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Lorentz Invariance and the Interpretation of $SU(6)$ Theory

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Conclusions drawn so far from the $SU(6)$ theory are reviewed. A further discussion of the magnetic moments of long-lived baryons appears to render implausible the existence of an integrally charged sextet [$SU(3)$ triplet with spin $\frac{1}{2}$]. The question of Lorentz invariance is discussed in some detail. A "relativistic completion" procedure is developed by means of which one can implement the requirements both of Lorentz invariance and of $SU(6)$ invariance for effective S-matrix elements. This procedure applies equally well to an interaction Lagrangian but not to the full Lagrangian including the free-field terms. The group $SU(6)$ apparently has to be interpreted as a dynamical group which applies to one-particle states at zero momentum. Lorentz invariance provides a natural mechanism for the first-stage breakdown of $SU(6)$. Considering $SU(6)$ invariance as a strictly zero-momentum property, we arrive nevertheless at unique predictions to order v/c in virtue of Lorentz invariance. Baryon recoil will be treated in future papers.

I. REVIEW OF PREVIOUS RESULTS

It has recently been found that the description of strong and electromagnetic phenomena in terms of a group $SU(6)$ leads to a considerable number of conclusions which are in good agreement with experiment.[1–8] It is the purpose of this paper to discuss some conceptual questions which arise in this theory, notably the problem of a synthesis between the $SU(6)$ scheme and relativistic invariance. In this section we shall first review the results obtained so far with the $SU(6)$ theory. We shall then state the problem in Sec. II where also the further plan of the paper is outlined.

(I).

$SU(6)$ appears to provide a natural classification of baryons, mesons and low-lying resonances.[1–2] The 56-dimensional representation comprises the well-known baryon octet and decuplet and dictates at the same time the correct spins and relative parities. Likewise for the 35-dimensional representation for mesons which contains the spin-zero octet and the spin-one nonet with the same (odd) parity. The next small baryon representation with dimension 70 has also several candidates for occupation. It will be a very interesting test of the theory to see whether the **70** can be filled correctly.[2,9,10]

(II). Mass Formulas for Baryons and Mesons

It has been observed[2] that mass formulas are an essential tool for ascertaining whether $SU(3)$ multiplets are appropriately united to $SU(6)$ supermultiplets, because such a unification will make it natural to relate the mass splittings of the $SU(3)$ multiplets involved by means of simple assumptions on the nature of a broken $SU(6)$. For the **56**, mass formulas have been proposed by Gürsey and Radicati,[1] Pais,[2] Kuo and Yao,[5] and Bég and Singh.[6] It should be noted that, insofar as this particular representation is concerned, all these formulas are in fact equivalent in their prediction of a connection between octet splits and decuplet splits. This is due to the following two identities, valid for the **56** only.

$$2J(J+1)-C_2^{(3)}=-(9/2), \tag{1}$$

$$2J(J+1)-(1/\sqrt{6})C_3^{(3)}=3/2. \tag{2}$$

Here J is the spin, $C_2^{(3)}$ and $C_3^{(3)}$ are the Casimir operators of $SU(3)$ of degree two and three, respectively.[11] The existence of the identities (1) and (2) is essentially due to the fact that the **56** contains only two multiplets which are distinct both with respect to $SU(3)$ and to spin. In any case, the equidistance within the decuplet now becomes predictable from the octet mass parameters[12] and the result is in good agreement with experiment.[2,5,6]

For almost all other representations the situation is more complex. For the general case, a mass formula has been derived by Bég and Singh[6] by means of $SU(6)$ tensor-operator analysis. In particular a detailed discussion is given there of the general mixing problems which arise when states with the same isospin and hypercharge but which belong to distinct $SU(3)$ multiplets are united within one $SU(6)$ supermultiplet. As applied to the **35**, the $SU(3)$ rule[13] $4K^2-\pi^2=3\eta^2$ is of course obtained, but no further relations exist unless one makes more restrictive assumptions.[14] On the other

[1] F. Gürsey and L. Radicati, Phys. Rev. Letters **13**, 173 (1964).
[2] A. Pais, Phys. Rev. Letters **13**, 175 (1964).
[3] B. Sakita, Phys. Rev. **136**, B1756 (1964).
[4] F. Gürsey, A. Pais, and L. Radicati, Phys. Rev. Letters **13**, 299 (1964).
[5] T. K. Kuo and T. Yao, Phys. Rev. Letters **13**, 415 (1964).
[6] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964); **13**, E681 (1964).
[7] M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964).
[8] B. Sakita, Phys. Rev. Letters **13**, 643 (1964).
[9] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 509 (1964).
[10] I. Gyuk and S. F. Tuan, Phys. Rev. Letters **14**, 126 (1964).
[11] Equation (1) was given in Eq. (23) of Ref. 6. The relation with the notations of Ref. 2 is as follows. $C_2^{(3)}=2F_iF_i$, $C_3^{(3)}=(\frac{2}{3}\sqrt{6})d_{ijk}F_iF_jF_k$. $C_2^{(3)}=6$, 12 for the octet and decuplet, respectively. The corresponding values for $C_3^{(3)}$ are 0, $6\sqrt{6}$. There exist similar identities for the 20-dimensional (baryon) representation of $SU(6)$, namely $2J(J+1)+C_2^{(3)}=15/2$, $C_3^{(3)}=0$.
[12] In the analysis of Ref. 6 the possible contributions from the real representation **2695** have not been considered.
[13] K^2 denotes the square of the K mass, etc.
[14] An example of such a restrictive sum rule is given in Eq. (30) of Ref. 6.

hand, the very well satisfied relation $\rho^2-\pi^2=K^{*2}-K^2$ was obtained by Pais.[2] This rule is not incompatible with the general Bég-Singh formula, but is only obtainable from it by imposing special conditions which are not dictated by the tensor analysis alone. This indicates to us the need to supplement the general algebraic methods by more dynamical considerations. We hope to come back to this question. At any rate, it is to be hoped that further information on the **70** may help to clarify the situation.[9,10] It remains to be seen (a) whether $SU(6)$ works as well for higher energies,[2] (b) whether the **70** is indeed the next supermultiplet to be filled, and (c) whether interference effects *between* supermultiplets are negligible.

