, although the static model radically simplifies the three-body problem, the solution displays some interesting three-body features. A discussion of the physical interpretation of the solution is presented, especially with regard to the definition of a three-particle resonance. Such a resonance is found; rough values for the position and half-width are 1550 and 125 MeV, respectively. No arbitrary parameters are used in obtaining these numbers, which agree rather well with recently reported experimental values for a $\pi^+\pi^+p$ enhancement in the final state of $\pi^++p\rightarrow\pi^-+\pi^++\pi^++p$.

I. INTRODUCTION

FOR a long time there has been speculation concerning the possibility of a resonance in pion-nucleon systems with isotopic spin $T=\frac{5}{2}$ and angular momentum $J=\frac{5}{2}$. The simplest such system is $\pi^+\pi^+p$, which is pure $T=\frac{5}{2}$. The original basis for these speculations was the old strong-coupling calculations,[1] which predicted a squence of bound pion-nucleon states with $2J=2T=1, 3, 5, \cdots$. More recently, a number of authors[2–4] have considered this problem from different points of view and all find indications that a resonance should exist in the (5,5) state. Roughly, the reason for this is that the exchange of a nucleon or an N^* (i.e., the πN resonance at 1238 MeV) provides a strong attractive force in the (5,5) state. (See Fig. 1.) All of these calculations neglect nucleon recoil.

There has recently been reported the observation of a $\pi^+\pi^+p$ enhancement at 1560 MeV with a half-width of 110 MeV.[5] This has led to further speculations regarding resonances in similar systems.[6–9] The generalizations of both the strong coupling calculations[6] and the exchange calculations[7] using SU_3 symmetry leads one to expect the $\pi^+\pi^+p$ resonance to be a member of a 35-dimensional multiplet. Many interesting questions are associated with this possibility, but in this paper we shall ignore strange particles altogether.

The purpose of the present work is to present a simple calculation of $\pi^+\pi^+p$ scattering which goes further in dealing with the three-particle system than the earlier calculations. The usual practice in treating three-particle systems, in which a pair can resonate, is to use the isobar model[10] to reduce the problem to a two-body problem. While this is frequently a successful procedure, some of the interesting features of the problem are lost. In this calculation, no such approximation will be made; the pion energies will be allowed to vary over the entire range compatible with energy conservation. Furthermore, the two pions will be treated in a completely symmetric way and no aritficial symmetrization of the final result will be needed. Lacking a useful dynamical basis for treating three-particle field-theory problems, we too will work in the static-nucleon approximation. Only fairly low energies of the individual pions, energies for which the static model has already been successful in πN scattering, will therefore be considered. At such energies, direct $\pi\pi$ interactions are probably not too important and will be omitted from the calculation. These assumptions radically simplify the three-body problem. However, they do provide us with a soluble problem, the solution of which has several interesting features not present in the two body case. In particular, we will be able to discuss to some extent the definition of a three-particle resonance. By physical arguments, the scattering amplitude will be separated into a part which results from the initial- and final-state interactions, and a part which results from the simultaneous interaction of the three particles. The latter part will be shown to be ≤ 1 in absolute value. The energy for which the equality holds will be called a three-body resonance energy.[11] The solution will be

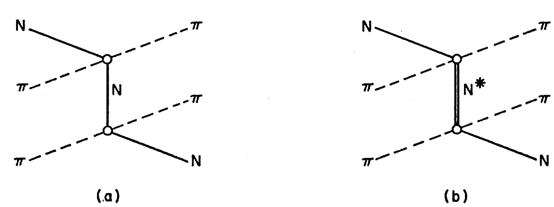

FIG. 1. $\pi^+\pi^+p$ scattering by way of (a) nucleon exchange, (b) (3,3) resonance exchange.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] For a survey and references to original work, see G. Wentzel, Rev. Mod. Phys. **19**, 1 (1947).

[2] W. N. Wong and M. Ross, Phys. Rev. Letters **3**, 398 (1959).

[3] A. Messiah, Phys. Letters **1**, 181 (1962).

[4] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[5] G. Goldhaber, S. Goldhaber, T. A. O'Halloran, and B. C. Sherr, University of California Radiation Laboratory Report No. UCRL-11445 (unpublished).

[6] G. Wentzel (unpublished).

[7] E. S. Abers, L. A. P. Balazs, and Y. Hara, Phys. Rev. (to be published).

[8] R. J. Oakes (unpublished).

[9] H. Harrari and H. J. Lipkin, Phys. Rev. Letters **13**, 345 (1964).

[10] R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. **109**, 1723 (1958); **123**, 333 (1961).

[11] No implication regarding associated poles in the nearby unphysical region is intended by this definition. For some interesting comments which bear on this question, see C. Goebel, Phys. Rev. Letters **13**, 143 (1964).

shown to display such a resonance in the $J=\frac{5}{2}$ state with parameters that agree quite well with the experimental values quoted above.

