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Impact-Parameter Expansion of High-Energy Elastic-Scattering Amplitudes*

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An impact-parameter expansion of high-energy scattering amplitudes is formulated, and the unitarity condition at large but finite energies is discussed. Applications are then made to high-energy πp and pp elastic scattering with particular reference to the Van Hove model. This model is shown to give a good description of $\pi^- p$ scattering at 17.0 BeV/c scattering momentum; the addition of a short-range real part gives a qualitative description of pp scattering at momenta between 12 and 20 BeV/c.

1. INTRODUCTION

CONSIDERABLE attention has been devoted recently to the behavior of elastic scattering of elementary particles at high energies ($\gtrsim 10$ BeV). Experimental data are available[1] for $\pi^\pm$, $K^\pm$, p, and $\bar{p}$ scattering from protons for small squared momentum transfers t [$\lesssim 1$ (BeV/c)^2] and in addition up to the largest possible momentum transfers for pp scattering.[2,3] For all of these processes the small momentum transfer region exhibits an exponentially sharp diffraction peak, energy-independent in the case of meson-proton scattering and shrinking slightly with energy for proton-proton scattering. Most theoretical discussion has been restricted to this region. The large momentum transfer proton-proton scattering cross sections exhibit a very strong energy dependence.[3]

In the present paper we apply a model proposed by Van Hove,[4] in which the elastic scattering is assumed to be pure imaginary shadow scattering corresponding to absorption into multiparticle inelastic channels. This model is shown to provide an essentially parameter-free fit to the πp data and, with the addition of a slowly varying real part, to reproduce the behavior of the p-p scattering at all momentum transfers.

In Secs. 2 and 3 we introduce the impact parameter expansion for the elastic-scattering amplitude showing that it provides a useful spatial description of the scattering without any reference to the existence of a local potential. In Sec. 4 the Van Hove model is reformulated in the impact parameter representation and the comparison given with the πp data. In Sec. 5 the addition of a real part is discussed and the results compared with the pp data.

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[1] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **11**, 425, 503 (1963).

[2] A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 111 (1962).

[3] G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubenstein, D. B. Scarl, W. F. Baker, E. W. Jenkins, and A. L. Read, Phys. Rev. Letters **11**, 499 (1963); W. F. Baker, E. W. Jenkins, A. L. Read, G. Cocconi *et al.*, *ibid.* **12**, 132 (1964).

[4] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).

2. THE IMPACT PARAMETER EXPANSION

Consider the elastic scattering of two spinless particles with center-of-mass momentum k, scattering through an angle θ. Let $A(s,t)$ be the invariant scattering amplitude. We define the impact parameter expansion of the amplitude by the relation

$$h(b)=\frac{1}{2kW}\int_0^\Lambda A(x)J_0(bx)xdx\,, \qquad (2.1)$$

where $W=\sqrt{s}$ is the total energy in the center-of-mass system, $\Lambda=\sqrt{2}k$ for identical particles and $2k$ for distinguishable particles, and $x^2=[2k\sin(\theta/2)]^2=-t$. We can now define the function

$$T(x)=2kW\int_0^\infty h(b)J_0(bx)bdb \qquad (2.2)$$

which has the property

$$\begin{aligned} T(x)&=A(x)\,, \quad 0\leq x\leq\Lambda \\ T(x)&=0\,, \qquad x>\Lambda\,. \end{aligned} \qquad (2.3)$$

This expansion is closely related to the eikonal approximation[5] for the scattering of a particle in a potential, and is almost identical with the amplitude $H(b^2,s)$ introduced by Blankenbecler and Goldberger[6] (BG) as a basis for dynamical calculations using dispersion relations. It differs from the former in that it is an exact definition, rather than a quantity occurring in an approximation, and from the latter in that, like a partial-wave amplitude, it involves only physical amplitudes.

The relation to the partial wave expansion is easily obtained. If we define $f_l=(e^{2i\delta_l}-1)/2i$ where δ_l is the partial-wave phase shift, then

$$f_l=\frac{k}{2W}\int_{-1}^{+1}A(\theta)P_l(\cos\theta)d\cos\theta \qquad (2.4)$$

(we restrict ourselves to the case of nonidentical particles).

[5] R. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1958), p. 315.

[6] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

Writing $z=\sin(\theta/2)$, $d\cos\theta=4zdz$, and using (2.2) and (2.3) we obtain, after integrating over z,

$$f_l=\int_0^\infty h(b)J_{2l+1}(2kb)d(2kb) \tag{2.5}$$

and, in a similar manner, the inverse relation

$$h(b)=2\sum_l(2l+1)f_l\frac{J_{2l+1}(2kb)}{2kb}. \tag{2.6}$$

We can see first of all that $h(0)=f_0$ from the definitions. Furthermore, provided that $h(b)$ falls off for large b at least as fast as some polynomial times e^{-b}, which is a reasonable requirement for a short-range interaction, one can expand (2.5) for large k[7]:

$$f_l=h\left(\frac{2l+1}{2k}\right)-\frac{1}{8k^2}\left(\frac{d^2h}{db^2}\right)_{b=(2l+1)/2k}+O\left(\frac{1}{k^4}\right). \tag{2.7}$$

The impact parameter expansion is particularly convenient for the discussion of the behavior of the high-energy elastic scattering. First of all, the invariant momentum transfer t is a much more natural variable than the center of mass scattering angle θ: indeed the experimental evidence indicates that asymptotically most elastic scattering become energy-independent functions of t (or at least functions of t which vary slowly with energy) indicating that the high-energy form of $h(b)$ is approximately energy-independent.