We now turn to a notion important for what follows, that of central mass of a supermultiplet.[15] Take the **56** as an example. For definiteness we write the corresponding mass formula for the broken $SU(6)$ as

$$M=M_{00}+M_1J(J+1)+M_2Y+M_3[I(I+1)-\tfrac{1}{4}Y^2]. \quad (3)$$

Here the coefficients M_{00}, M_1, M_2, M_3 are supposed to depend on the Casimir operators of $SU(6)$ only, that is, they are constants within a given supermultiplet. We now assume furthermore that $M=M_{00}$ when the $SU(6)$-breaking interaction is neglected. This last assumption is by no means self-evident. Its implication is that the $SU(6)$-breaking interaction does not generate any appreciable mass contribution which is independent of J, I, and Y. Whether or not this is true can only be found out by a more detailed knowledge of the dynamics than we have. With this forewarning, let us continue the argument. M_{00} is now the value to which all masses within the broken **56** tend in the strict $SU(6)$ limit. This we call the central mass. Equation (3) yields[16]

$$M_{00}=\tfrac{1}{4}(4\Lambda+\Sigma-Y^*)\simeq 1065 \text{ MeV}. \quad (4)$$

By a similar reasoning, the central mass of the 35-meson states is found to be[6]

$$\mu_{00}\simeq 615 \text{ MeV}. \quad (5)$$

(III). D/F Ratio for the Effective Coupling of the Pseudoscalar Octet to the Baryon Octet

This ratio was found to be[4]

$$(D/F)=\tfrac{3}{2}. \quad (6)$$

Making the customary assumption that this strong-interaction ratio is measurable in the axial-vector contributions to the semileptonic decays, we may compare Eq. (6) with the data analysis by Willis *et al.*[17] This shows that Eq. (6) agrees within the error with their solution A which gives a ratio 1.7 ± 0.35. We also recall[4] that according to this theory the vector-meson coupling to the baryon octet is pure F.

(IV). π-Nucleon Coupling Constant Versus ρ-Nucleon Coupling Constant

The study of the effective meson-baryon vertex at low energies within the framework of $SU(6)$ relates[4] the p-wave (pseudovector) π-nucleon constant g_A to the s-wave (vector) ρ-nucleon constant g by $g_A=5g/3$. The precise definitions of g_A and g were given in Ref. 4. From this relation it follows[4] that in the $SU(6)$ limit

$$\frac{g_{ps}^2}{4\pi}=\frac{25}{9}\left(\frac{2M_{00}}{\mu_{00}}\right)^2\frac{g^2}{4\pi}. \quad (7)$$

One can relate g to the rate for $\rho\to 2\pi$. Thus, g_{ps} is determined from Eqs. (4), (5), and[18]

$$g^2/4\pi\simeq\tfrac{1}{2}, \quad (8)$$

so that[6]

$$g_{ps}^2/4\pi\simeq 15, \quad (9)$$

remarkably close to the best value.[19]

While the precise values of the central masses M_{00} and μ_{00} are related to a more detailed interpretation of the mass formula,[16] it is nevertheless curious that one gets so close to the "experimental" value of g_{ps} by arguments that are only valid in the strict $SU(6)$ limit. As isovector current conservation is only broken by electromagnetic and weak interactions, Eq. (8) is also true for broken $SU(6)$ and $SU(3)$. Unless Eq. (9) is an accident, it is somewhat of a puzzle, however, why the M_{00}/μ_{00} ratio should be so closely reflected in the actual situation where neither $SU(6)$ nor $SU(3)$ invariance is manifest. This must be considered as a further dynamical clue, along with the good validity of the $SU(3)$ and $SU(6)$ mass formulas as *first*-order perturbations.

(V). Decuplet Decays

Also the strong transition rates decuplet $\to$ octet $+$ meson are determined by g and the central masses. With the usual treatment for phase-space corrections, the width Γ_{33} of the 33 resonance is given by[4]

$$\Gamma_{33}=\frac{12}{25}\frac{g_{ps}^2}{4\pi}\frac{k^3}{m_{33}^2}\left[\frac{m_N m_{33}}{M_{00}^2}\right]. \quad (10)$$

[15] From now on a representation of $SU(6)$ and of $SU(3)$ will be denoted respectively as supermultiplet and multiplet.

[16] See Ref. 6, Erratum. The following additional comment on Eqs. (3) and (4) should be made. Equation (3) is used in Refs. 1 and 6. For the **56**, the mass relation of Ref. 2 is equivalent to $M=M_{00}'+M_1'J(J+1)+M_2Y+M_3[I(I+1)-Y^2/4-C_2^{(3)}/6]$; see also footnote 11, with the assumption that the mass parameters are constant within the **56**. This relation is not strictly identical to Eq. (3), as it yields a central mass $M_{00}'=M_{00}+3(\Sigma-\Lambda)/8$, while $M_1'=M_1+(\Sigma-\Lambda)/6$. This difference in central mass changes the result of Eq. (9) below by $\sim$6%. Both equations give the same mass-split correlations between the octet and the decuplet. In the present state of the art there seems therefore not much point in arguing the relative merits of the two mass relations.

[17] W. Willis, H. Courant, H. Filthuth, P. Franzini *et al.*, Phys. Rev. Letters **13**, 291 (1964). We are grateful to Dr. Willis for communicating to us the error ±0.35 on the value 1.7.

[18] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[19] A. J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

TABLE I. Magnetic moments of long-lived baryons.

Particle	$SU(6)$ limit[a]	Mass-corrected
p	1	2.79
n	$-\frac{2}{3}$	-1.85
Λ	$-\frac{1}{3}$	-0.78
Σ^+	1	2.20
Σ^-	$-\frac{1}{3}$	-0.73
Ξ^0	$-\frac{2}{3}$	-1.32
Ξ^-	$-\frac{1}{3}$	-0.66
Ω^-	-1	-1.56

[a] The column "$SU(6)$ limit" gives the relative magnitudes of the magnetic moments as obtained in Ref. 7. In the column "mass-corrected" the magnetic moments are expressed in nuclear magnetons with the assumption on mass corrections described in Sec. I. μ_p is taken from experiment. As the proton-neutron ratio differs by $\approx 2.5\%$ from the experimental one, errors of at least this order must be anticipated in the other ratios.