II. CALCULATION OF THE $\pi^+\pi^+p$ WAVE FUNCTION

In this section, we shall construct an approximate wave function for the $\pi^+\pi^+p$ scattering state by a method which has been applied with some success in static-model calculations of the (3,3) resonance[12] and the Y_1^*.[13] The model which forms the basis for this calculation is the well-known static model of Chew, Low and Wick.[14] The interaction Hamiltonian is

$$H_{\text{int}}=(4\pi)^{1/2}f\int d^3x\,\boldsymbol{\sigma}\cdot\nabla[\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)]v(x)\,, \qquad (2.1)$$

where $\phi(x)$ denotes the pion fields, $v(x)$ the source function of the nucleon, and $f^2\cong 0.08$ with the pion mass $\mu=1$.

The calculation proceeds by constructing a trial state

$$|pq\rangle_+=\sum_{p''q''}a_{p''}^\dagger a_{q''}^\dagger|\ \rangle\chi(p'',q'';p,q)\,. \qquad (2.2)$$

Here $|pq\rangle_+$ represents an approximation to the correct scattering state which asymptotically consists of two pions in plane waves of momentum p and q plus scattered waves with the nucleon fixed at the origin; $a_k^\dagger$ is the creation operator for a π^+ meson with momentum k (only π^+ mesons will enter this calculation so isotopic indices are unnecessary); $|\ \rangle$ denotes a physical proton state with spin indices understood, and χ is a matrix in spin space. The zero of energy is chosen so that $H|\ \rangle=0$. The functions $\chi(p'q';pq)$ are to be determined by the condition

$$\frac{\delta}{\delta\chi(p'q';pq)}{}_+\langle pq|(H-\omega)|pq\rangle_+=0\,, \qquad (2.3)$$

where $\omega=\omega_p+\omega_q$, $\omega_p^2=p^2+1$. By means of the relations

$$[H,a_k^\dagger]=\omega_k a_k^\dagger+V_k\,, \qquad (2.4a)$$

$$\langle\ |V_k=-\langle\ |a_k^\dagger(H+\omega_k)\,, \qquad (2.4b)$$

Eq. (2.3) may be written

$$\begin{aligned}\chi(p',q')(\omega-\omega_{p'}-\omega_{q'})&=-\sum_{q''}\langle|a_{q''}^\dagger(H+\omega-\omega_{p'})a_{q'}|\rangle\chi(q'',p')\\&-\sum_{q''}\langle|a_{q''}^\dagger(H+\omega-\omega_{q'})a_{p'}|\rangle\chi(q'',q')\\&-\tfrac{1}{2}\sum_{p'',q''}\langle|a_{p''}^\dagger a_{q''}^\dagger(H+\omega)a_{q'}a_{p'}|\rangle\chi(q'',p'')\,. \qquad (2.5)\end{aligned}$$

The arguments p, q of χ are no longer written explicitly since they enter the calculations only parametrically.

In order to obtain a simply solvable equation for χ, the operator products are expanded over intermediate states. It is then assumed that all matrix elements $\langle n|a_k|\rangle$ may be neglected except when $|n\rangle$ represents the neutron state. This is the same approximation made in the calculation of the (3,3) resonance by this method. The result of this approximation is that Eq. (2.5) becomes

$$\begin{aligned}\chi(p',q')(\omega-\omega_{p'}-\omega_{q'})&=-\sum_{q''}\left\{\frac{\bar\omega_{p'}}{\omega_{q'}\omega_{q''}}\langle|V_{q''}|n\rangle\langle n|V_{q'}^\dagger|\rangle\chi(q'',p')\right.\\&\left.+\frac{\bar\omega_{q'}}{\omega_{p'}\omega_{q''}}\langle|V_{q''}|n\rangle\langle n|V_{p'}^\dagger|\rangle\chi(q'',q')\right\}\,, \qquad (2.6)\end{aligned}$$

where $\bar\omega_k\equiv\omega-\omega_k$ and

$$\langle|V_k|n\rangle=i\frac{f}{\pi}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{(2\omega_k)^{1/2}}u(k)\,; \qquad (2.7)$$

$u(k)$ is the cutoff function, the Fourier transform of $v(x)$. To get more useful equations, Eq. (2.7) is transformed to the angular momentum representation by way of

$$\chi_{JLM}(\omega_{p'},\omega_{q'})=\sum_{m,n,\mu}[(2J+1)(2L+1)]^{1/2}\begin{pmatrix}J&\mu&s\\M&L&\frac{1}{2}\end{pmatrix}\begin{pmatrix}L&m&n\\\mu&1&1\end{pmatrix}\int d\Omega_{q''}\int d\Omega_{p''}(p'\omega_{p'}q'\omega_{q'})^{1/2}\times Y_1^{m*}(\hat p')Y_1^{n*}(\hat q')\chi_s(p',q')\,. \qquad (2.8)$$

[Of course, there is a similar projection on the variables p and q. Since this is exactly the same for all members of Eq. (2.7), we may take it as understood.] In general, this leads to sets of coupled equations for the $\chi_{JLM}(\omega_{p'},\omega_{q'})$.

