Field-Theoretic Approach to the ρ Meson*

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A previously discussed method is used to improve the Born series for the π-π scattering amplitude in the $\lambda\phi^4$ model. Only the first two nonvanishing Born terms are considered in each instance: to order λ^2 for isospin 0 and 2, and to order λ^3 for isospin 1. The results may be interpreted in terms of the existence of an isospin-one P-wave resonance. Its position is adjusted to fit experiment, thereby determining all other parameters in the theory. Taking the pion mass as a unit, we find the energy width of the resonance to be $\Gamma=0.72$. The scattering lengths a_J^T come out as $a_0^0=-0.78$, $a_1^1=+0.032$, $a_0^2=-0.44$. The coupling constant, as defined by Chew and Mandelstam, is $\lambda=+0.24$. All these values are rather insensitive to the position of the resonance. The numerical calculations may be done by hand.

1. INTRODUCTION

THE so-called $\lambda\phi^4$ Lagrangian density

$$\mathfrak{L}=\tfrac{1}{2}Z[(\partial^\mu\phi^\zeta)(\partial_\mu\phi^\zeta)-\mu_0^2\phi^\zeta\phi^\zeta]-4\pi\lambda_0(\phi^\zeta\phi^\zeta)^2 \quad (1.1)$$

is the simplest one available for relativistic pion interactions. It is not absurd to suppose that its solution would provide a good description of actual pion systems at moderately low energies. This is because the manifestly neglected particles, such as kaons and baryons, have a high relative mass, and furthermore must be produced in pairs. Other particles, such as the ρ and other strangeness-zero mesons, are not manifestly neglected because they may be implied by the model. This hope is, in fact, the main motivation for studies of this kind.

The present paper deals with the two-pion system in the context of (1.1), with special attention to the production of the ρ meson as an isospin-one P-wave resonance. Our point of view is strictly field-theoretic.

A similar approach to the two-pion system has already been used by several authors.[1–5] Our treatment differs from theirs in the following respects: First, our calculations are interpreted in terms of a fit to the ρ meson, whose mass is used as the (single) adjustable parameter; and second, our technique is based on a recently discussed formula[6] designed to overcome the convergence difficulties of the Born series. We also take this opportunity to present a summary of relevant perturbation results which may be of use to other workers in the field.

2. RELATION BETWEEN SCATTERING AMPLITUDE AND GREEN'S FUNCTION

We shall consider the isospin-T amplitude $f_k{}^T(\theta)$ for $\pi-\pi$ scattering through an angle θ in the center-of-mass system with incoming center-of-mass momentum k. The partial-wave expansion for f is

$$f_k(\theta)=\sum_l \frac{2l+1}{2ik}(e^{2i\delta_l{}^T}-1)P_l(\cos\theta)\,, \quad (2.1)$$

where the summation runs only over even (odd) l if T is even (odd). We shall also consider Green's function

$$\gamma=\gamma(P_1,\cdots,P_4)\,,$$

where P_i stands for an energy–momentum p_i and corresponding isospin index ζ_i. The function γ is defined in terms of a time-ordered vacuum expectation value by

$$\langle 0|T(\phi^{\zeta_1}(x_1)\cdots\phi^{\zeta_4}(x_4))|0\rangle =\frac{1}{(2\pi)^{16}}\int\frac{d^4p_1}{p_1{}^2-1+i\epsilon}\cdots\int\frac{d^4p_4}{p_4{}^2-1+i\epsilon} \times e^{-ip_1\cdot x_1-\cdots-ip_4\cdot x_4}\gamma(P_1,\cdots,P_4)\,. \quad (2.2)$$

[We take the physical pion mass as a unit; $|0\rangle$ is the physical vacuum, and $\phi^\zeta(x)$ is normalized according to the asymptotic condition.[7]] The connection

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[1] S. Okubo, Phys. Rev. 118, 357 (1960), in which "chain diagrams" are summed exactly.

[2] M. Baker and F. Zachariasen, Phys. Rev. 118, 1659 (1960). This paper makes use of the determinantal method, discussed in Ref. 3.

[3] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).

[4] K. Smith and J. L. Uretsky, Phys. Rev. 131, 861 (1963).

[5] A. M. Saperstein and J. L. Uretsky, Phys. Rev. 133, B1340 (1964). In Refs. 4 and 5, a generalized potential is calculated by perturbation theory, the phase shifts being then obtained by dispersion methods.

[6] M. Wellner, Phys. Rev. 132, 1848 (1963).

[7] In the sense of H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955).

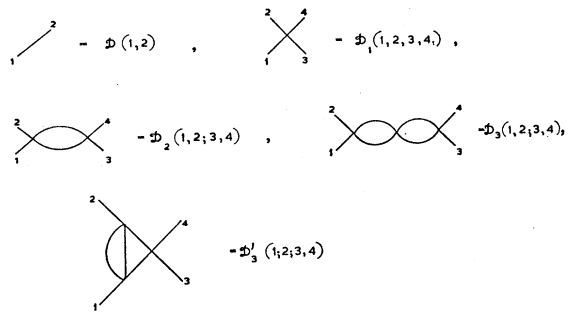

FIG. 1. The Feynman diagrams considered in this article and their designation in the text. The numbers 1, $\cdots$, 4 stand for the variables $P_1, \cdots, P_4$.

between γ and f may be obtained through the following prescription:

(a) Define some arbitrarily normalized isospin functions $\varphi_T{}^{\zeta_1\zeta_2}$ for 2-pion systems, for instance

$$\begin{aligned}\varphi_0{}^{\zeta_1\zeta_2}&=\delta_{\zeta_1\zeta_2},\\ \varphi_1{}^{\zeta_1\zeta_2}&=\epsilon_{3\zeta_1\zeta_2},\\ \varphi_2{}^{\zeta_1\zeta_2}&=\delta_{1\zeta_1}\delta_{2\zeta_2}+\delta_{1\zeta_2}\delta_{2\zeta_1}.\end{aligned}\tag{2.3}$$

