PHYSICAL REVIEW VOLUME 137, NUMBER 5B 8 MARCH 1965

Problem of Energy in an Expanding Universe

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(Received 10 August 1964)

It is proved that a plausible definition of the total energy of a body in Newtonian cosmology can be given by the condition that the energy is conserved, if the body is participating in the general cosmic expansion. The same formula for the total energy results also from a certain metric describing a relativistic model of a universe with a uniform and isotropic distribution of matter and distinguished by an interesting property: that the differences of space-like coordinates have in it an immediate metric meaning as the lengths measured by rigid rods. It is found that the cosmic expansion has either no effect, or quite imperceptible effects on the motion of planets in our solar system, but during the formation of local systems such as clusters of galaxies, the deviations from the exact validity of the conservation law of energy are of considerable magnitude. This process is to be studied on the basis of McVittie's model.

INTRODUCTION

THE concept of energy appears to be a very useful one in classical Newtonian dynamics as well as in the modern quantum theory of particles and wave fields. Its most important property, the conservation law, is in the pure gravitational field a consequence of the fact that the Newtonian absolute space is static and, because of this, the mean mass-density, taken over the infinite volume of the space, remains constant. The forces acting on a test particle moving in the field of an isolated system of celestial bodies can be therefore deduced from a scalar potential not depending explicitly on time.

The Newtonian absolute space is an idealization of reality. In fact, we are living in an expanding system of galaxies with a nonvanishing mass-density depending explicitly on time (with the exception of the steady-state universe in which the mean mass-density does not vary). In Newtonian cosmology we easily deduce from Newton's law of general gravitation and his second law of motion the dependence of the mean mass-density on time, but if in a simple model with a uniform and isotropic distribution of matter we assume that the force acting on a test particle participating in the general cosmic expansion equals the negative gradient of a scalar potential, we find that the sum of its kinetic and potential energy is not conserved because of the explicit dependence of the scalar potential on the time.

The aim of the present paper is to examine the significance of the concept of energy in an expanding universe. We shall show that also in Newtonian cosmology it is possible to define the Lagrangian and the total energy of a test particle in such a way that the latter is conserved for a certain standard motion and certain reference frames.[1] It is quite natural to choose the general cosmic expansion and the reference frames whose origin remains at rest relative to the expanding system of galaxies (which may be determined by the isotropy of the observed red shift) as the standard form of motion and as the standard reference frames, respectively. We shall see that the same expressions for the Lagrangian and for the total energy of a test particle follow also from a certain metric of relativistic cosmology. Thereafter we shall quantitatively investigate the motion of a test body and the change of its total energy in a very simplified model of a local system of celestial bodies and during its formation.

1. DEFINITION OF ENERGY IN NEWTONIAN COSMOLOGY

In a model filled with a uniformly and isotropically distributed cosmic "dust" the equation of motion of a test particle has in three-dimensional vector notation the form

$$d^2\mathbf{r}/dt^2 = -(4\pi/3)\gamma\rho_0\mathbf{r}\,, \tag{1.1}$$

γ being the Newtonian gravitational constant. The mean mass-density ρ_0 of the cosmic dust stands in a simple relation to Hubble's "constant" $H(t)$ and to the deceleration parameter $q(t)$:

$$(4\pi/3)\gamma\rho_0 = qH^2. \tag{1.2}$$

We have also

$$\dot{H} = -(1+q)H^2. \tag{1.3}$$

The dot indicates here and hereafter differentiation with respect to time.

