TABLE II. Summary of the transition probabilities determined in this work, and comparison with the predictions of the single-particle model (Ref. 16).

Nucleus	E_γ (MeV)	Transition	Transition probability (sec^{-1})	$B(E1)_d$ ($10^{-29}e^2$ cm^2)	$B(E1)_d / B(E1)_{sp}$
Sm^{148}	1.46	$1^- \to 0^+$	$(4.3\pm1.2)\times10^{12}$	0.9	5×10^{-4}
	0.91	$1^- \to 2^+$	$(2.7\pm0.8)\times10^{12}$	2.3	12×10^{-4}
Sm^{152}	0.96	$1^- \to 0^+$	$(11\pm1)\times10^{12}$	7.9	4×10^{-3}
	0.84	$1^- \to 2^+$	$(14\pm2)\times10^{12}$	15.4	8×10^{-3}

The upper limit established recently[17] for the $B(E1)$ of the 1.46-MeV level in Sm^{148} by a Coulomb excitation experiment with 43.5-MeV oxygen ions, $B(E1)_d \leq 3\times10^{-29}e^2$ cm^2, is consistent with our value for the reduced transition probability. As far as the lifetime of the 0.96-MeV state of Sm^{152} is concerned, our value falls into the range established by previous experiments.[2,9,18]

It is evident from Table II that the $B(E1)$ values change rather abruptly as one proceeds from the spherical nucleus Sm^{148} to the deformed Sm^{152} nucleus. Further measurements of transition probabilities of 1^- states in the region of the deformed rare-earth nuclei as well as below neutron number 90 will be necessary in order to establish whether this change in the $B(E1)$'s is accidental or whether it indicates a definite trend.

It might be worth pointing out that the ratio of the $B(E1)$'s for the corresponding transitions in Sm^{148} and Sm^{152} is approximately the same as the ratio of the $B(E2)$'s for the first 2^+ states. In addition, the excitation energies of the 1^- and the 2^+ states change by approximately the same amounts as the neutron number changes,[19] while the excitation energies of the 3^- states remain practically constant. Since the energies of the 2^+ states are much lower to start with, the fractional changes upon crossing neutron number 90 are much larger for the 2^+ states than for the 1^- states.[1] The relationships mentioned above may be of interest in view of the suggestion that some of the 1^- levels in even-even nuclei arise from the coupling of a quadrupole and an octupole collective excitation.[20]

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Alpha Clusters in a Harmonic-Oscillator Potential

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The probabilities of occurrence of alpha clusters are calculated for the case of four nucleons in a harmonic-oscillator potential, based on the simplifying approximation that the oscillator constants for alpha clusters and nucleons are equal. Some general observations as to how alpha-decay hindrance is affected by the overlap of wave function of a cluster and that of the constituent nucleons are made. The decay of Po^{211} is discussed in part.

I. INTRODUCTION

In the shell model the hindrance of alpha decay can be attributed to a number of factors. In this paper we shall study one of these, namely, the overlap of wave function of an alpha cluster and that of the constituent nucleons. Other factors include centrifugal barrier, configuration mixing, coefficient of fractional parentage, etc.

Consider the motion of two protons and two neutrons in a harmonic-oscillator potential. Their wave function can be written as a linear combination of wave functions, corresponding to various groupings of the nucleons, such as an alpha cluster; it has the form

$$\Psi = \sum_i a_i \Psi_i , \tag{1}$$

where Ψ_i are wave functions for the various groupings, and $|a_i|^2$ is the probability of occurrence of grouping i. We shall calculate the coefficients a_i for the alpha groupings.

II. CALCULATIONS

The Hamiltonian for four nucleons in a harmonic-oscillator potential is

$$H = \frac{1}{2m}\sum_{i=1}^{4} p_i^2 + \tfrac{1}{2}m\omega^2 \sum_{i=1}^{4} r_i^2 , \tag{2}$$

where $\mathbf{r}_i$ and $\mathbf{p}_i$ are the coordinates and momenta of the nucleons and m the nucleon mass.

The quantum numbers for orbital angular momentum and total angular momentum of nucleon i shall be denoted by l_i and j_i, respectively, and

$$\Psi(j_1j_2(J_{12})j_3j_4(J_{34})JM)$$

denotes a (unperturbed) normalized wave function in which j_1 and j_2 are coupled to J_{12}, j_3 and j_4 to J_{34}, and then J_{12} and J_{34} to J.