(b) Define the scalar amplitude γ_T as a function of two independent variables by forming the quantity (on the mass shell)

$$i\sum_{\zeta_3,\zeta_4}\varphi_T{}^{\zeta_3\zeta_4}[\gamma(P_1,\cdots,P_4)]_{p_1^2=\cdots=p_4^2=1} \equiv\varphi_T{}^{\zeta_1\zeta_2}\gamma_T(t,u)(2\pi)^4\delta(p_1+\cdots+p^4),\tag{2.4}$$

where t and u are two of the variables[8]

$$\begin{aligned}s&=(p_1+p_2)^2,\\ t&=(p_1+p_3)^2,\\ u&=(p_2+p_3)^2.\end{aligned}\tag{2.5}$$

(c) Set

$$\begin{aligned}s&=4(k^2+1),\\ t&=2k^2(\cos\theta-1),\\ u&=-2k^2(\cos\theta+1).\end{aligned}\tag{2.6}$$

(d) Then

$$f_k{}^T(\theta)=-\gamma_T(t,u)/16\pi s^{1/2}.\tag{2.7}$$

3. PERTURBATION THEORY

As a basis for this as well as future investigations of the $\lambda\phi^4$ model, we report in this section on some explicit results of perturbation theory.[9]

a. Feynman Diagrams

We list here some results connected with the three lowest order Feynman diagrams with four external lines. As far as these low-order diagrams are concerned it is consistent to set $Z\rightarrow 1$ and $\mu_0{}^2\rightarrow 1$ in (1.1), at least if the external lines are taken on the mass shell. Nor must we bother explicitly to renormalize λ_0, as this will be automatically accomplished later.[6] We therefore study the Born series as an expansion in λ_0, or, more conveniently, in

$$g_0\equiv 16\pi\lambda_0.\tag{3.1}$$

Setting

$$\gamma=\sum_{n=0}^{\infty}(g_0)^n\gamma^{(n)},\tag{3.2}$$

we obtain in terms of diagrams (see Fig. 1)

$$\begin{aligned}&\gamma^{(0)}=0,\quad \gamma^{(1)}=6\mathfrak{D}_1(1,2,3,4),\\ &\gamma^{(2)}=18[\mathfrak{D}_2(1,2;3,4)+\cdots]_{3\text{ terms}},\\ &\gamma^{(3)}=108[\mathfrak{D}_3'(1,2;3,4)+\cdots]_{6\text{ terms}}\\ &\qquad\qquad+54[\mathfrak{D}_3(1,2;3,4)+\cdots]_{3\text{ terms}}.\end{aligned}\tag{3.3}$$

In (3.3), those diagrams which do not contribute to scattering, either because they are disconnected or because they amount to zero on the mass shell, have been omitted. The notation $[\ ]_{n\text{ terms}}$ indicates a symmetrization with respect to the variables $P_1,\cdots,P_4$. As regards factors of i, 2π, etc., the notation employed in Fig. 1 is best reconstituted by the reader by noting that the line stands for

$$\mathfrak{D}(1,2)=i\delta_{12}(p_1{}^2-1)(2\pi)^4\delta(p_1+p_2),\tag{3.4}$$

while the vertex is

$$\mathfrak{D}_1(1,2,3,4)=-(i/3)(\delta_{12}\delta_{34}+\delta_{13}\delta_{24}+\delta_{14}\delta_{23}) \times(2\pi)^4\delta(p_1+\cdots+p_4).\tag{3.5}$$

The Kronecker delta subscripts 1, 2, etc. stand for ζ_1, ζ_2, etc.

Explicitly, one has

$$\begin{aligned}&\mathfrak{D}_2(1,2;3,4)\\ &\quad=\tfrac{1}{9}(7\delta_{12}\delta_{34}+2\delta_{13}\delta_{24}+2\delta_{14}\delta_{23})I((p_1+p_2)^2)\\ &\qquad\times(2\pi)^4\delta(p_1+\cdots+p_4),\\ &\mathfrak{D}_3(1,2;3,4)\\ &\quad=(i/27)(39\delta_{12}\delta_{34}+4\delta_{13}\delta_{24}+4\delta_{14}\delta_{23})I^2((p_1+p_2)^2)\\ &\qquad\times(2\pi)^4\delta(p_1+\cdots+p_4),\\ &\mathfrak{D}_3'(1,2;3,4)\\ &\quad=((19/27)\delta_{12}\delta_{34}+\tfrac{1}{3}\delta_{13}\delta_{24}+\tfrac{1}{3}\delta_{14}\delta_{23})J((p_1+p_2)^2)\\ &\qquad\times(2\pi)^4\delta(p_1+\cdots+p_4).\end{aligned}\tag{3.6}$$

In the last formula, the dependence on a single variable $(p_1+p_2)^2$ is correct if $p_1{}^2=p_2{}^2=1$.