[12] B. Bosco, S. Fubini, and A. Stanghellini, Nucl. Phys. **10**, 663 (1959).
[13] D. Amati, A. Stanghellini, and B. Vitale, Nuovo Cimento **13**, 1143 (1958); T. L. Trueman, Phys. Rev. **127**, 2240 (1962).
[14] See, for example, S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), Chap. 12.

However, for the case of primary interest $J=\frac{5}{2}$, a very simple uncoupled equation results:

$$(\omega-\omega_{p'}-\omega_{q'})\chi_{5/2,2,5/2}(\omega_{p'},\omega_{q'})=-\frac{4}{3}f^2\frac{1}{\pi}\int_1^\infty d\omega''u(q'')\frac{q''^{3/2}}{\omega_{q''}}\times\left\{\frac{q'^{3/2}}{\omega_{q'}}u(q')\bar{\omega}_{p'}\chi_{5/2,2,5/2}(\omega_{q''},\omega_{p'})+\frac{p'^{3/2}}{\omega_{p'}}u(p')\bar{\omega}_{q'}\chi_{5/2,2,5/2}(\omega_{q''},\omega_{q'})\right\}. \quad (2.9)$$

(From now on, the subscripts will be dropped since this is the only amplitude to be considered.)

The boundary conditions are incorporated into Eq. (2.9) by rewriting it as

$$\chi(\omega_{p'},\omega_{q'})=\delta(\omega_{p'}-\omega_p)\delta(\omega_{q'}-\omega_q)+\delta(\omega_{q'}-\omega_p)\delta(\omega_{p'}-\omega_q)+\frac{1}{\pi}\frac{\lambda}{\omega_{p'}+\omega_{q'}-\omega-i\epsilon}\left\{u(q')q'^{3/2}\frac{\bar{\omega}_{p'}}{\omega_{q'}}K(\omega_{p'})+u(p')p'^{3/2}\frac{\bar{\omega}_{q'}}{\omega_{p'}}K(\omega_{q'})\right\}, \quad (2.10)$$

where $\lambda=\frac{4}{3}f^2$ and

$$K(\omega_{p'})=\int_1^\infty d\omega_{q''}\frac{q''^{3/2}u(q'')}{\omega_{q''}}\chi(\omega_{q''},\omega_{p'}). \quad (2.11)$$

At this point, it is convenient to introduce the following functions:

$$D(\omega_{p'})=1-\omega_{p'}\frac{\lambda}{\pi}\int_1^\infty d\omega_{q''}\frac{q''^3u^2(q'')}{\omega_{q''}^2(\omega_{q''}-\omega_{p'}-i\epsilon)}, \quad (2.12)$$

$$L(\omega_{p'})=K(\omega_{p'})D(\bar{\omega}_{p'})\frac{\omega_{p'}}{u(p')p'^{3/2}}-\frac{u(\bar{p}')}{u(p')}\left(\frac{\bar{p}'}{p'}\right)^{3/2}\frac{\omega_{p'}}{\bar{\omega}_{p'}}[\delta(\omega_{p'}-\omega_p)+\delta(\omega_{p'}-\omega_q)]. \quad (2.13)$$

It follows from Eqs. (2.10)–(2.13) that $L(\omega_{p'})$ satisfies

$$L(\omega_{p'})=\frac{\lambda}{\pi}\frac{(pq)^{3/2}u(p)u(q)}{\omega_p\omega_q}\left[\frac{\omega_q}{D(\omega_q)(\omega_{p'}-\omega_q-i\epsilon)}+\frac{\omega_p}{D(\omega_p)(\omega_{p'}-\omega_p-i\epsilon)}\right]-\frac{1}{\pi}\int_1^\infty d\omega_{q''}\frac{\mathrm{Im}D(\omega_{q''})}{D(\bar{\omega}_{q''})}\frac{\bar{\omega}_{q''}}{\omega_{q''}}\frac{L(\omega_{q''})}{\omega_{q''}-\bar{\omega}_{p'}-i\epsilon}. \quad (2.14)$$

$\bar{p}'$ is shorthand for $[(\omega-\omega_{p'})^2-1]^{1/2}$. $D(\omega_{p'})$ will be recognized as the denominator function of the πN scattering amplitude in the (3,3) state, in the one-meson approximation with the crossed πN cut neglected. Explicitly,

$$e^{i\delta(\omega_p)}\sin\delta(\omega_p)=u^2(p)\lambda p^3/\omega_p D(\omega_p), \quad (2.15)$$

where $\delta(\omega_p)$ denotes the (3,3) phase shift.

Thus, the integral equation for $\chi(\omega_{p'},\omega_{q'})$ is reduced to a fairly simple single-variable integral equation for $L(\omega_{p'})$. An equation similar to this was obtained by Källén and Pauli[15] in the problem of constructing $|V\theta\rangle$ states in the Lee model; the significant difference is the inhomogeneous term in Eq. (2.14), which has a pole in the region of integration. Recently, Kenshaft and Amado[16] have presented a solution of the Källén-Pauli equation. Rather than try to adapt their method to Eq. (2.14), we present a different technique, which seems more direct, for obtaining a solution to such equations.