A. Two Equivalent Protons and Two Equivalent Neutrons

Let $l_1=l_2=l$, $l_3=l_4=l'$, $j_1=j_2=j$, and $j_3=j_4=j'$. A normalized wave function for two equivalent protons and two equivalent neutrons is

$$\Psi(j^2(J_{12})j'^2(J_{34})JM) = \sum_{M_{12},M_{34}} (J_{12}M_{12}J_{34}M_{34}|J_{12}J_{34}JM)\left[\sum_{S_{12},L_{12}} (2j+1)[(2S_{12}+1)(2L_{12}+1)]^{1/2}\begin{Bmatrix}\frac{1}{2} & l & j\\ \frac{1}{2} & l & j\\ S_{12} & L_{12} & J_{12}\end{Bmatrix} R_{nl}(r_1)R_{nl}(r_2)\right.$$
$$\left.\times\sum_{\mu_{12},\lambda_{12}}(S_{12}\mu_{12}L_{12}\lambda_{12}|S_{12}L_{12}J_{12}M_{12})\mathcal{Y}_{L_{12}}{}^{\lambda_{12}}\chi_{S_{12}}{}^{\mu_{12}}\right]\left[\sum_{S_{34},L_{34}}(2j'+1)[(2S_{34}+1)(2L_{34}+1)]^{1/2}\begin{Bmatrix}\frac{1}{2} & l' & j'\\ \frac{1}{2} & l' & j'\\ S_{34} & L_{34} & J_{34}\end{Bmatrix}\right.$$
$$\left.\times R_{n'l'}(r_3)R_{n'l'}(r_4)\sum_{\mu_{34},\lambda_{34}}(S_{34}\mu_{34}L_{34}\lambda_{34}|S_{34}L_{34}J_{34}M_{34})\mathcal{Y}_{L_{34}}{}^{\lambda_{34}}\chi_{S_{34}}{}^{\mu_{34}}\right]. \quad (3)$$

Here $(J_{12}M_{12}J_{34}M_{34}|J_{12}J_{34}JM)$ is a Clebsch-Gordan coefficient; $\begin{Bmatrix}\frac{1}{2} & l & j\\ \frac{1}{2} & l & j\\ S_{12} & L_{12} & J_{12}\end{Bmatrix}$ a $9-j$ symbol[1]; $R_{nl}(r_1)$ the radial part of a harmonic-oscillator wave function;

$$\mathcal{Y}_{L_{ij}}{}^{\lambda_{ij}}=\sum_{m,m'}(lmlm'|llL_{ij}\lambda_{ij})Y_l{}^m(\theta_i\varphi_i)Y_l{}^{m'}(\theta_j\varphi_j), \quad (4)$$

$$\chi_{S_{ij}}{}^{\mu_{ij}}=\sum_{m_s,m_s'}(\tfrac{1}{2}m_s\tfrac{1}{2}m_s'|\tfrac{1}{2}\tfrac{1}{2}S_{ij}\mu_{ij})\chi_{m_s}(i)\chi_{m_s'}(j), \quad (5)$$

where $Y_l{}^m(\theta_i\varphi_i)$ is a spherical harmonic for the ith nucleon, $\chi_{m_s}(i)$ a spin wave function for the ith nucleon, etc. J_{12} and J_{34} satisfy $0\leqslant J_{12}\leqslant 2j$ and $0\leqslant J_{34}\leqslant 2j'$, respectively, and take even values. And J satisfies $|J_{12}-J_{34}|\leqslant J\leqslant J_{12}+J_{34}$.