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[9] An improved understanding of the model may well depend on extending our store of such results. Some additional formulas may be found in Refs. 2, 4, and 5.

b. Feynman Integrals

The functions I and J are given by the integrals

$$I(p^2)=\frac{1}{(2\pi)^4}\int\frac{d^4q}{(q^2-1+i\epsilon)[(p+q)^2-1+i\epsilon]},$$

$$J((p_1+p_2)^2)=\frac{i}{(2\pi)^4}\int d^4q \times\frac{I(q^2)}{[(q-p_1)^2-1+i\epsilon][(q+p_2)^2-1+i\epsilon]} \quad (p_1{}^2=p_2{}^2=1). \tag{3.7}$$

These integrals may be thought of as depending also on a large cutoff parameter M through the regularization of the propagator $1/(p^2-1)$ wherever it appears:

$$\frac{1}{(p^2-1)}\to\frac{1}{(p^2-1)}-\frac{1}{(p^2-M^2)}. \tag{3.8}$$

The integrals $I(s)$ and $J(s)$, which are analytic in the upper half-plane of s, may be evaluated in terms of elementary functions. The results are

$$\begin{aligned}I(s)&=(i\pi^2/(2\pi)^4)[\ln M^2-2a(s)],\\ J(s)&=(-i/8(2\pi)^4)[\tfrac{1}{4}(1+\ln M^2)^2-(1+\ln M^2)a(s)\\&\quad+b(s)+(11\pi^2/54)-\tfrac{3}{4}].\end{aligned} \tag{3.9}$$

For the purpose of specifying the functions a and b it is convenient to introduce the auxiliary variables ν and $\bar{\nu}$, both defined between 0 and 1:

$$\begin{aligned}\nu&=-\tfrac{1}{2}s+1-(\tfrac{1}{4}s^2-s)^{1/2} \quad (s\leqslant 0),\\ \bar{\nu}&=\tfrac{1}{2}s-1-(\tfrac{1}{4}s^2-s)^{1/2} \quad (s\geqslant 4).\end{aligned} \tag{3.10}$$

Then

$$\begin{aligned}a&=-\tfrac{1}{2}[(1+\nu)/(1-\nu)]\ln\nu \quad (s\leqslant 0),\\ a&=\left(\arcsin\frac{s^{1/2}}{2}\right)\left(\frac{4}{s}-1\right)^{1/2} \quad \begin{pmatrix}0\leqslant\arcsin\leqslant\pi/2,\\ 0\leqslant s\leqslant 4\end{pmatrix},\\ a&=-\tfrac{1}{2}[(1-\bar{\nu})/(1+\bar{\nu})](i\pi+\ln\bar{\nu}) \quad (s\geqslant 4);\end{aligned} \tag{3.11}$$

$$\begin{aligned}b&=\tfrac{1}{6}(\nu/(1-\nu^2))(\pi^2\ln\nu+\ln^3\nu)+\tfrac{1}{4}\ln^2\nu \quad (s\leqslant 0),\\ b&=\frac{1}{6}\frac{1}{(s-\frac{1}{4}s^2)^{1/2}}\left[-\pi^2\arcsin\frac{s^{1/2}}{2}+4\left(\arcsin\frac{s^{1/2}}{2}\right)^3\right]\\&\quad-\left(\arcsin\frac{s^{1/2}}{2}\right)^2 \quad \begin{pmatrix}0\leqslant\arcsin\leqslant\pi/2,\\ 0\leqslant s\leqslant 4\end{pmatrix},\\ b&=\frac{i\pi}{2}\left(\ln\bar{\nu}-\frac{\bar{\nu}}{1-\bar{\nu}^2}\ln^2\bar{\nu}\right)+\frac{1}{6}\frac{\bar{\nu}}{1-\bar{\nu}^2}(2\pi^2\ln\bar{\nu}-\ln^3\bar{\nu})\\&\quad+\tfrac{1}{4}(\ln^2\bar{\nu}-\pi^2) \quad (s\geqslant 4).\end{aligned} \tag{3.12}$$

c. Isospin Components

We now list the isospin components γ_T as defined by (2.4). One finds, for $T=\begin{bmatrix}0\\1\\2\end{bmatrix}$,

$$\gamma_T{}^{(1)}(t,u)=\begin{bmatrix}10\\0\\4\end{bmatrix}, \tag{3.13}$$

$$\gamma_T{}^{(2)}(t,u)=\begin{bmatrix}i[50I(s)+30I(t)+30I(u)]\\ i[10I(t)-10I(u)]\\ i[8I(s)+18I(t)+18I(u)]\end{bmatrix}, \tag{3.14}$$

$$\gamma_T{}^{(3)}(t,u)=\begin{bmatrix}i[600J(s)+440J(t)+440J(u)]\\ \quad -250I^2(s)-110I^2(t)-110I^2(u)\\ i[80J(t)-80J(u)]-70I^2(t)+70I^2(u)\\ i[144J(s)+224J(t)+224J(u)]\\ \quad -16I^2(s)-86I^2(t)-86I^2(u)\end{bmatrix}. \tag{3.15}$$

d. The P-Wave Amplitude

We next turn to partial-wave calculations. The only such results used in this paper concern the $T=1$ P-wave amplitude. For this one finds

$$\begin{aligned}(e^{i\delta}\sin\delta)^{(1)}&=0,\\ (e^{i\delta}\sin\delta)^{(2)}&=-\frac{5}{16(2\pi)^3}\frac{k}{(k^2+1)^{1/2}}P(k^2),\\ (e^{i\delta}\sin\delta)^{(3)}&=-\frac{5}{8(2\pi)^5}\frac{k}{(k^2+1)^{1/2}}\\&\quad\times[Q(k^2)+(\mathrm{const})\cdot P(k^2)],\end{aligned} \tag{3.16}$$

where the constant multiplying $P(k^2)$ in the last expression will turn out to be irrelevant to our scheme, and where

$$\begin{aligned}P(k^2)&=\int_{-1}^{1}zdz\; a(2k^2(z-1)),\\ Q(k^2)&=\int_{-1}^{1}zdz\left[b(2k^2(z-1))+\frac{7}{4}a^2(2k^2(z-1))\right].\end{aligned} \tag{3.17}$$