The function L can be defined by Eq. (2.14) for complex energy z, with $L(z)\to L(\omega_{p'})$ as $z\to\omega_{p'}-i\epsilon$. $L(z)$ has poles at ω_p, ω_q and a branch cut from $-\infty$ to $\omega-1$. The main difficulty in solving this equation is that the discontinuity across the cut at $\omega_{p'}$ is related to $L(\omega-\omega_{p'})$, not to $L(\omega_{p'})$. Hence, some relation between $L(z)$ and $L(\omega-z)$ is necessary in order to proceed further.[17] This is provided by the simple relation between the discontinuity of $L(\omega_{p'})$ and the discontinuity of $D(\bar{\omega}_{p'})$. Form the symmetric combination

$$M(z)=(1/z)L(z)D(z)+[1/(\omega-z)]L(\omega-z)D(\omega-z),$$

and observe that $M(z)$ has no discontinuity across the

[15] G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

[16] R. P. Kenshaft and R. D. Amado, J. Math. Phys. **5**, 1340 (1964).

[17] Equations of the sort solved by R. D. Amado [Phys. Rev. **122**, 696 (1961)] and P. K. Srivastava [Phys. Rev. **131**, 464 (1963)] also relate the discontinuity at $\omega_{p'}$ to the value of the function at $\omega-\omega_{p'}$. However, their integral equations have a simple symmetry under $z\leftrightarrow\omega-z$ which provides the necessary relation immediately.

real axis. Furthermore, $M(z)$ does not have poles at ω_p and ω_q. [The first point is trivial; the second, though straightforward, requires a little care since the integral in Eq. (2.14) contributes a pole to $L(z)$ on one side of the cut.] $M(z)$ does have poles, in general, at 0 and ω. Thus,

$$(1/z)L(z)D(z)+[1/(\omega-z)]L(\omega-z)D(\omega-z) = b/z(\omega-z)\,, \qquad (2.16)$$

where b is a constant; i.e., b may depend on ω_p and ω_q, but not on z. At this point, the familiar sort of ambiguity arises: an arbitrary polynomial can be added to the right-hand side of Eq. (2.16) without changing the required analytic properties. As is usual in this situation, we choose the simplest alternative and assume that no polynomial is present. [The high-energy behavior of the cutoff function and the implicitly assumed convergence of the basic integrals in Eq. (2.9) can limit the order of the polynomial. However, we know of no pyhsical reason for specifying the precise behavior of $u(k)$ and so prefer to state the assumption simply, as above.] Then Eq. (2.16) provides the needed symmetry relation. With this relation, it is routine to obtain a solution of Eq. (2.14) by the Omnés method.[18] b is then determined by the condition

$$b=\omega L(0)\,. \qquad (2.17)$$

The result of this calculation is

$$L(z)=\frac{\lambda}{\pi}\frac{(pq)^{3/2}u(p)u(q)D(\omega-z)}{\omega_p\omega_q D(\omega_p)D(\omega_q)}\left\{\frac{\omega_p}{z-\omega_p}+\frac{\omega_q}{z-\omega_q}+\frac{2\omega D(\omega)A(z,\omega)}{1+\omega D(\omega)A(0,\omega)}\right\}, \qquad (2.18)$$

where

$$A(z,\omega)=-\frac{1}{\pi}\int_1^\infty d\omega''\left[\mathrm{Im}\frac{1}{\omega''D(\omega'')}\right]\times\frac{1}{D(\bar{\omega}'')}\frac{1}{\omega''+z-\omega}\,. \qquad (2.19)$$

When this result is combined with Eqs. (2.10)–(2.13), the expression for the wave function is obtained:

$$\chi(\omega_{p'},\omega_{q'})=\delta(\omega_{p'}-\omega_p)\delta(\omega_{q'}-\omega_q)+\delta(\omega_{p'}-\omega_q)\delta(\omega_{q'}-\omega_p)+\frac{1}{\pi}\frac{\lambda}{\omega_{p'}+\omega_{q'}-\omega-i\epsilon}\left\{\left[\frac{\lambda u^2(q)q^3}{\omega_q D(\omega_q)}+\frac{\lambda u^2(p)p^3}{\omega_p D(\omega_p)}+2i\frac{\lambda^2u^2(p)u^2(q)p^3q^3}{\omega_pD(\omega_p)\omega_qD(\omega_q)}\right][\delta(\omega_{p'}-\omega_p)+\delta(\omega_{p'}-\omega_q)]+\frac{\lambda^2u(p)u(q)u(p')u(q')(p'q'pq)^{3/2}}{\pi\omega_pD(\omega_p)\omega_qD(\omega_q)}\times\left[-\frac{2\omega}{\omega_{p'}\omega_{q'}}+\frac{2\omega D(\omega)[(1/\omega_{p'})A(\omega_{p'},\omega)+(1/\omega_{q'})A(\omega_{q'},\omega)]}{1+\omega D(\omega)A(0,\omega)}\right]\right\}. \qquad (2.20)$$

Within the brackets { } we have set $\omega_{p'}+\omega_{q'}=\omega$ since only terms satisfying this condition contribute to the asympotic wave function.