Secondly, when a large number of partial waves are present the integrals (2.1) and (2.2) are more convenient to handle than the less symmetrical sum and integral involved in the partial-wave expansion. Indeed the usual procedure is to use Macdonald's expansion to express the Legendre polynomial as a Bessel function and replace the discrete sum over l by an integral. Equation (2.7) above exhibits this approximation directly.

Let us define

$$h(b)=h_R(b)+ih_I(b). \tag{2.8}$$

$h_R(b)$ and $h_I(b)$ are the real and imaginary parts of $h(b)$. From the optical theorem we obtain immediately a formula for the total cross section σ_T

$$\sigma_T=\frac{4\pi}{Wk}\operatorname{Im}T(0)=8\pi\int_0^\infty h_I(b)bdb. \tag{2.9}$$

Also, the total elastic cross section σ_{el} is

$$\sigma_{\mathrm{el}}=\frac{1}{W^2}\int_0^\Lambda |A|^2d\Omega$$

$$=\frac{2\pi}{k^2W^2}\int_0^\infty |T(x)|^2xdx \tag{2.10}$$

[7] C. F. Curtis, J. Math. Phys. **5**, 561 (1964).

$$\sigma_{\mathrm{el}}=8\pi\int_0^\infty[h_R^2(b)+h_I^2(b)]bdb. \tag{1.11}$$

The principal disadvantage of this expansion, as compared with the partial-wave expansion, is that the amplitudes $h(b)$ do not, strictly speaking, satisfy a simple unitarity condition. However, in the high-energy limit, as pointed out by BG,[6] the unitarity condition of $h(b)$ does become simple, as is clearly the case for $b=(2l+1)/2k$ from Eq. (2.7). In the next section we show that the corrections to order k^{-2} to this unitarity condition can be calculated for arbitrary b. This enables us to say something about the approach to the asymptotic region.

Some care has to be exercised on account of the finite upper limit on the integral in (2.1). The functions $h(b)$ are restricted to the class of Hankel transforms of functions which vanish for $x>\Lambda$. Such a sharp cutoff in $T(x)$ will induce rapid oscillations in $h(b)$.

If $T(x)$ were to be replaced by a function $\tilde{T}(x)$ which is cut off sufficiently smoothly above $x=\Lambda$, we could obtain a function $\tilde{h}(b)$ without the oscillating transient. [$\tilde{T}(x)$ will in general bear no relation to the true analytic continuation of $A(x)$ outside the physical region.] The relative error in $h(b)$ if we use $\tilde{T}(x)$ is

$$\frac{\Delta h(b)}{h(b)}=\int_0^\infty[T(x)-\tilde{T}(x)]J_0(bx)xdx\Big/\int_0^\infty T(x)J_0(bx)xdx. \tag{2.12}$$

In the discussion of high-energy elementary particle elastic scattering, we will be concerned with amplitudes $A(x)$ which are exponentially small at $x=\Lambda$. Since $\tilde{T}(x)=T(x)$ except for $x\gtrsim\Lambda$ and is a smooth function, this error will be of order e^{-k} and can be neglected in any systematic expansion in inverse powers of k. However, this does mean that the expansion cannot be trusted in the immediate neighborhood of $x=\Lambda$.

3. THE UNITARITY CONDITION

The derivation in this section is essentially the one given by BG, though the details are different and, we hope, more transparent. In addition, we include the contribution of inelastic unitarity in a phenomenological way and are able to discuss how the asymptotic limit is approached for large but finite energies.

We start with the unitarity condition for an elastic-scattering amplitude:

$$\operatorname{Im}\langle f|T|i\rangle=\frac{k}{4\pi W}\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle, \tag{3.1}$$

where the sum runs over all possible intermediate states that conserve four-momentum. For the states $|n\rangle$ that correspond to inelastic channels $\langle i|T|n\rangle$ is the appropriately normalized production amplitude. The contribution to the sum of all such inelastic states can be considered separately from the elastic scattering contribution and Eq. (3.1) can be written

$$\operatorname{Im}T(x)=\frac{kW}{4\pi}\sigma_{\rm in}g(x)+\frac{k}{4\pi W}\int T^{\dagger}(x')T(x'')d\Omega', \quad (3.2)$$

where

$$x'=2k\sin(\theta'/2);\quad x''=2k\sin(\theta''/2);$$
$$d\Omega'=\sin\theta' d\theta' d\phi'=(1/k^2)x'dx'd\phi'.$$