The equation of motion (1.1) which follows directly from Newton's second law of motion and his law of general gravitation may be deduced also from the Lagrangian

$$\mathfrak{L} = m\{\tfrac{1}{2}\dot{\mathbf{r}}^2 - (2\pi/3)\gamma\rho_0\mathbf{r}^2\}\,, \tag{1.4}$$

where m denotes the mass of the test particle. Since the mean density ρ_0 depends explicitly on time, the total energy Λ of our test particle, defined by the usual formula

$$\Lambda = \textstyle\sum_j(\partial\mathfrak{L}/\partial\dot{q}_j)\dot{q}_j - \mathfrak{L}\,, \tag{1.5}$$

is not conserved, for

$$d\Lambda/dt = -\partial\mathfrak{L}/\partial t = (2\pi/3)\gamma\dot{\rho}_0 m\mathbf{r}^2 \neq 0. \tag{1.6}$$

It is well known that the Lagrangian is not uniquely determined by the equation of motion, for we may add to the former the total time derivative of any function

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[1] Compare in this connection: H. Bondi, in *Recent Developments in General Relativity* (Pergamon Press Ltd., London, 1962), p. 47 ff.

of coordinates and time without changing the latter. Since Λ given by Eq. (1.5) may be certainly interpreted as the total energy of a particle, if its kinetic energy is a quadratic function of velocities and if the potential depends linearly on velocities,[2] let us suppose the Lagrangian in the form

$$\mathfrak{L}=m\{\tfrac{1}{2}\dot{\mathbf{r}}^2-[\varphi(\mathbf{r},t)-\dot{\mathbf{r}}\cdot\mathbf{Q}(\mathbf{r},t)]\}. \qquad (1.7)$$

The scalar potential φ and the vector potential $\mathbf{Q}$ are to be determined by the condition that the total energy Λ, defined by Eq. (1.5), is conserved during a certain standard motion.

From the Lagrangian (1.7) we obtain the equation of motion

$$d^2\mathbf{r}/dt^2=-\operatorname{grad}\varphi-\partial\mathbf{Q}/\partial t+\dot{\mathbf{r}}\times\operatorname{curl}\mathbf{Q}, \qquad (1.8)$$

the expression for the total energy

$$\Lambda=m\{\tfrac{1}{2}\dot{\mathbf{r}}^2+\varphi\}, \qquad (1.9)$$

and its total time derivative

$$d\Lambda/dt=m\{\partial\varphi/\partial t-\dot{\mathbf{r}}\cdot\partial\mathbf{Q}/\partial t\}. \qquad (1.10)$$

The energy in Eq. (1.9) is conserved if

$$\varphi=\varphi_0-\tfrac{1}{2}\dot{\mathbf{r}}^2, \quad \varphi_0=\text{const}. \qquad (1.11)$$

Putting this function into (1.10), we find after integration that

$$\mathbf{Q}=-\dot{\mathbf{r}}+\mathbf{A}(\mathbf{r}). \qquad (1.12)$$

The left-hand side of Eq. (1.8) may be written in the form

$$d^2\mathbf{r}/dt^2=\partial\dot{\mathbf{r}}/\partial t+\tfrac{1}{2}\operatorname{grad}\dot{\mathbf{r}}^2-\dot{\mathbf{r}}\times\operatorname{curl}\dot{\mathbf{r}}. \qquad (1.13)$$

Substituting this relation and Eqs. (1.11) and (1.12) into (1.8), we get

$$\dot{\mathbf{r}}\times\operatorname{curl}\mathbf{A}(\mathbf{r})=0. \qquad (1.14)$$

This equation is satisfied if

$$\mathbf{A}=\operatorname{grad}a(\mathbf{r}), \qquad (1.15)$$

$a(\mathbf{r})$ being an arbitrary scalar field not depending explicitly on time.

In Newtonian cosmology we choose the general cosmic expansion described by Hubble's law

$$\dot{\mathbf{r}}=H\mathbf{r} \qquad (1.16)$$

as the standard motion, during which the numerical value of the total energy equals zero. Because of it

$$\varphi_0=0.$$

Without changing the generality of our computation we may put

$$a(\mathbf{r})=0,$$

for this scalar field appears neither in the equation of motion (1.8) nor in Eqs. (1.9) and (1.10). Both unknown fields φ and $\mathbf{Q}$ are then described by the functions

$$\varphi=-\tfrac{1}{2}H^2\mathbf{r}^2, \quad \mathbf{Q}=-H\mathbf{r}. \qquad (1.17)$$

The equation of motion (1.8), with φ and $\mathbf{Q}$ given by (1.17), reduces with the help of Eqs. (1.3) and (1.2) to (1.1).