III. DISCUSSION

Equation (2.20) contains all the information we need in order to calculate the $\pi^+\pi^+p$ scattering amplitude. Thus, the S matrix for the scattering of two π^+ mesons with energies ω_p, ω_q into two π^+ mesons with energies $\omega_{p'}$, $\omega_{q'}$, in the $J=\frac{5}{2}$ state, is given by

$$S(\omega_{p'},\omega_{q'};\omega_p,\omega_q)=\left\{\tfrac{1}{2}e^{2i[\delta(\omega_p)+\delta(\omega_q)]}[\delta(\omega_{p'}-\omega_p)+\delta(\omega_{p'}-\omega_q)]-2\pi i\frac{e^{i\delta(\omega_p)}\sin\delta(\omega_p)}{p^{3/2}}\frac{e^{i\delta(\omega_q)}\sin\delta(\omega_q)}{q^{3/2}}\frac{e^{i\delta(\omega_{q'})}\sin\delta(\omega_{q'})}{q'^{3/2}}\frac{e^{i\delta(\omega_{p'})}\sin\delta(\omega_{p'})}{p'^{3/2}}T(\omega)\right\}, \qquad (3.1)$$

with

$$T(\omega)=\frac{\omega}{\pi^2\lambda^2}\frac{D(\omega)}{1+\omega D(\omega)A(0,\omega)}\,. \qquad (3.2)$$

The structure of Eq. (3.1) is just what one would expect on rather general grounds: the first term corresponds to combinations of simple two-body processes (Fig. 2) while the second term corresponds to "true" three-particle scattering. The S matrix is explicitly symmetric under the interchange of the two initial or final mesons, by Bose statistics, and under the interchange of the initial and final particles, as required by time reversal invariance. The factors $e^{i\delta}\sin\delta/k^{3/2}$ reflect initial- and final-state interactions.[19] $T(\omega)$ has the expected analytic

[18] R. Omnés, Nuovo Cimento **8**, 316 (1958).
[19] R. F. Peierls and J. Tarski, Phys. Rev. **129**, 981 (1963).

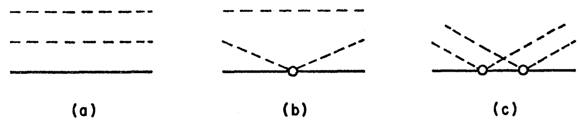

FIG. 2. These diagrams represent the contributions to the first term of Eq. (3.1). There is a diagram for each possible way of attaching initial and final pion energies to the dashed lines.

properties, viz. a branch point at $\omega=2$ and, if $D(\omega)$ has a zero at ω^* on its second sheet, a branch point at ω^*+1 on its second sheet.[20,21]

Below the threshold for producing additional mesons, the unitarity condition is $(\omega_{p'}+\omega_{q'}=\omega)$:

$$\int_1^{\omega-1} d\omega_{p''} S^*(\omega_{p'},\omega_{q'}\,;\,\omega_{p''},\bar{\omega}_{p''})S(\omega_{p''},\bar{\omega}_{p''}\,;\,\omega_p,\omega_q) =\tfrac{1}{2}[\delta(\omega_p-\omega_{p'})+\delta(\omega_{p'}-\omega_q)]. \quad (3.3)$$

For an S matrix of the form given in Eq. (3.1), this implies that

$$\operatorname{Im}\frac{1}{T(\omega)}=\pi\int_1^{\omega-1} d\omega_k \frac{\sin^2\delta(\omega_k)}{k^3}\frac{\sin^2\delta(\bar{\omega}_k)}{\bar{k}^3}\equiv\pi\kappa(\omega). \quad (3.4)$$

This equation is closely related to the two particle unitarity condition $\operatorname{Im}[1/T(\omega_k)]=\pi k$. It may be explicitly verified for the function given by Eq. (3.2).

The factor $T(\omega)$ has a resonance structure, as will be discussed in the next section. However, because of the other parts of S one cannot conclude immediately that this can properly be called resonance scattering. Since, in any observation of a $\pi^+\pi^+p$ state, the initial state will not be prepared in plane waves, it is clear that whether or not a bump is seen will depend strongly on how the initial state is prepared. Before discussing this further it is useful to note an elementary aspect of the problem.