The angles are related by the equation

$$\cos\theta''=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'.$$

$\sigma_{\rm in}$ is the total inelastic cross section and $W^2\sigma_{\rm in}g(x)$ is the inelastic contribution to the sum in Eq. (2.1). Since the point $x=0$ corresponds to forward scattering, the optical theorem implies that

$$g(0)=1.0.$$

We define the impact parameter expansion of $g(x)$ by

$$\rho(b)=\frac{\sigma_{\rm in}}{2\pi}\int_0^{\Lambda}g(x)J_0(bx)xdx,$$

$$g(x)=\frac{2\pi}{\sigma_{\rm in}}\int_0^{\infty}\rho(b)J_0(bx)bdb, \quad (3.3)$$

and

$$g(x)=0\quad\text{for}\quad x>\Lambda.$$

$g(x)$ will in general be energy-dependent. However, in considering the unitarity condition, which is a condition on the amplitude at fixed energy, we do not exhibit the energy dependence explicitly.

By expressing $T(x'')$ in the impact parameter representation Eq. (3.2) can be written

$$\operatorname{Im}T(x)=\frac{Wk}{4\pi}\sigma_{\rm in}g(x)+\frac{1}{2\pi}\int_0^{\Lambda}T^{\dagger}(x')x'dx'\times\int_0^{\infty}h(b'')b''db''\int_0^{2\pi}J_0(b''x'')d\phi'. \quad (3.4)$$

In order to perform the ϕ' integration we note that

$$b''x''=(z^2+z'^2-2zz'\cos\phi')^{1/2},$$

$$z=b''x\left(1-\frac{x'^2}{4k^2}\right)^{1/2}, \quad (3.5)$$

$$z'=b''x'\left(1-\frac{x^2}{4k^2}\right)^{1/2}.$$

Then[8]

$$J_0(b''x'')=\sum_{n=-\infty}^{+\infty}J_n(z)J_n(z')e^{in\phi}, \quad (3.6)$$

whence

$$\operatorname{Im}T(x)=\frac{Wk}{4\pi}\sigma_{\rm in}g(x)+\int_0^{\Lambda}T^{\dagger}(x')x'dx'\times\int_0^{\infty}h(b'')b''db''J_0(z)J_0(z'). \quad (3.7)$$

We want to consider the limit of large k. In the limit $k\to\infty$ the arguments of the Bessel functions in Eq. (3.7) become simply $b''x$ and $b''x'$. The x' integration then gives, using Eq. (2.3),

$$\operatorname{Im}T(x)=\frac{Wk}{4\pi}\sigma_{\rm in}g(x)+(2kW)\int_0^{\infty}|h(b'')|^2J_0(b''x)db''b'' \quad (3.8)$$

and taking the Hankel transform of both sides of this equation yields the asymptotic limit of the unitarity condition for $h(b)$

$$\operatorname{Im}h(b)=\tfrac{1}{4}\rho(b)+|h(b)|^2. \quad (3.9)$$

The corrections to Eq. (3.9) to order k^{-2} can be calculated in a straightforward manner and the details are given in the Appendix. We obtain the result

$$\operatorname{Im}h(b)=\tfrac{1}{4}\rho(b)+|h(b)|^2+\frac{1}{4k^2}\frac{d}{db^2}\left(b^2\left|\frac{dh}{db}\right|^2\right)+O(k^{-4}). \quad (3.10)$$

It is of interest to note that these corrections to order k^{-2} come only from the k dependence of the integrand in Eq. (3.7), not from the upper limit of the integration region. Equation (3.10) is therefore valid for both identical and nonidentical particles.

4. TOTALLY ABSORBING MODELS

Let us consider first the case of a purely imaginary amplitude, $h_R(b)=0$. We then have for the unitarity condition in the asymptotic limit $[h_I^0=\lim_{k\to\infty}h_I(b,k)]$

$$h_I^0=\tfrac{1}{4}\rho+(h_I^0)^2, \quad (4.1)$$

$$h_I^0=\tfrac{1}{2}[1-(1-\rho)^{1/2}]. \quad (4.2)$$

The choice of sign for the square root is dictated by the requirement that h_I should go to zero with ρ, since in the absence of inelastic scattering a vanishing real part implies no scattering at all. One can immediately see that we have the restrictions

$$\rho\leq 1,$$
$$h_I\leq\tfrac{1}{2}.$$

[8] *Bateman Manuscript Project, Highter Transcendental Functions*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, Eq. 7.15 (31).

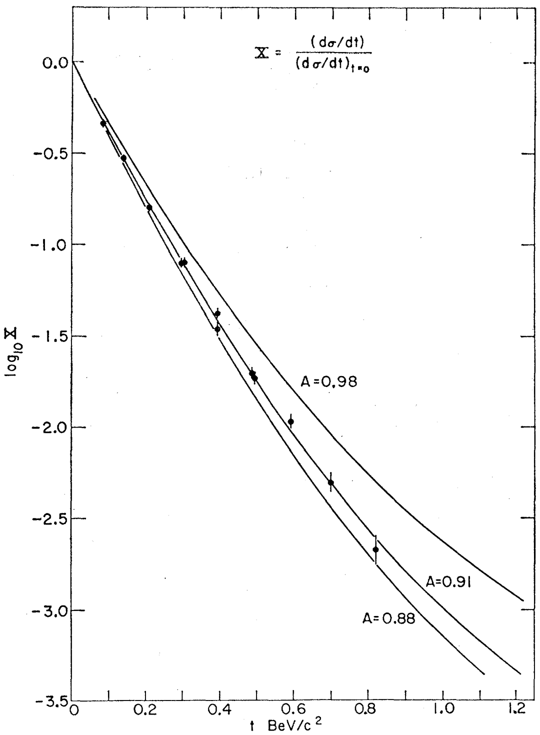

FIG. 1. $\pi^- p$ elastic scattering at 17.0 BeV/c compared to the predictions of the Van Hove model. The experimental points from Ref. 1.