We may thus conclude that in Newtonian cosmology energy recovers its fundamental property of being conserved during the general cosmic expansion, which we choose as the standard motion in the sense of Bondi's considerations,[1] only if the potential is composed additively by the scalar potential φ, and by the scalar product of a vector potential $\mathbf{Q}$ and the velocity of the particle.

[2] See, for instance, J. W. Leech, *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1958), Chap. 5.

2. NEWTONIAN COSMOLOGY AS A LIMITING CASE OF RELATIVISTIC COSMOLOGY

Since the field equations of general relativity contain as unknown functions the gravitational potentials, we may expect that a slight arbitrariness in the choice of the Lagrangian will be reduced if we compute it also by the usual limiting process from a certain preferred metric of relativistic cosmology.

In regions sufficiently small compared with the radius of the universe, the Robertson-Walker line element may be written in the form

$$ds^2=-[G(t)/G_0]^2(d\bar{x}^2+d\bar{y}^2+d\bar{z}^2)+c^2dt^2. \qquad (2.1)$$

This metric has some important features: The time-like coordinate is orthogonal to the space-like coordinates. It is identical with the cosmic time. The distance l_1 between the points $(\bar{x}_1,0,0,t_1)$ and $(\bar{x}_0,0,0,t_1)$, measured by a rigid rod, is given by the formula

$$l_1=(\bar{x}_1-\bar{x}_0)[G(t_1)/G_0]. \qquad (2.2)$$

After the transformation of coordinates

$$x_j=\bar{x}_j[G(t)/G_0], \quad (x_j=x,y,z), \qquad (2.3)$$

the metric (2.1) takes the form

$$ds^2=-(dx^2+dy^2+dz^2)+(c^2+2\Phi)dt^2, \qquad (2.4)$$

where in the three-dimensional vector notation

$$\Phi=\varphi(\mathbf{r},t)-\dot{\mathbf{r}}\cdot\mathbf{Q}(\mathbf{r},t),$$
$$\varphi(\mathbf{r},t)=-\tfrac{1}{2}H^2\mathbf{r}^2, \quad \mathbf{Q}(\mathbf{r},t)=-H\mathbf{r}. \qquad (2.5)$$

Here H also indicates Hubble's "constant"

$$H=\dot{G}/G. \qquad (2.6)$$

We emphasize that the metric (2.4) is not diagonal, for $\dot{\mathbf{r}}$ in Eqs. (2.5) does not indicate a velocity, but it is a symbol for the derivative with respect to t:

$$\dot{\mathbf{r}}=d\mathbf{r}/dt.$$

In contradistinction to the metric (2.1), the time-like coordinate in (2.4) is not orthogonal to the space-like coordinates (with the only exception of the origin of coordinates, where all four axes are orthogonal to each other). If a test body moves through the cosmic space with the velocity of the general cosmic expansion given by Eq. (1.16), its proper time becomes identical with the cosmic time. As follows from Eqs. (2.2) and (2.3), the distance between the same two points as in Eq. (2.2), measured by a rigid rod, exactly equals the difference (x_1-x_0) of the new coordinate x

$$l_1=x_1-x_0. \tag{2.7}$$

The metric (2.4) has thus a very preferred position, for the differences of the space-like coordinates have in it an immediate metric meaning as the lengths measured by rigid rods.