A potential analog of this problem is the following: two particles of coordinates r_1 and r_2 interacting with a center of force by way of a short-range potential of the form

$$V(r_1,r_2)=V_1(r_1)+V_2(r_2)+V_{12}(r_1,r_2)$$

where, because there is no direct $\pi\pi$ interaction, $V_{12}(r_1,r_2)$, is nonzero only when both r_1 and r_2 are small. Then if ρ is the range of the potentials and R is the radius of the spherical quantization volume, the effect of $V_{12}(r_1,r_2)$ is smaller by a factor of order (ρ/R) than $V_1(r_1)$ and $V_2(r_2)$ when plane-wave boundary conditions are imposed. This is quite clear in our results; for example, if the wave function [Eq. (2.20)] is transformed to coordinate space, the asymptotic form is just the symmetrized product of two π^+p, $J=\frac{3}{2}$ wave functions. In order to see any effects of the true three particle scattering, it is necessary to form pion wave packets which both reach the proton at nearly the same time. Of course, this is what is expected in practice when $\pi^+\pi^+p$ is produced as a final state in reactions such as

$$\pi^++p\rightarrow\pi^-+\pi^++\pi^++p\,,$$
$$p+p\rightarrow\pi^-+n+\pi^++\pi^++p\,.$$

In order to avoid discussing the details of how the final state is created, let us consider the following situation, which can be realized in principle if not in practice: initially two pions are in spherical wave packets, impinging on the proton, with total angular momentum $J=\frac{5}{2}$. By choosing the two wave packets to be identical, we ensure that they peak at the proton simultaneously. In general, some of the scattering results from the first term of the S matrix, Eq. (3.1). Since this is just combinations of simple two-body scattering, it is not of direct interest. This effect can easily be avoided by using the following property of two-body scattering: if the initial wave packet is given by $g(\omega_k)e^{-i\delta(\omega_k)}$ with $g(\omega_k)$ real, then the wave packet has the same shape asymptotically before and after the scattering. [The final wave packet is given by $g(\omega_k)e^{i\delta(\omega_k)}$.] Thus, if we calculate the $\pi^+\pi^+p$ S-matrix between initial and final wave packets of the form $g(\omega_p)g(\omega_q)e^{-i\delta(\omega_p)-i\delta(\omega_q)}$ and $g(\omega_{p'})g(\omega_{q'})e^{i\delta(\omega_{p'})+i\delta(\omega_{q'})}$, respectively, the simple two-body effects are removed. Then

$$S_g=\int d\omega\int_1^{\omega-1} d\omega_p\int d\omega'\int_1^{\omega'-1} d\omega_{p'}\,e^{-i[\delta(\omega)+\delta(\bar{\omega}_p)+\delta(\omega_{p'})+\delta(\bar{\omega}_{p'})]}S(\omega_{p'},\omega_{q'}\,;\,\omega_p,\omega_q)\delta(\omega'-\omega)g(\omega_{p'})g(\bar{\omega}_{p'})g(\omega_p)g(\bar{\omega}_p)$$
$$=\int d\omega\, G(\omega)[1-2\pi i\kappa(\omega)T(\omega)\eta^2(\omega)], \quad (3.5)$$

where

$$G(\omega)=\int_1^{\omega-1} d\omega_p g^2(\omega_p)g^2(\bar{\omega}_p)\,, \quad (3.6a)$$

and

$$\eta^2(\omega)=\left[\int_1^{\omega-1} d\omega_p \frac{\sin\delta(\omega_p)}{p^{3/2}}\frac{\sin\delta(\bar{\omega}_p)}{\bar{p}^{3/2}}g(\omega_p)g(\bar{\omega}_p)\right]^2\Big/\kappa(\omega)G(\omega)\leq 1. \quad (3.6b)$$

[20] R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961); D. Zwanziger, *ibid.* **131**, 888 (1963).

[21] The same expression for the S matrix can be obtained by dispersion-theory methods similar to those used in Refs **17**. There are, however, some subtle points which are not encountered in those papers.

The physical meaning of $\eta(\omega)$ is quite simple; it is the relative value at the origin of that part of the wave packet with energy ω, with the two-particle scattering taken into account. $\eta(\omega)$ takes its maximum value of 1 only when $g(\omega_p)\propto\sin\delta(\omega_p)/p^{3/2}$. This strongly suggests that the quantity $-\pi\kappa(\omega)T(\omega)$ plays the same role in three-body scattering as $e^{i\delta}\sin\delta$ plays in two-body scattering.[22]

This conclusion may be reached in another way, without the use of wave packets. Notice that if the second term in S, which is proportional to $T(\omega)$, were absent, the *exact* wave function of the $J=\frac{5}{2}$ state would be the symmetrized product of two $J=\frac{3}{2}$, π^+p wave functions. Let this wave function be denoted by $\Phi(\omega_p,\omega_q)$. Then the amplitude for finding the system in a different state with wave function $\Phi(\omega_{p'},\bar{\omega}_{p'})$, $\omega_{p'}\neq\omega_p$, $\omega_{p'}\neq\omega_q$, would vanish were it not for the second term in S; i.e. in terms of the potential analog, $V_{12}(r_1,r_2)$ is entirely responsible for transitions $\Phi(\omega_p,\omega_q)\to\Phi(\omega_{p'},\bar{\omega}_{p'})$. Because V_{12} is not zero, there is a nonzero probability for finding $\Phi(\omega_{p'},\bar{\omega}_{p'})$ in the distant future when the state contains pions of energies ω_p and ω_q in the distant past. The amplitude is given by