Introducing the auxiliary variable

$$\alpha=2k^2+1-2(k^4+k^2)^{1/2}, \tag{3.18}$$

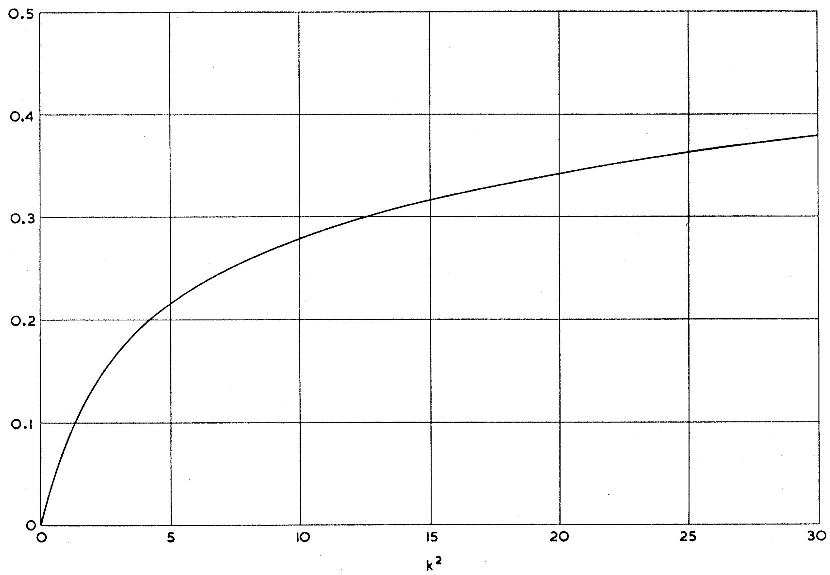

FIG. 2. The function $-P$, plotted against k^2. [See Eqs. (3.17), (3.19), and (3.20).]

we find for P and Q

$$P(k^2)=\frac{1}{2k^2}\left[1-k^2+\left(1+\frac{1}{k^2}\right)^{1/2}\ln\alpha+\left(\frac{1}{2}+\frac{1}{4k^2}\right)\ln^2\alpha\right],$$

$$Q(k^2)=\frac{7}{4k^2}\left(\int_\alpha^1 dz\frac{\ln^2 z}{1-z}-\frac{1}{6}\ln^3\alpha\right)$$
$$-\frac{1}{48k^2}\left(1+\frac{1}{k^2}\right)\ln^4\alpha-\frac{1}{6k^2}\left(1+\frac{1}{k^2}\right)^{1/2}\ln^3\alpha \quad (3.19)$$
$$+\left(\frac{\pi^2}{24}-\frac{11}{32}\right)\frac{1}{k^2}\ln^2\alpha+\frac{11}{8}\left(1+\frac{1}{k^2}\right)^{1/2}\ln\alpha$$
$$-\left(\frac{\pi^2}{6}+\frac{3}{8}\right)-\left(\frac{\pi^2}{3}+\frac{39}{8}\right)P(k^2).$$

For small k, P and Q may be expanded as Taylor series in k^2. The leading terms are

$$P(k^2)=-\tfrac{1}{9}k^2+2k^4/45+\cdots,$$
$$Q(k^2)=-\frac{1}{18}\left(\frac{\pi^2}{3}+11\right)k^2+\frac{1}{270}\left(4\pi^2+\frac{49}{2}\right)k^4+\cdots. \quad (3.20)$$

It will be convenient to introduce the function

$$X(k^2)=Q(k^2)-(\pi^2/6+11/2)P(k^2), \quad (3.21)$$

which has a higher order behavior,

$$X(k^2)\approx(-1/135)(83/4-\pi^2)k^4 \quad (3.22)$$

at the origin. The functions $-P$ and $-X$ are plotted in Figs. 2 and 3.

4. THE IMPROVED-CONVERGENCE FORMULA

Suppose a physical quantity[6] $G(g;s_1,s_2,\cdots)$, depending on a real adjustable parameter g and on some continuous variables s_1, s_2, $\cdots$, possesses the perturbation expansion

$$G=gG^{(1)}+g^2G^{(2)}+\cdots. \quad (4.1)$$

The usefulness of the present considerations will depend on the assumption that this expansion converges poorly or not at all. A simple heuristic reasoning leads then to the conclusion that, if we want to approximate G by using its two leading terms only, we must write

$$G=\frac{G^{(1)}}{\kappa-G^{(2)}/G^{(1)}}, \quad (4.2)$$

where κ depends on g but not on s_1, s_2, $\cdots$. This dependence on g is not supplied by our prescription; one can only say that $\kappa\to g^{-1}$ as $g\to 0$, in which case the Born series is recovered.

Similarly, if the perturbation expansion has the form

$$G=g^2G^{(2)}+g^3G^{(3)}+\cdots, \quad (4.3)$$

then the best approximation becomes

$$G=\frac{G^{(2)}}{(\kappa-\frac{1}{2}G^{(3)}/G^{(2)})^2}. \quad (4.4)$$

The rearrangements (4.2), (4.4) are most likely to be successful under the following important conditions: The variables s_1, s_2, $\cdots$ are in a region such that

(a) $G(s_1,s_2,\cdots)$ is differentiable;

(b) the complex phase of G is independent of the parameter g. (4.5)