A test particle moves in general relativity along a geodesic. In the Newtonian approximation it is described by the relation

$$d^2x_j/dt^2=\left\{\begin{matrix} j \\ 4\ \ 4\end{matrix}\right\}, \quad (x_j=x,y,z), \tag{2.8}$$

where, in the case of the metric (2.4),

$$\left\{\begin{matrix} j \\ 4\ \ 4\end{matrix}\right\}=-\frac{\partial g_{j4}}{\partial t}+\frac{1}{2}\frac{\partial g_{44}}{\partial x_j}=-\dot{H}x_j-H^2x_j. \tag{2.9}$$

Equation (2.8) with (2.9) is identical with the equation of motion (1.1). It can be also deduced from the Lagrangian

$$\mathfrak{L}=m(\tfrac{1}{2}\dot{\mathbf{r}}^2-\Phi). \tag{2.10}$$

The potential Φ is determined by Eqs. (2.5) in analogy with Einstein's statement[3] that $(\frac{1}{2}g_{44})$ plays the role of the gravitational potential in the Newtonian approximation. In our case this analogy is, of course, quite formal, for Φ contains, besides the nonconstant part of $(\frac{1}{2}g_{44})$, also the g_{j4} multiplied by $\dot{\mathbf{r}}$ which is now to be interpreted as the velocity of the test particle. However, the identification of $(\frac{1}{2}g_{44})$ with the gravitational potential as well as the identification of the variable part of the coefficient standing at dt^2 with the generalized gravitational potential, is justified in both cases by the identity of the equation of motion obtained from the Lagrangian containing $(\frac{1}{2}g_{44})$, or Φ, respectively, with the corresponding equation of a geodesic in Newtonian approximation.

Comparing Eqs. (2.5) with (1.17) we find that the metric (2.4), with its important properties mentioned above, gives us exactly the same Lagrangian that has been deduced in the foregoing section by the condition that the total energy of a test particle has to be conserved, if the particle is participating in the general cosmic expansion.

Till now we have considered an idealized model of the universe. In regions sufficiently small compared with the radius of the universe the actual local inhomogeneities and anisotropy in the distribution of matter are taken into account in the metric[4]

$$ds^2=-(1-2\Psi/c^2)(dx^2+dy^2+dz^2)+(c^2+2\Phi+2\Psi)dt^2 \tag{2.11}$$

by the scalar potential Ψ, which is given by the Poisson equation

$$\nabla^2\Psi=4\pi\gamma(\rho-\rho_0) \tag{2.12}$$

and by the conditions $\Psi=0$, grad$\Psi=0$ at the boundary of a sufficiently large region of the universe within which the mean mass-density equals the mean mass-density ρ_0 of the whole universe. By ρ we denote the actual mass-density. In the presence of the field Ψ the differences of spatial coordinates do not equal exactly the lengths measured by rigid rods, but, similar to the case in the Schwarzschild's metric, the influence of Ψ may usually be neglected.

Comparing the metric (2.4) with (2.11), we replace the Lagrangian (2.10) by the following one:

$$\mathfrak{L}=m[\tfrac{1}{2}\dot{\mathbf{r}}^2-(\Phi+\Psi)]. \tag{2.13}$$

The corresponding equation of motion,

$$d^2\mathbf{r}/dt^2=-qH^2\mathbf{r}-\mathrm{grad}\Psi, \tag{2.14}$$

is, of course, identical with the Newtonian approximation of the equation of a geodesic belonging to the metric (2.11). The total energy is expressed by the relation

$$\Lambda=m[\tfrac{1}{2}\dot{\mathbf{r}}^2+\varphi+\Psi], \tag{2.15}$$

and its total time derivative by

$$d\Lambda/dt=m[\dot{H}\mathbf{r}\cdot(\dot{\mathbf{r}}-H\mathbf{r})+\partial\Psi/\partial t]. \tag{2.16}$$

After having justified the choice of the Lagrangians (2.10) and (2.13), we shall apply the latter in the following two sections to weigh the real significance of the concept of energy in an expanding universe.