Using Eqs. (4) and (9) one finds $\Gamma_{33} \simeq 60$ MeV. The question of other decuplet widths has been commented on earlier.[4]

(VI). Magnetic Moments

It has been shown[7,8] that the magnetic moments of baryons are uniquely expressible in terms of μ_p, the proton moment, if we assume that the effective electromagnetic current transforms according to the 35-representation of $SU(6)$. The results are summarized in Table I. In this table we also give a "mass-corrected" value which is obtained under the assumption that the $SU(6)$ ratio (μ_B/μ_p) between the magnetic moment μ_B of a baryon B and μ_p is corrected by the true mass ratio (m_p/m_B) if we go from $SU(6)$ to broken $SU(3)$. In view of the successful description of mass splits as first-order effects we consider this a reasonable guess for this correction.

It has been emphasized earlier[7] that the $SU(6)$ results of Table I correspond to the assumption that the charge operator Q is given by

$$Q = F_3 + (F_8/\sqrt{3}). \qquad (11)$$

A more general definition has been proposed, namely[20]

$$Q(q_0) = F_3 + (F_8/\sqrt{3}) + (q_0 - \tfrac{2}{3})t, \qquad (12)$$

where q_0 is a number [$SU(3)$ scalar] and t is the triality quantum number.[21] $t=0$ for the usual baryons and mesons, $t=+1$ for the fundamental triplet (**3**) representation of $SU(3)$, $t=-1$ for **3***. We have $Q(\frac{2}{3}) \equiv Q$. According to Eq. (12) the three members of the fundamental triplet have charges q_0, q_0-1, q_0-1, respectively. Corresponding to Eq. (12) we have a magnetic moment operator

$$\mathbf{M}(q_0) = \mu_0 Q'(q_0)\mathbf{J}, \qquad (13)$$

where $\mathbf{J}$ is the relevant spin, μ_0 is a scale factor and $Q'(q_0)$ has the same $SU(3)$-transformation properties as $Q(q_0)$. Also for this general case the magnetic-moment ratios are unique and by the methods of Ref. 7 we find in particular that

$$\mu_n : \mu_p : \mu_\Lambda = (3q_0 - 4) : (3q_0 + 1) : (3q_0 - 3). \qquad (14)$$

Evidently $q_0 \neq \frac{2}{3}$ spoils the good agreement which was found previously for the proton/neutron ratio. We are therefore strongly committed to the expression (11) for the charge operator. This leads us to make the following comments on various triplet models which have been discussed recently.[22]

(a). Regardless of the definition of the charge operator, we cannot have as a separate representation a fundamental triplet with spin other than $\frac{1}{2}$, unless $SU(6)$ is considerably enlarged, because the sextet of $SU(6)$ has of course the $SU(3) \otimes SU(2)$ content (3,2).

(b). If the fundamental sextet has a charge parameter $q_0 \neq \frac{2}{3}$, attractive features of $SU(6)$ get lost. In particular we see no compelling reason to assume that the fundamental sextet has integral charged members. Under these circumstances, the only acceptable sextet is the straight extension to $SU(6)$ of the quark model discussed by Gell-Mann[23] and elaborated by Zweig.[24]

(c). At the same time we reiterate[7] that we do not read in the results reviewed so far and to be obtained below any additional evidence for or against the existence of quarks. We return to this point in the concluding Sec. V.

(VII). Transition Magnetic Moments

Those between decuplet and octet are also uniquely expressible in μ_p. In particular the relation[7] $\langle N^{+*}|\mu|p\rangle = 2\sqrt{2}\mu_p/3$ was found to be in qualitative agreement with other estimates. Also for the **35** there exist new relations for transition moments.[7]

Thus we hopefully take the following position. On the basis of $SU(3)$ alone, the interpretation of an experimental result concerning the Λ magnetic moment (for example) is obscured by two problems: (a) the question which is the appropriate charge operator, (b) what are the corrections due to broken $SU(3)$. Taking the neutron/proton ratio as a guide, we opt for Eq. (11) for the charge operator. As a result, the deviation from $\frac{1}{2}$ of the ratio μ_Λ/μ_n should be a broken $SU(3)$ effect. [*Note added in proof.* Our mass-corrected value for μ_Λ agrees within the error with the value -0.77 ± 0.31 reported by T. F. Kycia, Bull. Am. Phys. Soc. **10**, 101 (1965).]

II. STATEMENT OF THE PROBLEM

These encouraging results have several general characteristics in common.

[20] See, for example, M. Nauenberg, Phys. Rev. **135**, B1047 (1964).

[21] See e.g., G. Baird and L. Biedenharn, in *Operator Structures in SU_3 with an Application to Triplets, Proceedings of the Coral Gable Conference, 1964* (W. H. Freeman & Company, San Francisco, to be published), p. 58.

[22] F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. **135**, 467 (1964); T. D. Lee, CERN report 9425/Th 467 (unpublished).

[23] M. Gell-Mann, Phys. Letters **3**, 214 (1964).

[24] G. Zweig, CERN report (unpublished).

(a). They are all properties of *effective* matrix elements, either so-called two-point functions (like the results about masses) or of vertices (like the results about coupling constants, decay rates, magnetic moments).

(b). They are all characteristic low-energy parameters. This leads us to ask, as the first question, whether and how one can look upon these low-energy effective matrix elements as nonrelativistic limits of matrix elements with the proper Lorentz covariance properties. In other words, we ask for the synthesis of $SU(6)$ and of Lorentz invariance in an S-matrix theory. We show in this paper how this can be done by a process which we call the "relativistic completion" of $SU(6)$. For clarity we first do this (Sec. III) for the case of the effective vertex between the fundamental sextet of $SU(6)$ and the 35-meson representation. We show in particular how the completion of the **35** leads to a description in terms of a 12×12 matrix rather than by a 6×6 matrix as is the case for zero energy.[4] This doubling is intimately connected with the physical requirement of a **35** with *prescribed* parity. It will be recalled that, in the sense of labeling representations, parity is a label extraneous to $SU(6)$ or in other words, the parity operation commutes with all generators of $SU(6)$. In Sec. IV we discuss the same completion problem for the vertex of the **56** in interaction with the **35**.