For simplification we make the approximation that the oscillator constants for alpha clusters and nucleons are equal. This approximation is perhaps good enough for calculations using harmonic-oscillator potentials, because harmonic oscillator wave functions are not very sensitive to small variations in the oscillator constants.[2–4] Then a normalized wave function for an alpha grouping is

$$\Psi_\alpha(LM_L)=\psi(\mathbf{R})\varphi(\xi,\eta,\zeta)\chi_0{}^0(1,2)\chi_0{}^0(3,4), \quad (6)$$

where

$$\psi(\mathbf{R})=R_{NL}(R)Y_L{}^{M_L}(\mathbf{R}/R), \quad (7)$$

$$\varphi(\xi,\eta,\zeta)=(m\omega/\pi\hbar)^{9/4}\exp[-(m\omega/2\hbar)(\xi^2+\eta^2+\zeta^2)], \quad (8)$$

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4),\quad \xi=(\mathbf{r}_1-\mathbf{r}_2)/\sqrt{2},$$
$$\eta=(\mathbf{r}_3-\mathbf{r}_4)/\sqrt{2},\quad \zeta=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2-r_3-\mathbf{r}_4), \quad (9)$$

and $\chi_0{}^0(i,j)$ is the singlet spin wave function for nucleons i and j.

The coefficient for an alpha grouping in the state $\Psi((j^2(J_{12})j'^2(J_{34})JM)$ is given by

$$a=\sum_{\text{spin coordinates}}\int\Psi_\alpha{}^*(LM_L)\Psi(j^2(J_{12})j'^2(J_{34})JM)\times d\xi d\eta d\zeta d\mathbf{R}, \quad (10)$$

where the integration over R is from some positive value R, less than the nuclear radius to ∞, so as to account for the possibility that "clustering" occurs mainly in the nuclear surface. By means of the trans-

[1] H. Matsunobu, Progr. Theoret. Phys. (Kyoto) **14**, 589 (1955).

[2] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), p. 41.

[3] H. D. Zeh and H. J. Mang, Nucl. Phys. **29**, 529 (1962). Their Table 3 shows that the relative transition probabilities of Po^{211} are not very sensitive to small variations in the oscillator constants.

[4] K. Harada, Progr. Theoret. Phys. (Kyoto) **26**, 667 (1961). Harada has used two different oscillator constants for alpha clusters and nucleons, and consequently a summation over N appears in his Eq. (10). As a result of our approximation the evaluation of the overlap integrals, such as the one in our Eq. (10), does not involve such a summation.

formation brackets[5,6] it can be shown that

$$a=(2j+1)(2J_{12}+1)^{1/2}\begin{Bmatrix}\frac{1}{2} & l & j\\ \frac{1}{2} & l & j\\ 0 & J_{12} & J_{12}\end{Bmatrix}(2j'+1)(2J_{34}+1)^{1/2}\begin{Bmatrix}\frac{1}{2} & l' & j'\\ \frac{1}{2} & l' & j'\\ 0 & J_{34} & J_{34}\end{Bmatrix}$$
$$\times\langle 00,N_{12}J_{12},J_{12}|nl,nl,J_{12}\rangle\langle 00,N_{34}J_{34},J_{34}|n'l',n'l',J_{34}\rangle$$
$$\times\langle 00,NJ,J|N_{12}J_{12},N_{34}J_{34},J\rangle\int_{R_0}^{\infty}[R_{NJ}(R)]^2R^2dR\delta(J,L)\delta(M,M_L). \quad (11)$$

The coefficient vanishes unless the following conditions are satisfied:

$$\begin{aligned}4n+2l&=2N_{12}+J_{12},\\ 4n'+2l'&=2N_{34}+J_{34},\\ 2N_{12}+2N_{34}+J_{12}+J_{34}&=2N+J.\end{aligned} \quad (12)$$

B. Two Equivalent Neutrons (Protons) and Two Nonequivalent Protons (Neutrons)

Let $l_3=l_4=l'$ and $j_3=j_4=j'$. The coefficient for an alpha grouping in the state $\Psi(j_1j_2(J_{12})j'^2(J_{34})JM)$ is

$$a=\frac{1}{\sqrt{2}}[1+(-1)^{l_1+l_2-J_{12}}][(2j_1+1)(2j_2+1)(2J_{12}+1)]^{1/2}\begin{Bmatrix}\frac{1}{2} & l_1 & j_1\\ \frac{1}{2} & l_2 & j_2\\ 0 & J_{12} & J_{12}\end{Bmatrix}(2j'+1)(2J_{34}+1)^{1/2}\begin{Bmatrix}\frac{1}{2} & l' & j'\\ \frac{1}{2} & l' & j'\\ 0 & J_{34} & J_{34}\end{Bmatrix}$$
$$\times\langle 00,N_{12}J_{12},J_{12}|n_1l_1,n_2l_2,J_{12}\rangle\langle 00,N_{34}J_{34},J_{34}|n'l',n'l',J_{34}\rangle$$
$$\times\langle 00,NJ,J|N_{12}J_{12},N_{34}J_{34},J\rangle\int_{R_0}^{\infty}[R_{NJ}(R)]^2R^2dR\delta(J,L)\delta(M,M_L), \quad (13)$$