3. MOTION OF A TEST PARTICLE AROUND A CENTRAL BODY

In this section we examine the question whether the general cosmic expansion has some effect on the motion of planets in our solar system.

At first we simplify the given problem by assuming that a test particle moves around a single central body in an expanding universe filled throughout with a homogeneously and isotropically distributed cosmic dust. The motion is governed by Eq. (2.14), into which we put

$$\Psi=-\gamma m_0/r, \tag{3.1}$$

m_0 being the mass of the central body. We get

$$d^2\mathbf{r}/dt^2=-qH^2\mathbf{r}-\gamma m_0\mathbf{r}/r^3. \tag{3.2}$$

We now integrate this equation by the method of successive approximations. For this purpose we expand the function qH^2 into a Taylor series, restricing ourselves to the linear terms. In the Friedman universe

[3] A. Einstein, Ann. Physik **49**, 769 (1916).

[4] J. Pachner, Acta Phys. Polon. **25**, 735 (1964).

we have

$$qH^2=q_1H_1^2[1-3H_1(t-t_1)]+\cdots. \qquad (3.3)$$

In the first approximation we neglect the time-dependent term and obtain from (3.2) the following formula for the angular velocity ω of a test particle moving in a circular orbit around the central body:

$$\omega^2=(\gamma m_0/r_1^3)+q_1H_1^2. \qquad (3.4)$$

In the further approximation we suppose the initial conditions $r=r_1$, $\dot{r}=0$ for $t=t_1$, and find the radius of the circular orbit to be increasing by the function

$$r=r_1[1+\tfrac{1}{2}q_1H_1^3(t-t_1)^3]. \qquad (3.5)$$

This immeasurably small increase of the radius is not apparent, but it does really exist.[5] It is a consequence of the decrease of the total mass M within a sphere of radius r

$$M=m_0+(4\pi/3)r^3\rho_0=m_0+qH^2r^3/\gamma$$

resulting from the cosmic expansion ($\dot{\rho}_0<0$) and causing, by Newton's law of general gravitation,

$$\oint \mathbf{g}\cdot d\mathbf{S}=-4\pi\gamma M,$$

the decrease of the intensity $\mathbf{g}$ of the gravitational field. The energy of our test body,

$$\Lambda=\tfrac{1}{2}m[-(\gamma m_0/r)+(q_1-1)H_1^2r_1^2 +2(1+q_1)H_1^3r_1^2(t-t_1)+\cdots], \qquad (3.6)$$

is therefore very slowly increasing[6]:

$$d\Lambda/dt\cong m(1+q_1)H_1^3r_1^2. \qquad (3.7)$$

This result seems to be contradictory to the conclusion of Einstein and Straus[7] that "in the planetary realm everything behaves as if there existed no cosmic expansion or curvature." In fact, the vacuole model and the metric (2.11), representing with Ψ given by (3.1) the Newtonian approximation of McVittie's metric,[8] describe two quite different physical situations. In the former all the mass within the vacuole is concentrated into a single central body with the mass m_0, while in the latter, besides the central body, all the space is filled with the uniformly and isotropically distributed cosmic dust of density ρ_0.

There is now the question which of these two models, that of Einstein and Straus, or that of McVittie, depicts reality better. In the case just considered we may answer that the vacuole model should be preferred, if the major part of the matter in our universe consists of massive bodies and nebulae. However, if the major, or at least a considerable, part of the matter consists of a uniformly distributed radiation, such as neutrinos, then we have to prefer McVittie's model.