As in any S-matrix description, the Lorentz-invariant effective matrix elements are not unique, owing to the occurrence of form factors. A group like $SU(3)$ which has no spin among its generators restricts the number of independent form factors for given space-time transformation properties. In the $SU(6)$ theory further constraints exist between form factors with different space-time properties, as will be seen in the next two sections.

Insofar as the strong-interaction vertex functions are concerned, we can choose the form factors such that the vertex is actually equivalent to a local interaction Lagrangian, invariant under $SU(6)$ in a sense to be fully specified, and invariant under the Lorentz group.[25] This is possible for the sextet interaction[1] as well as for the **56** interaction.[4] It must be emphasized, however, that our procedure of relativistic completion which serves to give meaning to $SU(6)$ invariance concurrent with Lorentz invariance is in no accepted sense of the word an extension of the Lorentz group.[26]

In fact, not even the process of completion which we will outline below can be extended to the *full* Lagrangian, including the free-field terms. It will be shown that the free kinetic-energy terms do not submit to our completion. These terms act as "spurions" from the $SU(6)$ point of view but there is nothing dynamically spurious about them as they are dictated by the Lorentz group.

Here we are at the root of the incompatibility which was noted in Ref. 7. Clearly, if we drop the kinetic-energy term of a free particle with nonvanishing spin, we cannot generate the "normal" magnetic moment which accompanies the recoil terms in the free Lagrangian.

We are therefore led to look upon $SU(6)$ as a "dynamical group" which interlocks the purely internal $SU(3)$ variables with ordinary spin, in such a way that a leading approximation to the dynamics emerges which so far seems to "work." Moreover, the possibility now arises of a prescribed "first-stage" breakdown[2] of $SU(6)$ by the Lorentz group itself. Thus part of broken-symmetry theory may be due to a clash between the "dynamical" group $SU(6)$ and the kinematical Lorentz group.

As has been noted earlier,[27] the present picture seems at least superficially to have some elements in common with old strong-coupling ideas, where diagonalization of the interaction takes precedence over that of the "free" Hamiltonian. It has always been a dark point how to include recoil in such a theory in a systematic way. We may have to face the same problem here too. However, we now leave further questions of interpretation till Sec. V and first turn to some mathematical details of the completion procedure.

III. INTERACTION BETWEEN THE SEXTET AND THE 35 MESONS

In order to illustrate the nature of the problems mentioned above, and as a prelude to the discussion of meson-baryon couplings, we first consider the interactions associated with the fundamental six-dimensional representation (sextet) of $SU(6)$. The occupants of the sextet are an $SU(3)$ triplet, each member having spin $\frac{1}{2}$. As regards their charge, we may think of them as quarks,[23,24] for definiteness (although this is not crucial to the argument). We consider the sextet coupling with the 35-dimensional adjoint representation of $SU(6)$, in accordance with

$$\mathbf{6}^*\otimes\mathbf{6}=\mathbf{1}\oplus\mathbf{35}. \qquad (15)$$

We now take as the systematic starting point that $SU(6)$ gives definite information about the structure of wave functions at zero momentum. For the sextet this information is trivial. We denote its zero-momentum wave function by $u_+{}^\alpha(0)$, $\alpha=1,\cdots,6$. We also write $\alpha=\imath, A$, where $\imath=1,2$ is a spin *state* index and $A=1,2,3$ is the $SU(3)$ index. We have

$$u_+{}^\alpha(0)=t^A\chi^\imath,\quad \chi^1=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi^2=\begin{pmatrix}0\\1\end{pmatrix}. \qquad (16)$$

[25] By Lorentz group we always mean the homogeneous extended Lorentz group.

[26] We do not encounter in our work the noncompact groups $\mathcal{G}_4$ and $\mathcal{G}_6$ mentioned in Ref. 1.

[27] See also Ref. 2, footnote 6, and Ref. 7, footnote 16.

t^A denotes the $SU(3)$ triplet. Thus for each α we have a two-component function $u_+{}^\alpha(0)$. We denote its adjoint wave function by $u_{(+)\alpha}{}^\dagger(0)=t_A\chi_i{}^\dagger$.

For the meson wave functions, denoted at zero energy by $M_\alpha{}^\beta(0)$ we have[4]

$$M_\alpha{}^\beta(0)=P_A{}^B(0)\delta_i{}^j+V_A{}^B(0)(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})_i{}^j \tag{17}$$

with $\beta=j$, B and $\alpha=i$, A. $P_A{}^B$ denotes the pseudoscalar octet, $V_A{}^B$ the vector nonet. We define

$$\begin{aligned}
&P_1{}^1=\frac{\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}}, \quad P_2{}^2=-\frac{\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}}, \quad P_3{}^3=-\frac{2\eta}{\sqrt{6}},\\
&P_1{}^2=\pi^-, \qquad P_1{}^3=K^-, \qquad P_2{}^1=\pi^+,\\
&P_3{}^1=K^+, \qquad P_3{}^2=K^0, \qquad P_2{}^3=\bar{K}^0,
\end{aligned} \tag{18}$$

and[28]

$$\begin{aligned}
&V_1{}^1=\frac{\rho^0}{\sqrt{2}}+\frac{\omega^0}{\sqrt{6}}+\frac{\phi^0}{\sqrt{3}}, \quad V_2{}^2=-\frac{\rho^0}{\sqrt{2}}+\frac{\omega^0}{\sqrt{6}}+\frac{\phi^0}{\sqrt{3}},\\
&V_3{}^3=-\frac{2\omega^0}{\sqrt{6}}+\frac{\phi^0}{\sqrt{3}}, \quad V_1{}^2=\rho^-, \quad V_1{}^3=K^{-*},\\
&V_2{}^1=\rho^+, \quad V_3{}^1=K^{+*}, \quad V_3{}^2=K^{0*}, \quad V_2{}^3=\bar{K}^{0*}.
\end{aligned} \tag{19}$$

Note that

$$\text{Trace } M=0 \text{ as Trace}P=0,\ \text{Trace}(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})=0. \tag{20}$$

$(\boldsymbol{\varepsilon}V)_A{}^B$ denotes a vector meson with polarization vector $\boldsymbol{\varepsilon}$. This trace condition is of course what is needed to have 35 independent components. Furthermore

$$\text{Trace }(M^\dagger M)=P_A{}^BP_B{}^A+(\boldsymbol{\varepsilon}V)_A{}^B(\boldsymbol{\varepsilon}V)_B{}^A \tag{21}$$

provides a quadratic form invariant under the operations of $SU(6)$. The relative weight of vector and pseudoscalar terms in Eq. (17) has just been chosen in accordance with this requirement.