As we saw in Sec. 2, for very high energies the $h(b)$ become partial-wave amplitudes, for which we can add the restriction $0 \leq h_I$.

We can now use Eq. (3.10) to obtain, by iteration, the first correction to this asymptotic form, still preserving the pure imaginary nature of h

$$h = i\{h_I^0 + h_I^1/4k^2 + O(k^{-4})\},$$
$$h_I^1 = \frac{1}{1-2h_I^0}\frac{d}{db^2}\left[b^2\left(\frac{dh_I^0}{db}\right)^2\right]. \tag{4.3}$$

Thus even if $\rho(b)$ is independent of k the unitarity condition requires a k dependence of this form.

Also, from Eq. (2.11), the elastic cross section is

$$\sigma_{el} = 2\pi \int_0^\infty \{1-[1-\rho(b)]^{1/2}\}^2 b\,db - \frac{\pi}{32k^2}\int_0^\infty \frac{b^2[d\rho/db]^3}{[1-\rho(b)]^{5/2}}db + O(1/k^4). \tag{4.4}$$

In all models in which $\rho(b)$ is a monotonically decreasing function of b, the $(1/k^2)$ contribution to the elastic cross section is a positive definite quantity. For example, this implies that if $\rho(b)$ is energy-independent, then the elastic differential cross section for small t will tend from above to its asymptotic limit as $k^2 \to \infty$.

Van Hove[4] has suggested that asymptotically elastic scattering amplitudes should become pure imaginary with a Gaussian form for the inelastic overlap function $g(x)$ [Eq. (3.2)]

$$g(x) = e^{-x^2/2\mu^2},$$

μ being a parameter with the dimensions of a mass. He obtains this form from an analysis of $g(x)$ under the assumption that the multiplicity of secondaries in the inelastic collisions is high and that they are only weakly correlated in momentum distribution. The Hankel transform of $g(x)$ gives for $\rho(b)$ [Eq. (2.3)]

$$\rho(b) = A e^{-b^2\mu^2/2}.$$

A and μ^2 can be determined from the elastic cross section σ_{el} and the total cross section σ_T. To first order in $1/k^2$

$$\frac{\sigma_{el}}{\sigma_T} = 1 - \frac{A}{4}\left\{[1-(1-A)^{1/2}] - \ln\left(\frac{2[1-(1-A)^{1/2}]}{A}\right)\right\}^{-1},$$
$$[1-(\sigma_{el}/\sigma_T)]\sigma_T = 2\pi A/\mu^2. \tag{4.5}$$

The total cross section can be used to determine the scale factor μ^2 and the ratio of the elastic to total cross sections to determine the parameter A. The parameter A is a measure of the amount of absorption at small impact parameters b^2 (or equivalently, low partial waves). The maximum value of A corresponding to total absorption is $A = 1.0$. This value of A gives the maximum elastic to total ratio

$$(\sigma_{el}/\sigma_T)_{max} = 0.1845. \tag{4.6}$$

We have used the Van Hove model to try to fit the data for πp elastic scattering at high energies. Above 7.0 BeV the $\pi^\pm p$ total and differential cross sections are approximately equal and energy independent[1,9,10] [at least in the experimentally observed region $t<1$ (BeV/c)^2], and we take this as evidence that the small momentum transfer πp scattering has almost reached its asymptotic forms, so that it is reasonable to expect the real part of the scattering amplitude to be small.[4] Furthermore, the experimental forward elastic cross sections seem consistent, within experimental errors, with the optical theorem predictions. We discuss in detail the $\pi^- p$ elastic scattering at 17.0 BeV/c, the highest momentum with a well-measured differential cross section. At this momentum, the agreement between the optical theorem and the forward cross section seems particularly good.

The total cross section is taken to be 25.6 mb,[9,10] and the elastic to total ratio as 0.160±0.013.[9] This gives upper and lower estimates of 0.98 and 0.88 for the parameter A with μ^2 ranging from 0.113 to 0.0976 (BeV/c)^2. Corresponding to the variation of A the predicted dif-

[9] S. J. Lindenbaum, W. A. Love, J. A. Niederer, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters 7, 352 (1961).
[10] G. von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters 8, 173 (1962).

ferential cross sections lie on one of a family of curves. In Fig. 1 we show the curves for the upper and lower values of A and for the intermediate case $A=0.91$ which seems to give extremely good agreement with the data which are also shown. This gives a theoretical "measurement" of the elastic to total ratio of 0.155. Owing to possible errors in the total cross section, all of the curves might be shifted by a corresponding percentage change in the t scale.