It will be expedient now to compute the Lagrangian, and the relations deduced therefrom, describing the field within a vacuole of the radius r_0. The scalar potential Ψ is here given by the function

$$\Psi=-(\gamma m_0/r)+2\pi\gamma\rho_0(r_0^2-r^2/3), \qquad (3.8)$$

which follows from Eq. (2.12) and the boundary conditions $\Psi=0$, $\partial\Psi/\partial r=0$ at $r=r_0$. With the help of (1.2) and (1.3), the Lagrangian takes the form

$$\mathfrak{L}=m[\tfrac{1}{2}\dot{\mathbf{r}}^2+(\gamma m_0/r)-H\mathbf{r}\cdot\dot{\mathbf{r}}-\tfrac{1}{2}H^2\mathbf{r}^2 +\tfrac{3}{2}(\dot{H}+H^2)r_0^2]. \qquad (3.9)$$

Hence we obtain the equation of motion

$$d^2\mathbf{r}/dt^2=-\gamma m_0\mathbf{r}/r^3, \qquad (3.10)$$

the expression for the total energy of our test particle

$$\Lambda=m[\tfrac{1}{2}\dot{\mathbf{r}}^2-(\gamma m_0/r)-\tfrac{1}{2}(1+q)H^2\mathbf{r}^2+\tfrac{3}{2}qH^2r_0^2], \qquad (3.11)$$

and that for its total time derivative

$$d\Lambda/dt=m[-(1+q)H^2\mathbf{r}\cdot\dot{\mathbf{r}}+(1+\tfrac{5}{2}q)H^3\mathbf{r}^2 -\tfrac{9}{2}qH^3r_0^2], \qquad (3.12)$$

for in the Friedman universe

$$d^2H/dt^2=(2+5q)H^3. \qquad (3.13)$$

We compare these relations with the equations deduced from the Newtonian approximation of the metric within a vacuole as calculated by Schücking.[9] The Lagrangian belonging to his metric takes in our notation the form

$$\mathfrak{L}=m[\tfrac{1}{2}\dot{\mathbf{r}}^2+(\gamma m_0/r)-2(\gamma m_0/r_0)]. \qquad (3.14)$$

The radius r_0 of the vacuole is given by his formula

$$r_0=r_0(t)=(3m_0/4\pi\rho_0)^{1/3}. \qquad (3.15)$$

We now obtain

$$d^2\mathbf{r}/dt^2=-\gamma m_0\mathbf{r}/r^3, \qquad (3.16)$$

$$\Lambda=m[\tfrac{1}{2}\dot{\mathbf{r}}^2-(\gamma m_0/r)+2(\gamma m_0/r_0)], \qquad (3.17)$$

and, with the help of Eq. (1.2),

$$d\Lambda/dt=-2mH(\gamma m_0/r_0)=-2mqH^3r_0^2. \qquad (3.18)$$

From the standpoint of our investigation it is interesting that the motion of a test particle within a vacuole is not influenced by the cosmic expansion and may be computed by starting also from the Lagrangian

$$\mathfrak{L}=m[\tfrac{1}{2}\dot{\mathbf{r}}^2+(\gamma m_0/r)]. \qquad (3.19)$$

We have thus one equation of motion only, but three different Lagrangians and three different expressions for the total energy of the test particle and its conservation. With the same right we may say that the energy of a test particle moving within a vacuole is exactly conserved [if we assume the Lagrangian (3.19)], or it varies by Eq. (3.12) or (3.18), respectively. This indeterminateness concerning the energy is a con-

[5] Compare in this connection a contradictory opinion: R. H. Dicke and P. J. E. Peebles, Phys. Rev. Letters **12**, 435 (1964).

[6] J. Pachner, Phys. Rev. Letters **12**, 117 (1964).

[7] A. Einstein and E. G. Straus, Rev. Mod. Phys. **17**, 120 (1945); **18**, 148 (1946).

[8] G. C. McVittie, Monthly Notices Roy. Astron. Soc. **93**, 325 (1933). See also: J. Pachner, Phys. Rev. **132**, 1837 (1963).

[9] E. Schücking, Z. Physik **137**, 595 (1954).

sequence of the invariance of the field equations of the general relativity with respect to arbitrary transformations of coordinates.[10]

We may thus conclude: As long as the whole space is static, the law of conservation of energy does certainly hold, but in an expanding space with an inhomogeneous distribution of matter it is impossible to define the energy uniquely and in such a way that it is conserved everywhere and always.