An $SU(6)$-invariant coupling with all particles having zero momentum may be trivially written as $u_{+\alpha}{}^\dagger(0)M_\beta{}^\alpha(0)u_+{}^\beta(0)$. It is clear however that if the mesons are characterized by negative parity, this coupling is not invariant under space inversion. It leads e.g., to S-wave emission of pions. In order to write down meaningful $SU(6)$ couplings which conserve parity it is necessary to extrapolate the $u_+{}^\alpha(0)$, $M_\beta{}^\alpha(0)$, etc. to finite momenta via appropriate Lorentz transformations. Furthermore we may not just "boost" $u_+{}^\alpha(0)$ to a finite momentum via a transformation[29] of $SL(2,C)$; for then we will still have difficulty with parity conservation. It is necessary therefore to consider the particle and antiparticle states simultaneously and join the two-dimensional representations of the Lorentz group in the usual manner.

[28] ω^0, ϕ^0 are identical with ω_P, ϕ_P in Ref. 6.

[29] See, for example, R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (W. Benjamin Inc., New York, 1964), Chap. 1.

For the sextet this "completion" is obvious. We declare the completion of Eq. (16) to be

$$u_+{}^\alpha(q)=N(q)\begin{bmatrix}\chi^i\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{q}}{q_0+M}\chi^i\end{bmatrix}t^A \tag{22}$$

which doubles the number of (nonvanishing) *components* to four. $N(q)$ is a normalizing factor, $N(0)=1$. Furthermore we have to complete the particle with the antiparticle states. $SU(6)$ does not intrinsically realize the corresponding degeneracy. From the point of view of this group, the antiparticle states form a new representation $u_-{}^\alpha(q)$. We join particles and antiparticles by

$$u_-{}^\alpha(q)=\gamma_5u_+{}^\alpha(q), \quad \gamma_5=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{23}$$

where I, 0 are 2×2 unit and null matrices, respectively.

The completion is not so immediate for mesons. We proceed to show that here the extrapolation is uniquely determined by the requirement that meson-sextet couplings be invariant under the extended Lorentz group.[25] *Furthermore the requirement that an $SU(6)$ structure exist in the low-frequency limit gives constraints which are valid in the relativistic domain where the only linear invariance group to the best of our present knowledge is the direct product of $SU(3)$ and the Lorentz group.*

Let $\psi(x)$ be the field operator of a spin-$\frac{1}{2}$ particle, transforming as a triplet under $SU(3)$. Furthermore, let $\phi(x)$ and $V^\mu(x)$ denote the field operators of mesons transforming respectively as pseudoscalar and vector under the Lorentz group and as 8 and $1\oplus8$ under $SU(3)$. The interaction density

$$i\mathfrak{L}_I(x)=i\frac{g_1}{\mu_{00}}\bar{\psi}\gamma_5\gamma_\mu\partial^\mu\phi\psi+g_2\bar{\psi}\gamma_\mu[V^\mu-\tfrac{1}{3}\,\text{Tr}(V^\mu)]\psi+\tfrac{1}{3}g_3\bar{\psi}\gamma_\mu\,\text{Tr}(V^\mu)\psi \tag{24}$$

[where the indicated traces are over $SU(3)$ indices] is invariant under $L\otimes SU(3)$. This group, of course, tells us nothing about the relative magnitudes of g_1, g_2, and g_3.

Introduce the Fourier decomposition

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\sum_{k,i,A}\left(\frac{M}{E}\right)^{1/2}\{a_{k,i,A}t^Au_+{}^i(k)e^{ikx}+b_{k,i,A}{}^+t^Au_-{}^i(k)e^{-ikx}\} \tag{25}$$

with similar decompositions for V_μ and ϕ. A matrix element of $\mathfrak{L}_I(x)$ may be exhibited as

$$g_1\bar{w}_\lambda(p_2)\mathfrak{M}_\mu{}^\lambda(q)w^\mu(p_1), \tag{26}$$

where $q=p_2-p_1$ and λ, μ are labels that run from 1 through 12,

$$w^\lambda(p)=t^A u^{i,a}(p),\quad A=1,2,3;\ i=1,2;\ a=1,2.$$

Here a is the index which doubles the number of components. The matrix $\mathfrak{M}_\mu{}^\lambda$ is given by

$$\mathfrak{M}_\mu{}^\lambda(q)=\begin{pmatrix} N_\alpha{}^\beta(q) & -M_\alpha{}^\beta(q) \\ M_\alpha{}^\beta(q) & -N_\alpha{}^\beta(q)\end{pmatrix}, \tag{27}$$

where

$$N_\alpha{}^\beta(q)=P_A{}^B\frac{(\boldsymbol{\sigma}\cdot\mathbf{q})_i{}^j}{\mu_{00}}+\left(\frac{g_2}{g_1}\right)(V_A{}^B-\tfrac{1}{3}\delta_A{}^B V_C{}^C)\epsilon^0\delta_i{}^j+\left(\frac{g_3}{g_1}\right)\times\tfrac{1}{3}\delta_A{}^B V_C{}^C\epsilon^0\delta_i{}^j, \tag{28}$$

$$M_\alpha{}^\beta(q)=P_A{}^B\frac{q_0}{\mu_{00}}\delta_i{}^j+\left(\frac{g_2}{g_1}\right)(V_A{}^B-\tfrac{1}{3}\delta_A{}^B V_C{}^C)(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})_i{}^j+\left(\frac{g_3}{g_1}\right)\times\tfrac{1}{3}\delta_A{}^B V_C{}^C(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})_i{}^j, \tag{29}$$

$\boldsymbol{\epsilon}$ being the polarization vector of the vector mesons, satisfying the Lorentz condition $q_\mu\epsilon^\mu=0$.