We found that the $1/k^2$ correction term to the asymptotic form of the unitarity condition gave a negligible ($<1\%$) contribution to both the elastic to total cross section ratio and the differential cross section for all values of t. We believe that this justifies the neglect of all correction terms at these high energies.

The model considered above makes quite specific qualitative predictions about the behavior at large momentum transfer. To investigate this, it is instructive to consider the general case, where $h(b^2)$ is arbitrary, and write $T(x)$ in a somewhat different form. Suppose that $h(b^2)$ can be represented as a Laplace transform

$$h(b^2)=\int_0^\infty e^{-S^2b^2/2}L(S^2)dS^2.$$

Then substituting in (2.2) and carrying out the b integration, we obtain

$$T(x)=2kW\int_0^\infty \frac{L(S^2)}{S^2}e^{-x^2/2S^2}dS^2 \tag{4.7}$$

suggesting the general exponential behavior of the cross sections. For the pure Van Hove model considered above, we have

$$\begin{aligned} h_I(b)&=\tfrac{1}{2}[1-(1-Ae^{-b^2\mu^2/2})^{1/2}] \\ &=\sum_{n>1} c_nA^ne^{-nb^2\mu^2/2}, \end{aligned} \tag{4.8}$$

$$c_n=2(2n-3)!/[2^{2n}n!(n-2)!],\quad n\geq 2$$

so that in this case $L(S^2)$ has the form

$$L(S^2)=\sum_{n=1}^{\infty} c_nA_n\delta(S^2-n). \tag{4.9}$$

For large x the main contribution to the integral (4.7) must come from large S^2, i.e., from large values of n. Under these circumstances we can replace the summation over n by an integration, using the fact that $c_n\simeq 1/(4\pi^{1/2}n^{3/2})$ for large n:

$$\begin{aligned} T_I(x)&\cong\frac{2kW}{4\pi^{1/2}\mu^2}\int_0^\infty \frac{A^n}{n^{5/2}}e^{-(x^2/2n)\mu^2}dn \\ &=\frac{kW}{\sqrt{2}x\mu}\exp\left(-\frac{x}{\mu}\ln(1/A^2)\right)\left(\frac{[\ln(1/A^2)]^{1/2}}{x/\mu}+\frac{1}{x^2/\mu^2}\right). \end{aligned} \tag{4.10}$$

From this it is clear that as $A\to 1$, $T_I(x)$ becomes extremely sensitive to A. This is because the functional form of $h(b^2)$ for small impact parameters is very sensitive to the value of A near to the singularity in the square root function.

For pion scattering energies between 7.0 and 16.0 BeV the forward differential cross section is consistently somewhat larger than the optical theorem point.[9] This could be due to the experimental uncertainties in the absolute normalization of the measured diffraction curves and the uncertainties of the total elastic cross section. If we take the total cross section to be in fact close to the upper limit as given by experiment, and the normalization of the elastic diffraction to be close to the lower limit of its experimental value, then the data are consistent with a Van Hove model with no real part and $A\simeq 0.91$.

If the discrepancy with the optical-model prediction is genuine, then the data require the addition either of a real part or a spin dependence to the scattering amplitude.

5. PROTON-PROTON SCATTERING

The experimental data on proton-proton scattering in the energy range below 20 BeV cannot be explained by the Van Hove model in which the scattering amplitude has no real part. One reason for this is that the ratio of the elastic to total cross sections is large, equal to 0.244 ± 0.012 at 19.6 BeV/c, much larger than the maximum allowed ratio as given by Eq. (37). Serber[11] has shown that it is possible to construct a model for the p-p scattering in which the scattering amplitude is pure imaginary and consistent with the experimental data at the highest measured energies. In the notation of this reference

$$2h_I(b^2)=1-e^{-2\chi(\rho)},$$
$$b^2=\rho$$

$h_I(b^2)$ is not of the Van Hove form. The potential model of Serber can be interpreted as shadow scattering due to inelastic channels in which the overlap function $g(x)$ is not a Gaussian function.

We wish to consider here an alternative model in which we preserve the Gaussian overlap function, and obtain the departures from the behavior of the Van Hove model by the assumption that the real parts of the amplitudes are still significant. We assume that the spin dependence of the scattering amplitudes is negligible. Van Hove has shown,[4] using dispersion relations, that if at asymptotically large energies the p-p and $\bar{p}$-p cross sections tend to the same limits, then the real part of the scattering amplitude will tend asymptotically to zero. That the energy region below 20.0 BeV/c is not in the asymptotic region is indicated by the large difference between both the elastic p-p and $\bar{p}$-p differen-

[11] R. Serber, Rev. Mod. Phys. 36, 649 (1964).