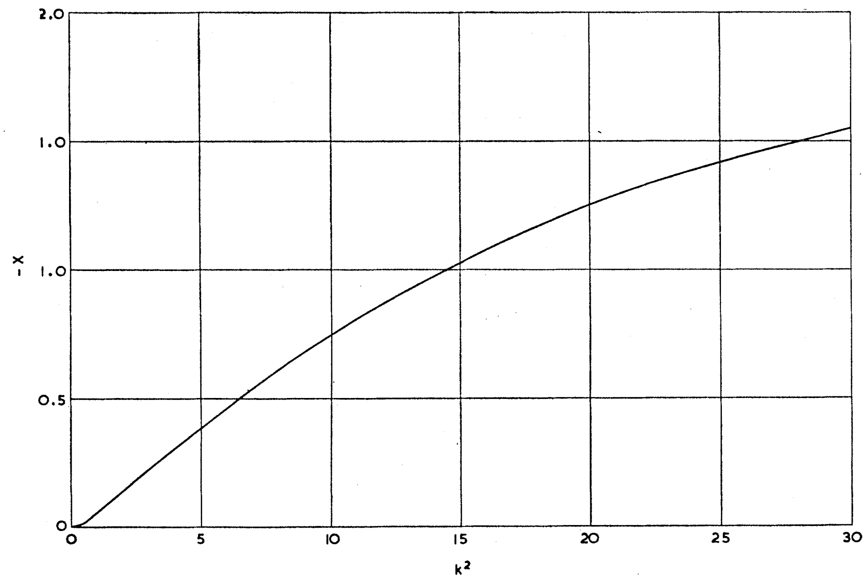

FIG. 3. The function $-X$ plotted against k^2. [See Eqs. (3.17), (3.19), and (3.20).]

A related set of favorable conditions occurs if the variables $s_1, s_2, \cdots$ are *near* a region where

(a′) $G(s_1,s_2,\cdots)$ is analytic (or the boundary value of an analytic function); (4.6)
(b) as before.

The question may arise as to why one should not approximate some simple function of G, say $F(G)$, rather than G itself: The result will in general be somewhat different. Actually, this is not a serious arbitrariness, for two reasons:

First, the end results are not very sensitive to the function F. For example, our two-term scheme yields *exactly* the same result whether we approximate G, any positive power G^N, or any constant multiple of these.

Second, the derivation of the method makes it likely that, in case of doubt, we must approximate the function whose expected behavior is smoothest for general g. For example, if δ is a phase shift, we should approximate δ rather than, say, $\tan\delta$.

We finally observe that it is not necessary to know the relation between κ and g. The latter will not occur in the results of any calculation, and what we are doing amounts to replacing one adjustable parameter by another. This is basically why no explicit coupling renormalization is needed.

5. ρ-MESON CALCULATIONS

a. The $T=1$ P-Wave Phase Shift δ and Scattering Length a_1^1

In order to describe physical scattering by our improved-convergence scheme, we are led to search for functions which satisfy the conditions (4.5) or (4.6) as far out as possible in the physical region. The best candidates are the phase shifts, which satisfy (4.6) as far as $k^2=3$, where inelasticity sets in, so that (4.6) breaks down only from here on. Experimentally,[10] the ρ meson occurs at about $k^2=6.3$. Whether this is still "close" enough in the sense of (4.6) is *a priori* unknown to us. This question may be answered to some extent, either by computing the next systematic correction and showing that it is small, or by assuming the validity of the model itself, appealing then to experiment, and finally assuming that any agreement is not fortuitous. Only the latter justification can be presented here. The former one, although apparently feasible, is laborious and will have to be left to the future.

To third order, for $T=1$, the Born expansion for δ is the same as that for $e^{i\delta}\sin\delta$ [Eqs. (3.16)]. Application of (4.4) gives then, with the use of (3.21),

$$\delta=-\frac{5\pi}{8}\frac{k}{(k^2+1)^{1/2}}\frac{P}{(\kappa-X/P)^2}, \tag{5.1}$$

κ being a real adjustable parameter. A resonance, if any, will occur at

$$\delta=\pi/2, \tag{5.2}$$

i.e., at the solution $k=k_\rho$ of

$$\kappa=\frac{X}{P}\pm\left(-\frac{5}{4}\frac{k}{(k^2+1)^{1/2}}P\right)^{1/2}. \tag{5.3}$$

Two cases must be distinguished.

(a) $\kappa<0$. Only the minus sign is then possible in (5.3). Numerically, one finds $k_\rho^2\lesssim 0.065$. Since our aim is to account for the ρ meson we shall reject the range $\kappa<0$ as incompatible with experiment.

[10] M. Roos, Rev. Mod. Phys. **35**, 314 (1963); see, however, M. H. Ross and G. L. Shaw, Phys. Rev. Letters **12**, 627 (1964), suggesting a lower width and a shifted position.

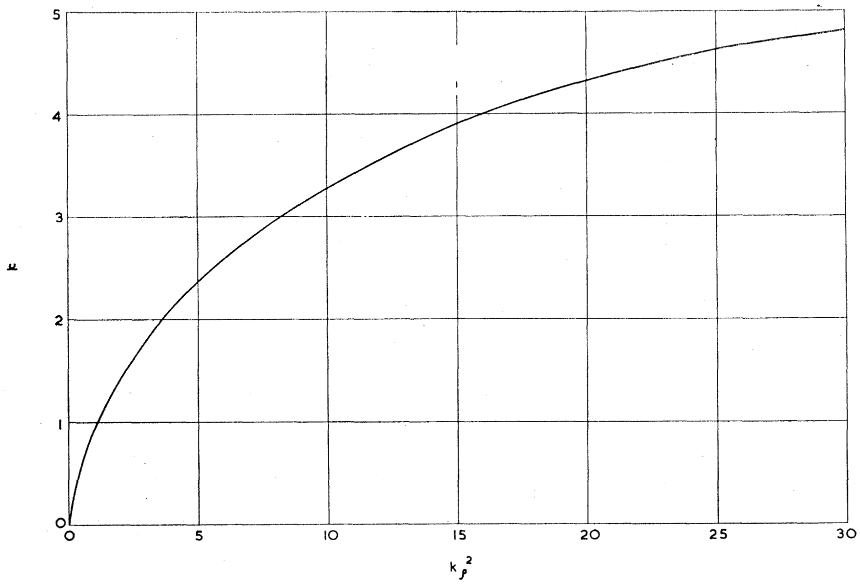

FIG. 4. The parameter κ, plotted against the position k_ρ^2 of the ρ meson. [See Eqs. (5.1) and (5.3).]