In the limit in which $\mathbf{q}=0$, we find

$$N_\alpha{}^\beta(0)=0, \tag{30}$$

$$M_\alpha{}^\beta(0)=P_A{}^B\delta_i{}^j+\left(\frac{g_2}{g_1}\right)(V_A{}^B-\tfrac{1}{3}\delta_A{}^B V_C{}^C)(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})_i{}^j+\left(\frac{g_3}{g_1}\right)\tfrac{1}{3}\delta_A{}^B V_C{}^C(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})_i{}^j. \tag{31}$$

If we now require that the theory have an $SU(6)$ structure in the low-frequency limit, we must identify $M_\alpha{}^\beta(0)$ with the meson-tensor Eq. (17). This identification requires that

$$g_2/g_1=g_3/g_1=1. \tag{32}$$

The requirement of an $SU(6)$ limit therefore tells us that the Lorentz invariant interaction of the fundamental sextet with the pseudoscalar octet, the vector octet and the vector singlet is characterized by a single coupling constant. It is this circumstance that leads us to assert that $SU(6)$ symmetry is a meaningful concept in the relativistic domain.

Clearly, the description of the **35** mesons by the 12×12 matrix Eq. (27) does not introduce any new fields, that is, new representations of $SU(6)$. Just as it is necessary, by completion, to double the number of nonvanishing *components* of a spinor when going from the zero-energy to the relativistic description, so the same is true for the mesons. At zero momentum a pseudoscalar (vector) meson has one (three) degrees of freedom corresponding to the spin. In the near-static limit ($q/\mu_{00}\ll1$) we may look upon the completion of $M_\beta{}^\alpha$ by $N_\beta{}^\alpha$ as the introduction of the "small components" of the meson field, in close analogy to the spin-$\frac{1}{2}$ case. Note also that the 12-trace $\mathrm{Tr}(\bar{\mathfrak{M}}\mathfrak{M})/2$ reduces to Eq. (21) in the limit $\mathbf{q}=0$. Here $\bar{\mathfrak{M}}=\gamma_4\mathfrak{M}^\dagger\gamma_4$.

In Sec. II we promised to treat the theory with relativistic form factors. This is now done at once, as follows. In momentum space introduce three form factors $g_i(q^2)$, $i=1,2,3$ with the property

$$\lim_{q^2\to0} g_i(q^2)=g_i, \tag{33}$$

while Eq. (32) remains enforced. This is the completion of $SU(6)$ in the general case. From this it is evident that one can not conclude from $SU(6)$ *alone* anything new about high-energy behavior *unless the $SU(6)$ theory is further supplemented by specific dynamical arguments. However, to order v/c the predictions of $SU(6)$ are evidently unique.*

We consider next the free Lagrangian, staying in momentum space. First consider the mass term of the quarks. This is of the form

$$M\bar{w}_\lambda(p)w^\lambda(p) \tag{34}$$

and is as invariant as is the interaction (26). We may look upon M as the (1,1) representation of $SU(6)$ and satisfy $SU(6)$ by Eq. (15), and also satisfy Lorentz invariance.

Not so for the kinetic-energy term

$$\bar{w}_\lambda(p)[\gamma_\epsilon p^\epsilon]_\mu{}^\lambda w^\mu(p), \tag{35}$$

where, in a notation analogous to that of Eq. (27)

$$[\gamma_\epsilon p^\epsilon]_\mu{}^\lambda=\delta_A{}^B\begin{pmatrix} p_0\delta_i{}^j, & -(\boldsymbol{\sigma}\cdot\mathbf{p})_i{}^j \\ +(\boldsymbol{\sigma}\cdot\mathbf{p})_i{}^j, & -p_0\delta_i{}^j\end{pmatrix}. \tag{36}$$

This term has the same *value* as does Eq. (34) for free fields, or in the interaction representation, on the mass shell. However, for general dynamical considerations we can not confine ourselves to that special case. It is now quite obvious from (36) that we cannot give $[\gamma_\epsilon p^\epsilon]$ a completed $SU(6)$ meaning. This term corresponds in fact to the (1,3) part of a **35**, but is neither accompanied by (8,3) nor by (8,1). Similar arguments hold for the mass-versus-kinetic-energy terms in the free-meson Lagrangian.

This breakdown of the completed $SU(6)$ by the kinetic-energy term can be exemplified in an inexact but perhaps illuminating way, as follows. Let us take the complete Lagrangian, impose the conditions Eq. (32) and now calculate by naive perturbative field theory the second-order self-energy of a vector and of a pseudoscalar meson via a quark bubble, using the same cutoff for both integrals. The results are distinct which indicates (but does not prove) that the mass degeneracies are broken. Now drop from the integrand the γp term in the quark propagator. Then the formal expressions *are* the same. Likewise for multibubble diagrams.

We conclude that the apparent success of the $SU(6)$ ansatz must have important implications for the effective damping of such integrals at high-frequency virtual states. We comment further on this in Sec. V, but now turn first to the baryon-meson vertex.

IV. BARYON (56)-MESON (35) INTERACTION

The completion of the $SU(6)$ invariant meson-sextet coupling, in order to incorporate the physically indispensable requirement of invariance under the extended Lorentz group, opens the road for a similar completion of couplings associated with other $SU(6)$ representations. To the extent that one can overcome, bypass or ignore the possible difficulties associated with quantization of fields with higher spin, one does not encounter any new problems that are more than technical in nature.