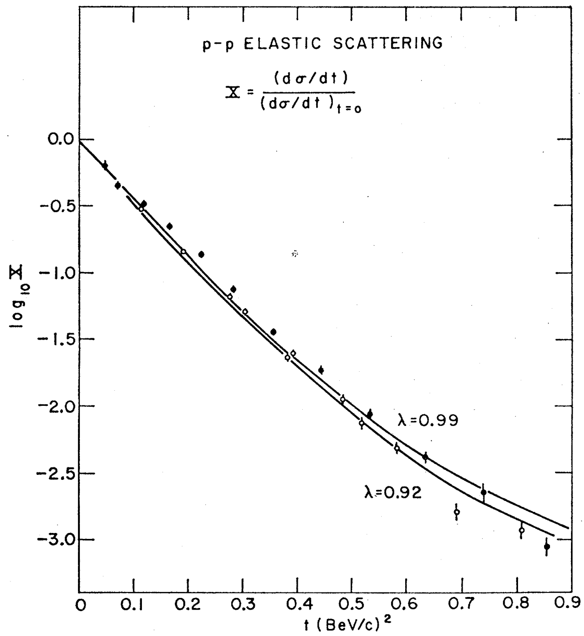

FIG. 2. Small-angle elastic pp scattering at 12.8 and 19.6 BeV/c. The experimental points are taken from Ref. 1. The theoretical curves are from the model discussed in the text. For $\lambda=0.99$ $\sigma_{\rm el}/\sigma_t=0.258$ (from experiment $\sigma_{\rm el}/\tau_t=0.276\pm0.008$ at 12.8 BeV/c) and the optical model discrepancy is 23%. For $\lambda=0.92$ $\sigma_{\rm el}/\sigma_t=0.238$ (from experiment $\sigma_{\rm el}/\sigma_t=0.244\pm0.12$ at 19.6 BeV/c), the optical-model discrepancy is 21%.

tial cross sections and the large difference ($\approx 20\%$) in their total cross sections.[12] While significant differences in the p-p and $\bar{p}$-p cross sections persist, significant real parts of the scattering amplitude can be expected. Support for this claim can be found in a paper by Söding.[13] He estimates the real parts of the scattering amplitudes at $t=0$ using the p-p and $\bar{p}$-p forward dispersion relations. The real parts in this calculation are repulsive and large in the energy range that we consider.

As a specific model we have considered the possibility that the inelastic overlap function $g(x)$ is of the Van Hove form but that superimposed upon this at small impact parameters is a repulsive $h_R(b^2)$. The reason for expecting the real part to be of short range is that if it is associated with the difference between the p-p and $\bar{p}$-p cross sections, and if this difference is due to the possibility of annihilation reactions in $\bar{p}$-p collisions, then this difference will occur at small impact parameters.

We wish to construct a model in which $h_R(b^2)$ is of short range and where, as in the πp case, there is much inelastic absorption at small impact parameters ($A\leq 1.0$). In order to ensure that our model will give large values for the ratio of the elastic to total cross sections, and also large ($\simeq 20\%$) optical model discrepancies at $t=0$[1], we must make our function more square than a simple Gaussian function. We have investigated models in which $g(b^2)=Ae^{-b^2/2}$ and $h_R(b^2)$ is the difference of two Gaussian functions:

$$h_R(b^2)=-\tfrac{1}{2}\lambda e^{-sb^2/2}\{[\alpha+(1-A)^{1/2}]-\alpha e^{-sb^2/2}\}. \quad (5.1)$$

The unitarity condition is now, for large energies,

$$h_I{}^0(b^2)=\tfrac{1}{2}\{1-[1-\rho(b^2)-4h_R{}^2(b^2)]^{1/2}\}; \quad (5.2)$$

unitarity requires $|\lambda|\leq 1.0$.

We have not made a systematic search for the best fits to the experimental data with such a model. However, we find that many of the features of the data between 12 and 20 BeV/c can be reproduced with values of the parameters

$$\mu^2=0.0662\ \text{BeV}^2,\quad A=0.80,$$
$$\alpha=0.45,\quad S=2.25,$$

with $\lambda=0.99$ at 12.0 BeV/c scattering momentum decreasing to $\lambda=0.92$ at 20 BeV/c (see Figs. 2 and 3). As in the πp case we found that the $1/k^2$ correction to the asymptotic form was negligible for small momentum transfers (Fig. 2). At the largest momentum transfers shown (Fig. 3) the correction term was small but not negligible. It reached a maximum value of $\sim\frac{1}{2}$ of the leading term for $\lambda=0.92$ at the maximum $|t|$ value. The correction term was negligible almost everywhere and dominant nowhere. We believe that except for $|t| \simeq |t_{\max}|$ we are justified in neglecting all other correction terms for the model we have presented. It should be noted that this model does not give the expected zero slope of $d\sigma/dt$ at $|t|=|t_{\max}|$. This is because we have not imposed any symmetry condition on $T(x)$.