$$\begin{aligned}\lim_{t\to\infty}&\langle\Phi(\omega_{p'},\omega_{q'})|\Psi(\omega_p,\omega_q)\rangle_+\,e^{-i(\omega-\omega')t}\\&=\tfrac{1}{2}\delta(\omega'-\omega)\int d\omega_{p''}\int d\omega_{q''}S(\omega_{p''},\omega_{q''};\omega_p,\omega_q)e^{-2i\delta(\omega_{p'})-2i\delta(\omega_{q'})}[\delta(\omega_{p''}-\omega_{p'})\delta(\omega_{q''}-\omega_{q'})+\delta(\omega_{p''}-\omega_{q'})\delta(\omega_{q''}-\omega_{p'})]\\&=\delta(\omega'-\omega)S(\omega_{p'},\omega_{q'};\omega_p,\omega_q)e^{-2i\delta(\omega_{p'})-2i\delta(\omega_{q'})}\\&\equiv\delta(\omega'-\omega)\{\tfrac{1}{2}[\delta(\omega_{p'}-\omega_p)+\delta(\omega_{p'}-\omega_q)]+2iR(\omega_{p'},\omega_{q'},\omega_p,\omega_q)\}.\end{aligned}\tag{3.7}$$

$|\Psi(\omega_p,\omega_q)\rangle_+$ denotes the exact state with outgoing wave boundary conditions. The transition amplitude is thus

$$R(\omega_{p'},\omega_{q'};\omega_p,\omega_q)=-\pi T(\omega)\frac{e^{-i\delta(\omega_{p'})}\sin\delta(\omega_{p'})}{p'^{3/2}}\frac{e^{-i\delta(\omega_{q'})}\sin\delta(\omega_{q'})}{q'^{3/2}}\frac{e^{i\delta(\omega_p)}\sin\delta(\omega_p)}{p^{3/2}}\frac{e^{i\delta(\omega_q)}\sin\delta(\omega_p)}{q^{3/2}}.\tag{3.8}$$

Because R is separable, it can easily be diagonalized by states of the form

$$|\Psi_\lambda(\omega)\rangle_+=\int_1^{\omega-1}d\omega_p f_\lambda(\omega_p,\omega)|\Psi(\omega_p,\bar{\omega}_p)\rangle_+.\tag{3.9}$$

One simply requires that

$$\begin{aligned}R_{\mu\lambda}(\omega)\equiv\int_1^{\omega-1}d\omega_{p'}\int_1^{\omega-1}&d\omega_p f_\mu^*(\omega_{p'},\omega)f_\lambda(\omega_p,\omega)\\&\times R(\omega_{p'},\bar{\omega}_{p'};\omega_p,\bar{\omega}_p)\\=\delta_{\mu\lambda}r_\lambda(\omega),&\end{aligned}\tag{3.10}$$

where $\{f_\lambda(\omega_p,\omega)\}$ is assumed to be a complete orthonormal set of functions on the interval 1 to $\omega-1$. The solution of this problem is trivial; the results are

$$\begin{aligned}r_0(\omega)&=-\pi\kappa(\omega)T(\omega),\\r_\lambda(\omega)&=0,\quad\lambda\neq0.\end{aligned}\tag{3.11}$$

The corresponding eigenfunctions are

$$\begin{aligned}f_0(\omega_p,\omega)=e^{-i\delta(\omega_p)}&\sin\delta(\omega_p)e^{-i\delta(\bar{\omega}_p)}\\&\times\sin\delta(\bar{\omega}_p)/\kappa(\omega)^{1/2}p^{3/2}\bar{p}^{3/2},\end{aligned}\tag{3.12}$$

with the remaining functions arbitrary, subject only to the completeness and orthonormality conditions imposed.

Equations (3.11) and (3.12) clearly separate the two- and three-body aspects of the problem. The two-body scattering and the initial- and final-state interactions are contained in the eigenfunctions $f_\lambda(\omega_p,\omega)$. The true three-body effects are all contained in $r_\lambda(\omega)$, which are the three-body transition amplitudes analogous to $e^{i\delta}\sin\delta$ of the two-body problem. In particular, from Eq. (3.14), $|r_\lambda(\omega)|\leq1$. Clearly, one should interpret $r_\lambda(\omega_R)=i$ as a three-body resonance at $\omega=\omega_R$. It should be emphasized that an enhancement of this sort is a result of the simultaneous interaction of all three particles; it should not be confused with an enhancement due to the overlap of two π^+p resonances. The latter type of enhancement comes from the first term of the S matrix, Eq. (3.1). Incidentally, the entire S matrix is also diagonalized in the $\{f_\lambda\}$ basis:

$${}_-\langle\Psi_\mu(\omega')|\Psi_\lambda(\omega)\rangle_+=\delta(\omega-\omega')\delta_{\mu\lambda}S_\lambda(\omega).\tag{3.13}$$

In particular,

$$S_0(\omega)=1-2\pi i\kappa(\omega)T(\omega).\tag{3.14}$$

Note that $S_0(\omega)$ has the form $\exp(2i\alpha(\omega))$, with $\alpha(\omega)$ real, and that $\alpha(\omega_R)=\pi/2$.