We proceed to consider the coupling of the spin-$\frac{1}{2}$ baryon octet and the spin-$\frac{3}{2}$ decuplet, to the mesons in accordance with

$$\mathbf{56}^*\otimes\mathbf{56}=\mathbf{1}+\mathbf{35}+\mathbf{405}+\mathbf{2695}. \tag{37}$$

In the limit of zero momenta, this coupling is given by

$$B_{\alpha\beta\gamma}{}^{\dagger}B^{\alpha\beta\delta}M_{\delta}{}^{\gamma}(0), \tag{38}$$

where $M_\delta{}^\gamma(0)$ is the meson tensor mentioned earlier and $B^{\alpha\beta\gamma}$ is the completely symmetric baryon tensor, reducible under $SU(2)\otimes SU(3)$ as

$$B^{\alpha\beta\gamma}=\chi^{(ijk)}d^{ABC}+\frac{1}{3\sqrt{2}}[(2\epsilon^{ij}\chi^k+\epsilon^{jk}\chi^i)\epsilon^{ABD}b_D{}^C + (\epsilon^{ij}\chi^k+2\epsilon^{jk}\chi^i)\epsilon^{BCD}b_D{}^A]. \tag{39}$$

Here ϵ^{ij} and ϵ^{ABC} are the Levi-Civita symbols left invariant by the groups $SL(2)$ and $SL(3)$, respectively and thus under $SU(2)\subset SL(2)$ and $SU(3)\subset SL(3)$. The χ^i are defined in Eq. (16). The $\chi^{(ijk)}$ are the spin-$\frac{3}{2}$ spinors,

$$\chi^{(111)}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad \chi^{(112)}=\frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad \chi^{(122)}=\frac{1}{\sqrt{3}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad \chi^{(222)}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \tag{40}$$

$b_A{}^B$ is the baryon $SU(3)$-octet tensor,

$$\begin{gathered} b_1{}^1=\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}},\quad b_2{}^2=-\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}},\quad b_3{}^3=-\frac{2\Lambda}{\sqrt{6}},\\ b_2{}^1=\Sigma^+,\quad b_3{}^1=p,\quad b_1{}^2=\Sigma^-,\quad b_3{}^2=n,\\ b_1{}^3=\Xi^-,\quad b_2{}^3=-\Xi^0, \end{gathered} \tag{41}$$

d^{ABC} is the $SU(3)$-decuplet tensor,

$$\begin{gathered} d^{111}=N_{++}{}^*,\quad d^{112}=\frac{1}{\sqrt{3}}N_+{}^*,\quad d^{122}=\frac{1}{\sqrt{3}}N_0{}^*,\quad d^{222}=N_-{}^*,\\ d^{113}=\frac{1}{\sqrt{3}}Y_+{}^*,\quad d^{123}=\frac{1}{\sqrt{6}}Y_0{}^*,\quad d^{223}=\frac{1}{\sqrt{3}}Y_-{}^*,\\ d^{133}=\frac{1}{\sqrt{3}}\Xi_0{}^*,\quad d^{233}=\frac{1}{\sqrt{3}}\Xi_-{}^*,\quad d^{333}=\Omega^-. \end{gathered} \tag{42}$$

A relativistic completion can be obtained as follows. We treat the χ^i as before. In place of the $\chi^{(ijk)}$ we use eight component spinors constructed from the solutions of the Rarita-Schwinger[30] equations. For a spin-$\frac{3}{2}$ particle moving along the z axis with momentum p_3, we may write these spinors as $u_+{}^{(ijk)}(p_3)$, where

$$\begin{aligned} u_+{}^{(111)}(p_3)&=N_{3/2}(p_3)\begin{bmatrix}\chi^{(111)}\\ \dfrac{p_3}{E+M}\chi^{(111)}\end{bmatrix},\\ u_+{}^{(112)}(p_3)&=N_{3/2}'(p_3)\begin{bmatrix}\chi^{(112)}\\ \dfrac{p_3}{E+m}\dfrac{2E-m}{2E+m}\chi^{(112)}\end{bmatrix},\\ u_+{}^{(122)}(p_3)&=N_{3/2}'(p_3)\begin{bmatrix}\chi^{(122)}\\ \dfrac{-p_3}{E+m}\dfrac{2E-m}{2E+m}\chi^{(122)}\end{bmatrix},\\ u_+{}^{(222)}(p_3)&=N_{3/2}(p_3)\begin{bmatrix}\chi^{(222)}\\ -\dfrac{p_3}{E+m}\chi^{(222)}\end{bmatrix}. \end{aligned} \tag{43}$$

In order to have a complete set, we need in addition four negative-energy solutions; these are given by

$$u_-{}^{(ijk)}=\gamma_5 u_+{}^{(ijk)}. \tag{44}$$

The reader may satisfy himself of the correctness of these solutions by verifying that they satisfy the spin-$\frac{3}{2}$ wave equation written down by Moldauer and Case.[31]

[30] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

[31] P. A. Moldauer and K. M. Case, Phys. Rev. **102**, 279 (1956), see especially Eq. (2.19).

For arbitrary orientation of the momentum a set of helicity solutions may be generated by applying the matrix

$$\begin{pmatrix} \mathfrak{D}^{(3/2)}(-\phi,\theta,\phi), & 0 \\ 0, & \mathfrak{D}^{(3/2)}(-\phi,\theta,\phi) \end{pmatrix}, \tag{45}$$

where ϕ, θ specify the direction of $\mathbf{p}$ and

$$\mathfrak{D}^{(3/2)}(-\phi,\theta,\phi)=e^{iM_3\phi}e^{-iM_2\theta}e^{-iM_3\phi}, \tag{46}$$

M_1, M_2, M_3 being the 4-dimensional representations[32] of the three generators of $SU(2)$.

We can now write down a completion of Eq. (39) in an arbitrary Lorentz frame. For positive- or negative-energy solutions it is

$$\begin{aligned} B_{\pm}{}^{\alpha\beta\gamma}(p)=&u_{\pm}{}^{(ijk)}(p)d^{ABC} \\ &\oplus\frac{1}{3\sqrt{2}}[\{2\epsilon^{ij}u_{\pm}{}^{k}(p)+\epsilon^{jk}u_{\pm}{}^{i}(p)\}\epsilon^{ABD}b_D{}^C \\ &+\{\epsilon^{ij}u_{\pm}{}^{k}(p)+2\epsilon^{jk}u_{\pm}{}^{i}(p)\}\epsilon^{BCD}b_D{}^A]. \end{aligned} \tag{47}$$

Note added in proof. One cannot form a Lorentz invariant vertex from Eqs. (27) and (47). However, it turns out that if one neglects baryon recoil completely, the coupling

$$3\sqrt{2}B_{(+)\alpha\beta\gamma}{}^{\dagger}N_{\delta}{}^{\gamma}(q)B_{+}{}^{\alpha\beta\delta} \tag{48}$$

is a legitimate limit of the Lorentz invariant vertex.[33] We have meanwhile found a fully covariant description, see M. A. B. Bég and A. Pais, Phys. Rev. Letters (to be published).