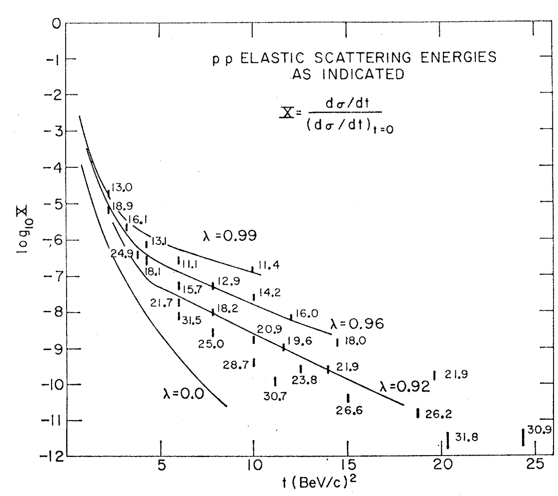

FIG. 3. Large-momentum-transfer pp scattering. The experimental points are taken from Ref. 2. The theoretical curves come from the same model as those in Fig. 2. The end points of the curves are maximum t values for momenta 12.0, 16.0, and 20.0 BeV/c, respectively. The $1/k^2$ corrections also were computed at these energies.

[12] S. J. Lindenbaum, W. A. Love, J. A. Niederer, S. Ozaki, *et al.*, Phys. Rev. Letters **7**, 185 (1961); K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, *et al.*, *ibid.* **11**, 503 (1963).

[13] P. Söding, Phys. Letters **8**, 285 (1964).

However, symmetrizing corrections should again be important only in the immediate neighborhood of $t=t_{\max}$. As can be seen from the curves, many of the features of p-p scattering in this energy range can be explained by the presence of a slowly decreasing short-range real part. Such a real part gives rise to a slowly decreasing forward diffraction peak. (However, the shrinkage is only one-half that observed experimentally.) Also, the very large energy variation of the cross section at large t is reproduced. The reason that a 10% decrease of λ can induce a change of an order of magnitude in the cross section for large t is the sensitivity of the amplitude at large t to the functional form of $h_I(b^2)$ for small b^2. For values of $\lambda\simeq 1.0$ when the right-hand side of the unitarity condition Eq. (5.2) is almost saturated, $h_I(b^2)$ as a function of b^2 has a square-root branch point near $b^2=0$. This qualitative behavior was already discussed for the pure Van Hove model ($\rho=Ae^{-b^2/2}$, $h_R=0$) and showed up as the occurrence of $[\ln(1/A^2)]^{1/2}$ in the exponent in Eq. (4.10). The value $\lambda=1.0$ in this case corresponds to the singularity for $A=1.0$ in the latter case.

We cannot, of course, argue that this effect of the real part must be the explanation of the observed cross section behavior. For example a real part linear in b near $b=0$ combined with a Gaussian $\rho(b)$ could give a dominantly real $T(t)$ for large values of t. Alternatively there might be no real part, and the observed tail could come from a varying imaginary part of the same form, as (5.2) corresponding to Serber's purely absorptive fit mentioned above. However, the attractive feature of the present model is that it allows us to preserve the simplicity of the Van Hove model and use a single slowly varying addition to describe simultaneously the departures from it.

6. CONCLUSIONS

We have found that the Van Hove model, modified where necessary by a suitable real part, seems capable of describing the behavior of high-energy elastic scattering experiments. Much of the qualitative agreement is independent of the exact details of the model. For π-p scattering, the main qualitative features of the predictions are that the cross sections fall off exponentially in $|t|$ for small $|t|$ and then flatten off until they are exponentials in $(|t|)^{1/2}$. The source of this behavior lies in the unitarity condition. In the impact parameter representation, we have

$$h_I(b)=\tfrac{1}{2}\{1-[1-\rho(b)]^{1/2}\}. \tag{6.1}$$

If $\rho(b)$ falls rapidly as a function of b, then except at very small b, $h_I\approx\frac{1}{4}\rho(b)$. For small values of $|t|\,(=x^2)$, the amplitude $T(x)$ has significant contributions from a wide range of values of b, so $T(x)$ has the same form as the transform of $\rho(b)$; i.e., its shape is that of $g(x)$. For large $|t|$, however, the oscillations of the Bessel function suppress all but the contributions from very small b which accounts for the very small magnitude of the amplitude at large momentum transfers. The shape is controlled almost entirely by the behavior of the square root.

The same result can be seen directly in the momentum transfer expression for the unitarity condition, Eq. (3.2). In the second (elastic unitarity) term the integral runs over momentum transfers x', x'' which can combine to give x. For small x, $g(x)$ is large and dominates the iteration solution which has contributions from all x. For large x, $g(x)$ falls off rapidly, the integral will dominate since it will have appreciable contributions from $x', x''\ll x$. Thus the quadratic nature of the elastic unitarity integral has the effect of continually broadening the tail of $T(x)$ beyond the extrapolation from lower momentum transfers.

The observed agreement of the Van Hove model with the πp data is evidence that $g(x)$ is Gaussian up to moderate momentum transfers, and that the predicted flattening of the cross section should continue as larger momentum transfers are observed.

The empirical form we have used for the pp data is mainly evidence of the qualitative behavior of $h(b)$ in this case. While the idea that the deviation from the Van Hove model is due to a slowly decreasing real part is attractive, we cannot exclude the possibility that in fact it is mainly $g(x)$ which is changing with energy. The test lies in finding whether the shrinkage will ease when eventually the $\bar{p}p$ and pp cross sections merge, and whether the elastic to total ratio falls to the expected region of about 0.18.