The deviations from the exact validity of the law of conservation of energy lie, in all the cases considered hitherto, so far below the limit of observability that it seems to be idle to discuss this problem. However, in the following section we shall see that in a certain case they amount to values which must not be neglected.

4. A MODEL OF FORMATION OF LOCAL SYSTEMS

Astronomical observations show that there exist large regions of the universe which are not influenced by the general cosmic expansion. If they are of a spherical shape, their radii can be computed by a formula following from a dimensional consideration: Since the motion of a test body within this local system is governed by the laws of Newtonian dynamics and outside it by Hubble's law (1.16), we may expect that the radius r_0 depends on the Newtonian gravitational constant γ, on the total mass m_0 of the local system, and on Hubble's "constant" H. The only expression, formed by γ, m_0, and H, which has the dimension of a length, is $(\gamma m_0/H^2)^{1/3}$. We suppose therefore that

$$r_0 = k(\gamma m_0/H^2)^{1/3}. \qquad (4.1)$$

The dimensionless factor of proportionality k is not determined by the dimensional consideration, but usually it lies nearby unity.

Schücking's formula (3.15) for the radius of a vacuole can be easily reduced with the help of Eq. (1.2) to (4.1) with

$$k = q^{-1/3}. \qquad (4.2)$$

Physically it states that the total mass of a vacuole and the intensity of the gravitational field at its surface reach the same values regardless of whether all the matter is concentrated into a single central body, or whether it is uniformly distributed with the density ρ_0.

In a model of a local system in which, besides the central body with the mass m_0, all the space is filled with uniformly and isotropically distributed cosmic dust of density ρ_0, we may determine its radius r_0 by the condition[6] that the intensity of the cosmic field $-qH\mathbf{r}$ equals here the intensity of the Newtonian field $-\gamma m_0 \mathbf{r}/r^3$ [see Eq. (3.2)]. Hence we obtain again the formula (4.1) with k given by (4.2). In an earlier paper[11] this author used the formula (4.1) with $k=1$ and found very good agreement with the results of astronomical measurements of clusters of galaxies. Though the coefficient $k=q^{-1/3}$ is better founded theoretically than $k=1$, in application we take the latter value because of the uncertainty in the empirical data on q.[12]

Both models under consideration differ substantially, however, in one important feature. The radius of a vacuole given by (3.15) is increasing with the velocity

$$\dot{r}_0 = Hr_0, \qquad (4.3)$$

for in Newtonian cosmology

$$\rho_0 \approx G^{-3}(t).$$

Since the system of galaxies is expanding with the same velocity, a body outside a vacuole never can become its member and, if we go back towards the beginning of the expansion, we cannot explain by the vacuole model why a member of a local system ceased at one time in the past to participate in the general cosmic expansion. The necessary retardation of its velocity can be caused only by a higher intensity of the gravitational field due to a local aggregation of matter above the cosmic average. The formation of local systems is thus to be studied on the basis of McVittie's model.

Investigating in McVittie's model the transition from the radial motion by Hubble's law (1.16) to the motion influenced mainly by the Newtonian force of the central body, we may restrict ourselves on the radial component of $\mathbf{r}$. In the equation of motion following from (3.2)

$$d^2r/dt^2 = -qH^2r - \gamma m_0/r^2 \qquad (4.4)$$

we put

$$q = \tfrac{1}{2}, \quad H = 2/3t. \qquad (4.5)$$

These relations (4.5) hold exactly in a flat space. In a curved Friedman universe, they may be applied with an accuracy that improves the more we approach the beginning of the cosmic expansion. By r_{01} we shall denote the radius of the local system at the moment t_1, i.e., the radius of the sphere where the cosmic force qH^2mr equals the Newtonian force $\gamma m_0 m/r^2$:

$$r_{01} = [(9/2)t_1^2\gamma m_0]^{1/2}. \qquad (4.6)$$

By means of the substitutions

$$t = xt_1, \quad r = yr_{01}, \qquad (4.7)$$

we reduce Eq. (4.4) into a dimensionless form

$$d^2y/dx^2 = -(2/9)(x^{-2}y + y^{-2}). \qquad (4.8)$$

As the initial conditions we choose

$$\text{for} \quad x=1: \quad y=y_1, \quad dy/dx = \tfrac{2}{3}y_1. \qquad (4.9)$$