V. CONCLUDING REMARKS

(a). Section IV describes how it was possible to give the covariant completion of the **56** representation of $SU(6)$. For this purpose we constructed local wave functions for both spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ components of this state with the appropriate Lorentz transformation properties. While it is an extremely useful mathematical tool for the construction of $SU(6)$ states (at zero energy) to build them up out of product states of three sextets with the right symmetry, this construction in itself is not predicated on the actual existence of such sextets. Furthermore it is not obvious how to make such an explicit and covariant construction for nonzero momentum. However this may be, the completion calculations once again provide no evidence for or against the existence of physical sextets. We feel impelled to repeat this point (see also Sec. II) in order to stress that the $SU(6)$ theory is not necessarily built upon an atomic-structure-type model.

(b). At the same time the impression is inescapable that the description of the low-lying baryons by the **56** is highly complex and that a simpler underlying description is called for, but we do not know what that is.

(c). At the $SU(3)$ level it is a puzzle why the Gell-Mann–Okubo mass formula works as well as it does. At the $SU(6)$ level this puzzle is magnified. Why does the $SU(6)$ mass formula work so well, at least for the **56**? Why are the several agreements with experiment reviewed in Sec. I so good when $SU(6)$ is badly broken in the real world? The suggestion has been made[22] that, insofar as the Gell-Mann–Okubo mass formula is concerned, its success may indicate the existence of triplets with a relatively high mass. To what extent this could also be helpful to explain some of the $SU(6)$ regularities remains a question for further study.

(d). $SU(6)$ is compatible with the existence of a conserved isovector-vector current. This follows from the formalism given in Secs. III and IV.

(e). If in Eq. (49) we replace $\mathfrak{M}$ by $(1+\lambda\gamma_5)\mathfrak{M}$, the $SU(6)$ structure remains unaffected and the same is of course true for the behavior under proper Lorentz transformations, but of course parity would not be conserved. In our opinion, $SU(6)$ invariance by itself sheds therefore no new light on the question of parity conservation in strong interactions.

(f). While quantities like rest mass, magnetic moment, coupling constants are all zero- (or low-) energy parameters, their effective values are codetermined by high virtual-frequency contributions. In view of what has been said about the violation of completed $SU(6)$ by kinetic-energy terms, one is led to surmise that, wherever an $SU(6)$ prediction works well, there is a strong effective damping involved in these high-energy contributions.

In this connection it should be noted that the interaction (49) is unrenormalizable in the conventional sense. However, the conjectured high-frequency damping may render such questions irrelevant, emerging as they do from naive perturbation theory.

Generally, through the notion of dynamical group, pure-symmetry arguments are intertwined with dynamical considerations. A main problem now appears to be to find the best dynamical (nonperturbative[34]) methods to cope with this situation.

(g). In this paper we have only considered explicitly the vector and pseudovector couplings of the **35**. In an S-matrix framework one should of course consider all possible form factors [we thank S. B. Treiman for emphasizing this point]. Also the nonminimal form factors[4] can easily be treated by the present methods.

[32] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed. p. 146.

[33] Instead of the factor $3\sqrt{2}$, a normalization factor 6 was used in Eq. (4) of Ref. 4. This is an inessential difference, due to a different normalization of the meson matrix which was used in Ref. 4. With the help of Eqs. (27) and (48), the reader will be able to check the results obtained in Ref. 4 and reviewed in Sec. I.

[34] See R. Jost, as quoted in p. 31 of Ref. 29 and public communication to the authors, 1962.

Note added in proof. After the completion of this work we received a preprint by K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee[35] which also deals with the synthesis of $SU(6)$ and relativistic field theory. Their group $W(6)$ is broken by the kinetic-energy as well as the mass terms. Unlike the present work, new mesons are necessitated. As in the present paper, a possible connection with strong coupling is also noted.

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Strong-Interaction Symmetries Based upon Rank-Three Lie Groups*

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We consider all rank-three simple Lie groups as possible candidates for a higher symmetry of strong interactions. All such groups imply the existence of a new quantum number X, the oddness, and of odd particles with nonzero values of X. Because of uncertainties in the experimental observation of these particles, we look for evidence of such symmetries in the properties of ordinary ($X=0$) particles. We give arguments to show that ϕ decay into ρ and π mesons is a particularly good place to look for such evidence. In all groups, we assign the vector mesons to the regular representation and derive mass formulas and decay rates for various assignments of the pseudoscalar mesons and the mass operator to representations of the group. We find that it is possible to formulate a general criterion, which can be applied to all rank-three Lie groups, for assigning these representations, and that with this assignment all such groups give the same mass formula and decay widths for the vector mesons, namely,

$$(3\omega+\rho-4K^*)(3\phi+\rho-4K^*)+8(\rho-K^*)^2=0$$

and

$$\Gamma(\phi\to\rho\pi)=0.3\text{–}0.6\text{ MeV},\ \Gamma(\phi\to K\bar{K})=2\text{ MeV},$$

in extremely good agreement with experiment. We summarize the main properties of rank-three Lie groups in appendices.

I. INTRODUCTION

THE octet model of $SU(3)$[1] has recently proved strikingly successful in correlating experimental information both in the strong and the weak interactions. There are, however, certain unexpected regularities between different $SU(3)$ multiplets which have led many authors[2–10] to consider the possibility of embedding $SU(3)$ in a higher symmetry group $\mathcal{G}$. Among these are the approximate degeneracy of the unmixed masses of the vector meson singlet and octet[11,12] and the anomalously small ratio of the $\phi\rho\pi$ to the $\omega\rho\pi$ coupling. There are also several relationships between multiplets of different spins which might be explained by extending the treatment we shall give here as has been done for $SU(3)$.[13,14] Furthermore, in looking for a dynamical basis for $SU(3)$ symmetry, one is naturally led to introduce triplets[15] of fundamental fields. The nonexistence of fractionally charged particles with masses less than 3 BeV seems well established experimentally,[16] however, and modifying the Gell-Mann–Nishijima relationship to allow for integral triplet charges in itself suggests the existence of a higher symmetry.[17,18]

We shall consider the case when $SU(3)$ is embedded in a simple Lie algebra of rank three. Including baryon conservation this means that the Lie algebra corre-

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