Note added in proof: Since this paper was written, experimental evidence has been presented[13a] indicating a forward real amplitude, about one-quarter to one-third the size of the forward imaginary amplitude, in both πp and pp scattering. This would fit in quite well with the conclusions above, the expectation being that in πp scattering the real part is of long range, while for pp scattering it is the short-range nature that leads to the effects discussed in Sec. 5.

APPENDIX: CALCULATION OF k^{-2} CORRECTIONS TO THE UNITARITY EQUATION

We wish to expand Eq. (3.7) in inverse powers of k^2. We have

$$\operatorname{Im}T(x)=\frac{Wk}{4\pi}\sigma_{\text{in}}g(x)+\int_0^{\Delta}T(x')x'dx'\int_0^{\infty}h^{\dagger}(b)J_0(bx(1-x'^2/4k^2)^{1/2})\times J_0(bx'(1-x^2/4k^2)^{1/2})bdb. \tag{A1}$$

[13a] S. J. Lindenbaum, Rapporteurs report at the 12th International Conference on High Energy Physics at Dubna, 1964 (unpublished).

We can expand the Bessel functions in the integrand of (A.1) using the result[14]

$$J_0(\lambda z)=\sum_{n=0}[\tfrac{1}{2}z(1-\lambda^2)]^n\frac{J_n(z)}{n!},\quad \text{real } \lambda>0. \quad \text{(A2)}$$

We then obtain, assuming that the integrals converge [if $h(b)$ is associated with the spatial distribution of a locally confined scattering center, then $h(b)$ will decrease exponentially at large b and the integrals will converge], that

$$\mathrm{Im}T(x)=\frac{Wk}{4\pi}\sigma_{\mathrm{in}}g(x)+\int_0^\infty h^\dagger(b)J_0(bx)bdb\int_0^\Lambda T(x')J_0(bx')x'dx'$$
$$+\frac{1}{8k^2}\int_0^\Lambda T(x')x'dx'\int_0^\infty\{xJ_1(bx)x'^2J_0(bx')+x'J_1(bx')x^2J_0(bx)\}h^\dagger(b)b^2db+O(1/k^4). \quad \text{(A3)}$$

Using the formulas

$$xJ_1(bx)=-(d/db)J_0(bx),\qquad \text{(A4)}$$
$$xJ_0(bx)=(1/b)(d/db)bJ_1(bx),$$

and Eq. (2.1), the x' integration can be performed, to give

$$\frac{\mathrm{Im}T(x)}{2kW}=\frac{\sigma_{\mathrm{in}}g(x)}{8\pi}+\int_0^\infty h(b)h^\dagger(b)J_0(bx)bdb$$
$$-\frac{1}{8k^2}\int_0^\infty h^\dagger(b)h'(b)b^2x^2J_0(bx)db$$
$$-\frac{1}{8k^2}\int_0^\infty h^\dagger(b)[h'(b)+bh''(b)]xJ_1(bx)bdb \quad \text{(A5)}$$
$$+O(1/k^4).$$

$$h'(b)=dh(b)/db,$$
$$h''(b)=d^2h/db^2.$$

The last two integrals in (A5) can be reduced by partial integration using (A4) leading to

$$\frac{\mathrm{Im}T(x)}{2kW}=\frac{\sigma_{\mathrm{in}}g(x)}{8\pi}+\int_0^\infty\left\{h(b)h^\dagger(b)+\frac{1}{8k^2}\frac{1}{b}\frac{d}{db}[b^2h'(b)h'^\dagger(b)]\right\}\times J_0(bx)bdb+O(1/k^4). \quad \text{(A6)}$$

[15] *Bateman Manuscript Project, Higher Transcendental Functions*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, Eq. 7.10.1(16).

The end-point contributions to (A6) from the partial integration vanish, provided that the integrals converge.

Equation (A6) is valid in the region $0<x<\Lambda$. We can now deduce that the unitarity equation implies that

$$\mathrm{Im}h(b)=\frac{\rho(b)}{4}+|h(b)|^2+\frac{1}{4k^2}\frac{d}{db^2}\left(b^2\left|\frac{dh}{db}\right|^2\right)+O(1/k^4)+c(b), \quad \text{(A7)}$$

where $c(b)$ is a function such that its Hankel transform vanishes in the region $0<x<\Lambda$. $c(b)$ is determined by the requirement that $T(x)$ vanish for $x>\Lambda$, which is not generally a property of any solution of (A7). In dealing with corrections of order k^{-2}, we proceed by iteration: accordingly, if $h_0(b)$ is a solution of (A7) to zero order in k^{-2} and with $c(b)=0$, the first approximation to $c(b)$ is

$$c(b)=\frac{1}{2kW}\int_\Lambda^\infty \mathrm{Im}T_0(x)J_0(bx)xdx,$$
$$T_0(x)=2kW\int_0^\infty h_0(b)J_0(bx)bdb. \quad \text{(A8)}$$

As in the discussion of the transient oscillating terms in Sec. 2 in considering high-energy elementary particle elastic scatterings, any reasonable first approximation will lead to terms $c(b)$ which are exponentially small and can therefore be neglected if we work to order k^{-2}.