with the conditions

$$\begin{aligned}2n_1+l_1+2n_2+l_2&=2N_{12}+J_{12},\\ 4n'+2l'&=2N_{34}+J_{34},\\ 2N_{12}+2N_{34}+J_{12}+J_{34}&=2N+J;\end{aligned} \quad (14)$$

otherwise the coefficient vanishes. J_{12} and J_{34} satisfy $|j_1-j_2|\leqslant J_{12}\leqslant j_1+j_2$ and $0\leqslant J_{34}\leqslant 2j'$, respectively, J_{34} takes even values, and J satisfies $|J_{12}-J_{34}|\leqslant J\leqslant J_{12}+J_{34}$.

C. Two Nonequivalent Neutrons and Two Nonequivalent Protons

The coefficient for an alpha grouping in the state $\Psi(j_1j_2(J_{12})j_3j_4(J_{34})JM)$ is

$$a=\tfrac{1}{2}[1+(-1)^{l_1+l_2-J_{12}}][1+(-1)^{l_3+l_4-J_{34}}][(2j_1+1)(2j_2+1)(2J_{12}+1)]^{1/2}$$
$$\times\begin{Bmatrix}\frac{1}{2} & l_1 & j_1\\ \frac{1}{2} & l_2 & j_2\\ 0 & J_{12} & J_{12}\end{Bmatrix}[(2j_3+1)(2j_4+1)(2J_{34}+1)]^{1/2}\begin{Bmatrix}\frac{1}{2} & l_3 & j_3\\ \frac{1}{2} & l_4 & j_4\\ 0 & J_{34} & J_{34}\end{Bmatrix}$$
$$\times\langle 00,N_{12}J_{12},J_{12}|n_1l_1,n_2l_2,J_{12}\rangle\langle 00,N_{34}J_{34},J_{34}|n_3l_3,n_4l_4,J_{34}\rangle$$
$$\times\langle 00,NJ,J|N_{12}J_{12},N_{34}J_{34},J\rangle\int_{R_0}^{\infty}[R_{NJ}(R)]^2R^2dR\delta(J,L)\delta(M,M_L), \quad (15)$$

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[5] M. Moshinsky, Nucl. Phys. **13**, 104 (1959).
[6] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets for Nuclear Shell-Model Calculations* (Universidad Nacional Autonoma De Mexico, Mexico, 1960).

TABLE I. Some ratios of probabilities of occurrence of an alpha cluster. J is the angular momentum quantum number of a cluster. The two neutrons (protons) couple to zero angular momentum and are represented by (a,b); the proton (neutron) pair by $(3s_{1/2})^2$, etc. For each value of J the numbers give the ratios among the probabilities of occurrence of which the largest one is taken to be 1.

J	$(a,b)(3s_{1/2})^2$	$(a,b)(2d_{3/2})^2$	$(a,b)(2d_{5/2})^2$	$(a,b)(1g_{7/2})^2$	$(a,b)(1g_{9/2})^2$	$(a,b)(0i_{11/2})^2$	$(a,b)(0i_{13/2})^2$
0	0.75	0.67	1.00	0.18	0.22	0.0062	0.0073
2		0.58	1.00	0.18	0.23	0.0066	0.0076
4			1.00	0.22	0.29	0.0085	0.0098
6				0.63	1.00	0.0300	0.0382

with the conditions

$$\begin{aligned} 2n_1+l_1+2n_2+l_2&=2N_{12}+J_{12}, \\ 2n_3+l_3+2n_4+l_4&=2N_{34}+J_{34}, \\ 2N_{12}+2N_{34}+J_{12}+J_{34}&=2N+J; \end{aligned} \tag{16}$$

otherwise the coefficient vanishes. J_{12}, J_{34} and J satisfy $|j_1-j_2|\leqslant J_{12}\leqslant j_1+j_2$, $|j_3-j_4|\leqslant J_{34}\leqslant j_3+j_4$ and $|J_{12}-J_{34}|\leqslant J\leqslant J_{12}+J_{34}$, respectively.