IV. THE (5,5) RESONANCE

In conclusion, we discuss the resonance behavior of the three-particle scattering amplitude. Equation (3.2)

[22] See, for example, the discussion of final-state interactions in M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, New York, 1964), p. 540.

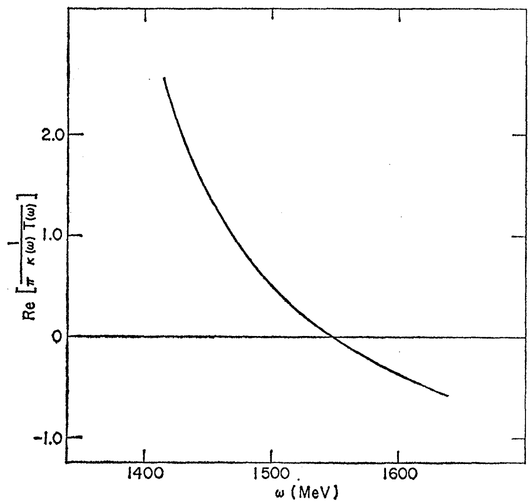

FIG. 3. $\mathrm{Re}[1/\pi\kappa(\omega)T(\omega)]$ as a function of ω the total energy.

may be rewritten in the form

$$\frac{1}{\pi\kappa(\omega)T(\omega)}=\frac{\lambda^2\pi}{\kappa(\omega)}\left\{\frac{1}{\omega D(\omega)}+\frac{1}{\pi}\int_1^\infty d\omega''\times\left[\mathrm{Im}\frac{1}{\omega''D(\omega'')}\right]\frac{1}{D(\bar{\omega}'')}\frac{1}{\omega-\omega''+i\epsilon}\right\}$$
$$=\frac{\lambda^2}{\kappa(\omega)}\int_1^\infty d\omega''\left[\mathrm{Im}\frac{1}{\omega''D(\omega'')}\right]\left[\frac{1}{\bar{\omega}''D(\bar{\omega}'')}-\frac{\omega''}{\bar{\omega}''\omega}\right].\qquad(4.1)$$

In the second form, the integrand no longer has a singularity at $\omega''=\omega$. This form is thus more suitable for numerical evaluation. In the integrand, we make the following approximations:

$$1/\omega'D(\omega')=e^{i\delta(\omega')}\sin\delta(\omega')/\lambda q'^3,\quad \omega'>1;\qquad(4.2a)$$
$$\omega'D(\omega')=\omega'[1-(\omega'/\omega_r)],\qquad \omega'<1,\qquad(4.2b)$$

where ω_r is the energy of the (3,3) resonance. These approximations are quite good for ω' not too large, and the contributions from large ω' are strongly damped out by the factor $\mathrm{Im}[1/\omega'D(\omega')]$. The resonance position is determined by $\mathrm{Re}[1/\pi\kappa(\omega_R)T(\omega_R)]=0$. From Eq. (4.1),

$$\mathrm{Re}[1/\pi\kappa(\omega)T(\omega)]=\frac{1}{\kappa(\omega)}\left[\int_1^{\omega-1}d\omega''\frac{\sin^2\delta(\omega'')}{q''^3}\left(\frac{\sin\delta(\bar{\omega}'')\cos\delta(\bar{\omega}'')}{\bar{q}''^3}-\frac{\lambda\omega''}{\bar{\omega}''\omega}\right)+\lambda\int_{\omega-1}^\infty d\omega''\frac{\sin^2\delta(\omega'')}{q''^3}\frac{(\omega_r+\omega'')}{(\omega_r-\bar{\omega}'')\omega}\right].\qquad(4.3)$$

The fit to the π^+p data given by Gell-Mann and Watson[23] is used in evaluating these integrals. The result is plotted in Fig. 3. The half-width is given by

$$\frac{\Gamma}{2}=-\left[\frac{d}{d\omega}\frac{1}{\pi\kappa(\omega)T(\omega)}\right]^{-1}_{\omega=\omega_R}.\qquad(4.4)$$

From Fig. 3, the following values of the resonance position and width are obtained:

$$\omega_R\approx4.35\approx1550\ \mathrm{MeV},$$
$$\Gamma/2\approx0.9\ \approx125\ \mathrm{MeV}.$$

Such good agreement with the experimental numbers is undoubtedly fortuitous. In addition to the obvious flaws in the model used, it is to be expected that the details of the process by which the $\pi^+\pi^+p$ final state is produced can have a substantial effect on the observed position and width. Nevertheless, the results encourage one to believe that the essential ingredients of the resonance are included in the model. It is, perhaps, unfortunate that the energy of the resonance is expected to be near $2\omega_r$; at that energy, there should be enhancement of the $\pi^+\pi^+p$ production from the overlap of the simple two-body scattering and it may be difficult to disentangle the two effects.

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[23] M. Gell-Mann and K. Watson, Ann. Rev. Nucl. Sci. 4, 219 (1954).