(b) $\kappa>0$. We must require that the zero of $\kappa-X/P$ occur to the right of the resonance, i.e., for $k>k_\rho$. Indeed, from Sec. 4 it follows that our approximation should get better towards the "favorable" region $k^2<3$. Hence the zero, which is a catastrophic breakdown of (5.1), should be towards higher energies. Thus we must choose the plus sign in (5.3). Alternatively, one may argue (to the same effect) that δ should increase at the resonance. A plot of κ as a function of the resonance position k_ρ^2 is shown in Fig. 4.

The $T=1$ P-wave scattering length $a_1{}^1$ follows directly from (5.1). We find

$$a_1{}^1\equiv\lim_{k\to0}\frac{\sin\delta}{k^3}=\frac{5\pi}{72}\frac{1}{\kappa^2}. \tag{5.4}$$

This result is plotted as a function of k_ρ^2 in Fig. 5.

b. Interpretation and Width

Some further remarks on the range of validity of (5.1) are in order. The phase shift δ exhibits no point of inflection near k_ρ^2, so that the resonance curve is somewhat distorted from the expected Breit-Wigner shape. In particular, the width may not be inferred from the slope of δ at the resonance, but must be calculated directly from a plot of δ. We interpret this as follows: In our approximation, the derivative of δ breaks down before δ itself as we go towards higher energies.[11] The best we can do in the present context is to accept $\delta=\frac{1}{2}\pi$ at face value, but we note that the distortion of the derivative at $k=k_\rho$ is symptomatic of the fact that calculations become unreliable above that point. Thus we are fortunately not allowed to take the further intersections of δ with $3\pi/2$, $5\pi/2$, etc. seriously. These spurious resonances occur because δ increases (monotonically) to infinity as k approaches the zero of $(\kappa-X/P)$. For example, if κ is such that $k_\rho^2=6.3$, then $\delta=3\pi/2$, ∞ at approximately $k^2=7.6$, 9.5, respectively.

In accordance with these remarks, we must compute the width Γ as twice the energy interval,

$$\Gamma=2\Delta(2(k^2+1)^{1/2}),$$

between the *left* half-point $\delta=\frac{1}{4}\pi$ and the center $\delta=\frac{1}{2}\pi$. The result is plotted as a function of k_ρ^2 in Fig. 6, together with the experimental point.[10]

c. Amplitude for $T=1$ in the Unphysical Region

From now on we shall deal only with the unphysical region $t=u$, $s<4$ and the threshold, $s=4$. Since the scattering amplitudes are real, there is no need to take partial waves in order to apply the improved convergence scheme. This allows us to use the crossing relations[4,8] at the symmetry point $s=t=u=\frac{4}{3}$ in order to fix the $T=0$, 2 parameters in terms of the already determined $T=1$ parameters.

First we apply (4.4) to (3.14) and (3.15):

$$\gamma_1(t,u)=\frac{5(2\pi)^2[a(t)-a(u)]}{\{\kappa_1-(7/4)[a(t)+a(u)]-[b(t)-b(u)]/[a(t)-a(u)]\}^2} \tag{5.5}$$

for some parameter κ_1. The latter can be determined by comparing γ_1 and δ at the elastic threshold $t=u=0$, $s=4$, where the higher partial waves are not relevant. In this neighborhood, (5.5) becomes

$$\gamma_1(t,u)=\frac{-5(2\pi)^2}{3(\kappa_1-11/2-\pi^2/6)^2}k^2\cos\theta \tag{5.6}$$

[11] A similar situation is clearly the case in Ref. 6, Fig. 1.

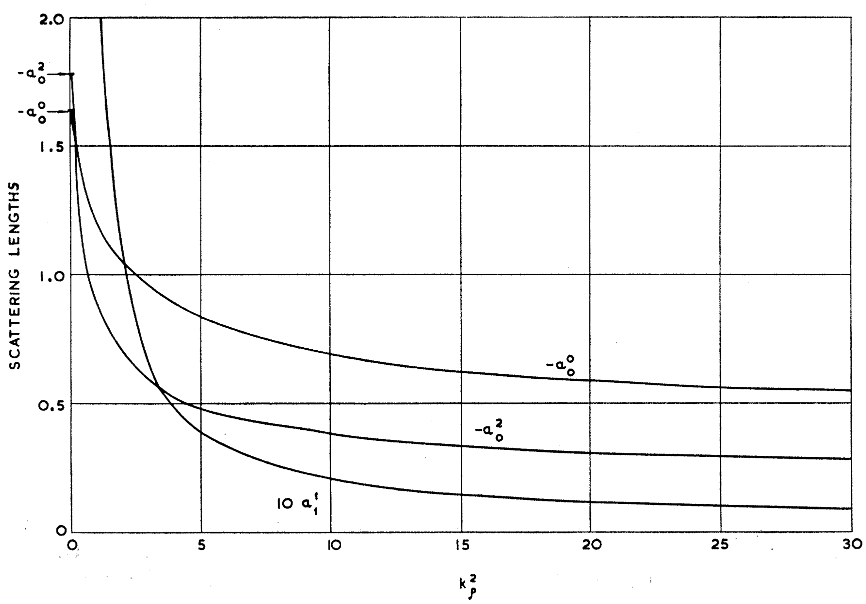

FIG. 5. The scattering lengths a_0^0, a_1^1, a_0^2, plotted against the resonance position k_ρ^2. Note the signs and the different scale for a_1^1. [See Eqs. (5.4) and (5.21).]