III. DISCUSSION

Some general observations can now be made.

(1) We expect that in each shell the overlap of wave function of an alpha cluster and that of the constituent nucleons of lower angular momenta is larger. As an illustration, Table I lists some ratios of probabilities of occurrence of alpha clusters formed from two protons (neutrons) coupling to zero angular momentum and a pair of neutrons (protons) in the shell with $2n+l=6$.

Zeh[7] has calculated the alpha reduced widths of even Po isotopes. It is noted that the ratios of the pure shell-model reduced widths for the neutron configurations $(1g_{9/2})^2$ (at neutron number 136), $(0i_{11/2})^2$ (at neutron number 138), and $(0i_{13/2})^2$ (at neutron number 114) are similar to the ratios of probabilities of occurrence of an alpha cluster of $J=0$ for the configurations $(a,b)(1g_{9/2})^2$, $(a,b)(0i_{11/2})^2$, and $(a,b)(0i_{13/2})^2$.

(2) In a nucleus, the factor $\int_{R_0}^{\infty}[R_{NJ}(R)]^2R^2dR$ is larger for nucleons of higher shells. For an alpha cluster formed from nucleons of higher shells has higher energy, and therefore the wave function $R_{NJ}(R)$ is larger in the nuclear surface.

(3) We expect that when the constituent nucleons are the same, the occurrence of alpha clusters of lower angular momenta is more probable. As an illustration, Table II lists, excluding the factor $\int_{R_0}^{\infty}[R_{NJ}(R)]^2R^2dR$, some ratios of probabilities of occurrence of alpha clusters formed from $(1p_{3/2})^2(0f_{7/2})^2$. Including the factor is likely to make the occurrence of alpha clusters of lower angular momenta even more probable, because the wave function $R_{NJ}(R)$ has, for a larger N, more nodes away from the origin.

[7] H. D. Zeh, Z. Physik **175**, 490 (1963).

TABLE II. Some ratios of probabilities of occurrence of an alpha cluster. The constituent nucleons are $(1p_{3/2})^2(0f_{7/2})^2$ with the first pair coupling to zero angular momentum. J and N are the angular momentum quantum number and radial quantum number of the cluster, respectively. The numbers in the third column give the ratios among the probabilities of occurrence of which the one for $J=6$ is taken to be 1.

N	J	Probability of occurrence
6	0	3.40
5	2	1.01
4	4	0.84
3	6	1.00

In the alpha decay of Po^{211},[8,9] the transition to the 569-keV level in Pb^{207} is slower than the transition to the 900-keV level in the latter nucleus. The ratio of the reduced transition probabilities, calculated by taking the lowest permitted alpha angular momentum, for the two transitions is about 16. We have found that the probability of occurrence of an alpha cluster of angular momentum $J=3$ formed from $(1g_{9/2},2p_{3/2})(0h_{9/2})^2$ is about 16.5 times that of an alpha cluster of the same angular momentum formed from $(1g_{9/2},1f_{5/2})(0h_{9/2})^2$. Thus it is possible that, when more exact wave functions are used, the difference in the overlap of wave function of an alpha cluster and that of constituent nucleons accounts for a large part of the difference in hindrance between the two transitions.

Finally, we wish to mention: First, in recent shell-model calculations[10] of alpha transition probabilities there is used a radius parameter, the choice of which is somewhat ambiguous. We have in this paper used overlap integrals, such as the one in Eq. (10), in the hope that our study may point to a way of avoiding the use of the radius parameter. Second, in applications to heavy nuclei the tables of transformation brackets[6] need to be extended to include higher quantum numbers.

[8] I. Perlman and J. O. Rasmussen, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 42, p. 186.

[9] Zeh and Mang (Ref. 3) have recently made detailed calculations on the relative transition probabilities of Po^{211}.

[10] See Refs. 3, 4, 7 and J. O. Rasmussen, Nucl. Phys. 44, 93 (1963).