(only an infinitesimal continuation beyond the threshold is involved). On the other hand, (2.7) becomes

$$3k^2a_1^1\cos\theta=-(1/32\pi)\gamma_1(t,u). \qquad (5.7)$$

Comparison of (5.6) and (5.7) gives

$$(\kappa_1-11/2-\pi^2/6)^2=\kappa^2. \qquad (5.8)$$

In solving (5.8), the sign must be chosen such that the first two terms of the Born series are recovered with correct relative sign as $|\kappa|\to\infty$ and $|\kappa_1|\to\infty$. This relative sign must be consistent as between δ [Eq. (5.1)] and γ_1 [Eq. (5.5)]. In this way we obtain

$$\kappa_1=\kappa+11/2+\pi^2/6. \qquad (5.9)$$

Equations (5.5) and (5.9) summarize the extension of the $T=1$ amplitude to the unphysical region. It is convenient to rewrite (5.5) as

$$\gamma_1(t,u)=\frac{5(2\pi)^2[a(t)-a(u)]}{[\kappa+c(t,u)]^2}, \qquad (5.10)$$

and to note for later reference the special result along $t=u$

$$c(t,t)=-\tfrac{7}{2}a(2-\tfrac{1}{2}s)-(b'(2-\tfrac{1}{2}s)/a'(2-\tfrac{1}{2}s))+\frac{11}{2}+\tfrac{1}{6}\pi^2. \qquad (5.11)$$

d. Amplitudes and Scattering Lengths for $T=0,\ 2$; Coupling Constant

Using (4.2) in connection with the first two Born terms (3.13) and (3.14), we obtain

$$\gamma_0(t,u)=\frac{10(2\pi)^2}{\kappa_0-[\frac{5}{2}a(s)+\frac{3}{2}a(t)+\frac{3}{2}a(u)]},$$
$$\gamma_2(t,u)=\frac{4(2\pi)^2}{\kappa_2-[a(s)+9a(t)/4+9a(u)/4]}, \qquad (5.12)$$

in the unphysical region. The parameters κ_0, κ_2 must still be determined. Crossing symmetry at $s=t=u=\frac{4}{3}$ implies

$$2\gamma_0(\tfrac{4}{3},\tfrac{4}{3})=5\gamma_2(\tfrac{4}{3},\tfrac{4}{3}), \qquad (5.13)$$

or

$$\kappa_0=\kappa_2. \qquad (5.14)$$

The connection with κ may also be found from crossing symmetry. We define the partial derivatives

$$\partial_\perp\equiv\left(\frac{\partial}{\partial t}\right)_{s=\text{const.}},$$
$$\partial_{\parallel}\equiv\left(\frac{\partial}{\partial s}\right)_{t-u=\text{const.}}, \qquad (5.15)$$

and then make use of either one of the relations

$$\partial_\perp\gamma_1=\partial_{\parallel}\gamma_0, \qquad (5.16)$$

(at the symmetry point).

$$\partial_\perp\gamma_1=-2\partial_{\parallel}\gamma_2. \qquad (5.17)$$

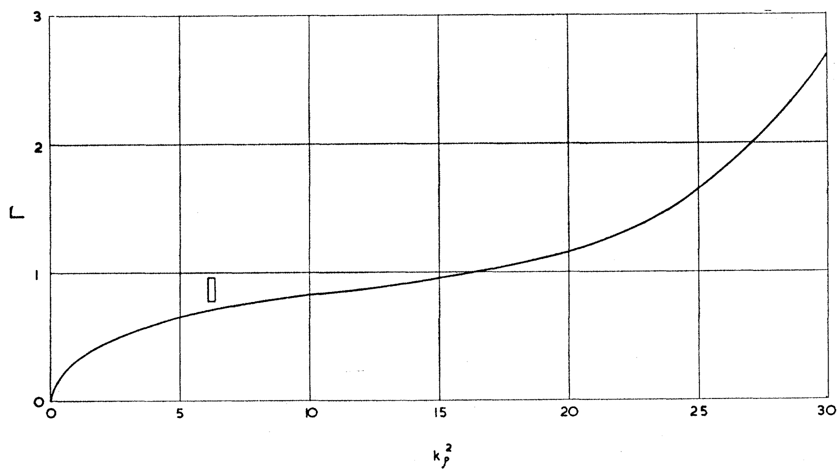

FIG. 6. The computed resonance energy width Γ, plotted against the resonance position k_ρ^2, together with the experimental value (small rectangle).

Also at the symmetry point, one finds from (5.10) and (5.12)

$$\partial_\perp\gamma_1=\frac{10(2\pi)^2a'(\frac{4}{3})}{[\kappa+c(\frac{4}{3},\frac{4}{3})]^2},\qquad \partial_{\parallel}\gamma_0=-2\partial_{\parallel}\gamma_2=\frac{10(2\pi)^2a'(\frac{4}{3})}{\left[\kappa_0-\frac{11}{2}a(\frac{4}{3})\right]^2}. \tag{5.18}$$

Equation (5.16) yields

$$[\kappa+c(\tfrac{4}{3},\tfrac{4}{3})]^2=\left[\kappa_0-\frac{11}{2}a(\tfrac{4}{3})\right]^2, \tag{5.19}$$

whence, by (5.11),

$$\kappa_0=\kappa+2a(\tfrac{4}{3})-b'(\tfrac{4}{3})/a'(\tfrac{4}{3})+\frac{11}{2}+\tfrac{1}{6}\pi^2, \tag{5.20}$$

where the ambiguous sign was fixed just as in (5.9). Remarkably enough in view of the fact that we are dealing with an approximation, both relations (5.16) and (5.17) yield exactly the same result in conjunction with (5.14).

For $s=4$, $t=u=0$ we obtain the S-wave scattering lengths [from (5.12)]

$$a_0^{0,2}\equiv\lim_{k\to0}\frac{1}{k}\sin\delta_0^{0,2}:\qquad a_0^0\approx\frac{-5\pi/4}{\kappa+2.37},\qquad a_0^2\approx\frac{-\frac{1}{2}\pi}{\kappa+0.88}. \tag{5.21}$$

These are plotted as functions of k_ρ^2 in Fig. 5. It is worth noting that the threshold unitarity condition

$$(f_k^2)_{k=0^+}=(\mathrm{Im}f_k/k)_{k=0^+} \tag{5.22}$$

is satisfied exactly by (5.12).

For comparison with the literature we report here our value of the coupling constant. We define the renormalized coupling constant[12] as

$$g=16\pi\lambda=\tfrac{1}{10}\gamma_0(\tfrac{4}{3},\tfrac{4}{3}). \tag{5.23}$$

Use of (5.12) and (5.20) then gives

$$g\approx\frac{(2\pi)^2}{\kappa+0.575}. \tag{5.24}$$

This is plotted in Fig. 7.

6. SUMMARY AND DISCUSSION

If the $\lambda\phi^4$ theory is solved to second nonvanishing order with the help of the improved convergence formula of Sec. 4, and if we assume this approximation to be reliable at a sufficiently high energy, then we obtain a $T=1$ P-wave resonance. Adjusting its position to be $k_\rho^2=6.3$ (the experimental ρ meson[10]), we find a width $\Gamma=0.72$ and scattering lengths a_J^T with the values

$$a_0^0=-0.78,\quad a_1^1=+0.032,\quad a_0^2=-0.44.$$

The coupling constant, as defined by Chew and Mandelstam, is $\lambda=0.24$. Inasmuch as these numbers are rather insensitive to k_ρ^2, we believe they are significant. We also want to stress that our method makes the numerical calculations quite amenable to slide-rule treatment.

As regards comparison with experiment, our value for Γ agrees fairly well with the measured 0.87±0.08. The coupling constant and scattering lengths are more controversial. On the whole, our values agree best with the analyses of τ and τ' decays.[13–16] The values closest

[12] Our λ is defined as in Ref. 8, and as −λ of Refs. 4 and 5; our g is defined as λ of Ref. 2.

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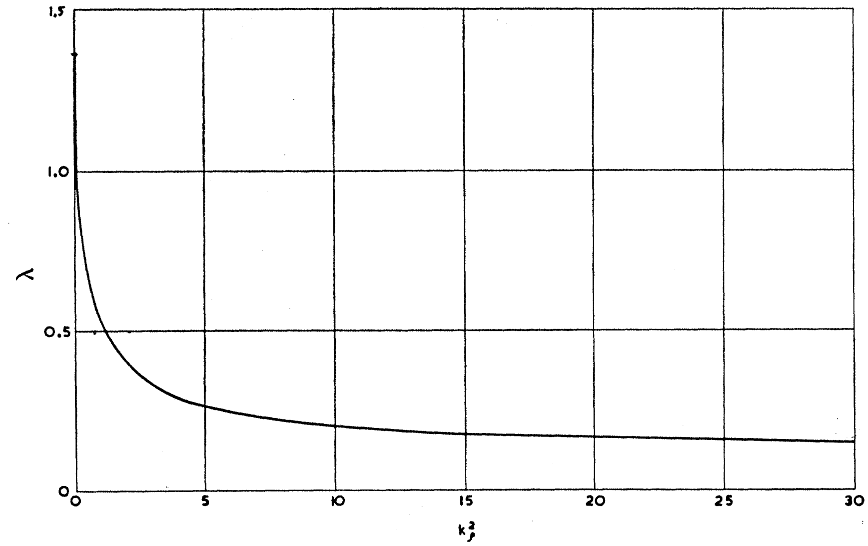

FIG. 7. The renormalized coupling constant $\lambda=g/16\pi$ as a function of the resonance position k_ρ^2. [See Eqs. (5.23) and (5.24).]

to ours are[14]

$$a_0{}^0=-0.8\,,\quad a_0{}^2=-0.48\,,\quad \lambda=+0.3\,.$$

The interpretation of other processes[17] has usually given attractive S-wave scattering lengths. Typical are[18]

$$a_0{}^0=+0.5\,,\quad a_1{}^1=+0.07\,,\quad a_0{}^2=+0.16\,.$$

An exception is the estimate $\lambda\approx+0.5$ from the nucleon isovector-moment form factor.[2]

This question of signs is worth a few additional comments. The field-theoretic investigations of the $\lambda\phi^4$ model seem to imply qualitatively[5] that a sufficiently attractive P-wave interaction is associated with repulsive S-wave scattering lengths. Dispersion-theoretic models are not unanimous on this point.[19–21] If, however, the $\lambda\phi^4$ model is taken seriously, and if the sign of the renormalized λ is indicative of that of the unrenormalized λ_0, then we must have $\lambda>0$ in order to have a positive-definite Hamiltonian.

The present study has little to say about S-wave phase shifts above threshold. The conflicting signs which are found in the literature for the scattering lengths might indicate a fairly complicated behavior not far above threshold. Hence it would be of great interest to extend the present treatment to S waves. This necessitates the inclusion of three perturbation terms. Indeed, it is quite easy to use the two-term formula (4.2) for these phase shifts, but the result is not reliable when $k^2\gtrsim 1$ for $T=0$ and when $k^2\gtrsim 2$ for $T=2$. This is deduced from the fact that Wigner's inequality[22]

$$d\delta/dk\gtrsim -1-1/2k \qquad (6.1)$$

(generously assuming a unit range for the force) breaks down badly soon after these points. This conclusion is not surprising, since the λ^3 term was needed in the $T=1$ case at moderate energies. In practice, use of a three-term formula would mean the numerical solution of a nonlinear differential equation,[6] and will have to be left to a future investigation. This should also be relevant to the problematical Abashian-Booth-Crowe[23] and Brown-Singer-Samios[24,25] particles.

ACKNOWLEDGMENT

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