We thus assume that until the moment t_1 the influence of the Newtonian force on the velocity of our test body may be fully neglected.

The differential equation (4.8) can be integrated by a numerical method. The author carried out the integra-

[10] Compare Ref. 9. Schücking remarks that his metric (34) [from which we have deduced the Lagrangian (3.14)] can be made static by a suitable transformation of coordinates. The Lagrangian (3.19) corresponds to this static metric.

[11] J. Pachner, Z. Astrophys. **55**, 177 (1962).

[12] A. Sandage, Astrophys. J. **133**, 355 (1961); J. Soc. Ind. Appl. Math. **10**, 781 (1962).

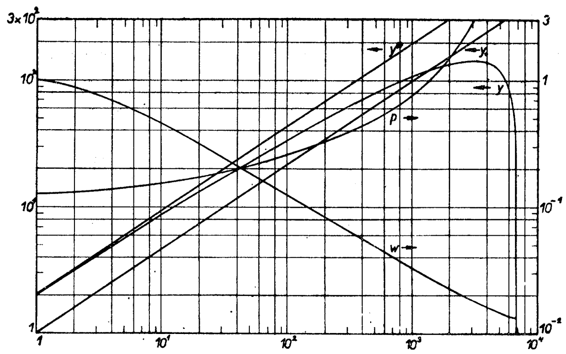

FIG. 1. Graphical representation of the functions $y(x)$, $y_0(x)$, $y^*(x)$ (scale on the left-hand side), and $p(x)$, $w(x)$ (scale on the right-hand side).

tion with the help of Milne's method XI,[13] supposing $y_1=2$. The result of this calculation is graphically represented in Fig. 1 by the curve y. The straight line y_0 depicts here the dependence of the radius r_0 of the local system on the time, for we have

$$r_0=r_{01}y_0=r_{01}x^{2/3}. \qquad (4.10)$$

The curve p represents the ratio of the Newtonian force $\gamma m_0 m/r^2$ to the cosmic force qH^2mr acting at the moment x on the test body:

$$p=(\gamma m_0 m/r^2)/qH^2mr=x^2y^{-3}. \qquad (4.11)$$

If McVittie's model of a local system were replaced by the vacuole model, the radius of the vacuole r_0 would increase by the function (4.10) too, but the test body would move, under the same initial conditions (4.9), according to the relation

$$r^*=r_{01}y^*=r_{01}y_1x^{2/3}. \qquad (4.12)$$

As we see in Fig. 1, the graph of the function $y^*(x)$ is exhibited by a straight line parallel to $y_0(x)$. Therefore, the test body moving outside the vacuole by Hubbles' law (1.16) never can enter into it.

The expression (2.15) for the total energy of our test body can be easily reduced by means of Eqs. (4.5)–(4.7) to the form

$$\Lambda=-(\gamma m_0 m/r_1)w(x), \qquad (4.13)$$

where $r_1=r_{01}y_1$ indicates the position of the test body at the moment t_1 and

$$w(x)=y_1[(1/y)+(y/x)^2-(9/4)(dy/dx)^2]. \qquad (4.14)$$

The energy Λ thus equals the total energy of the test body $-\gamma m_0 m/r_1$ at the moment t_1 multiplied by the dimensionless function $w(x)$. With regard to the chosen initial conditions (4.9), we have

$$w(1)=1. \qquad (4.15)$$

The function $w(x)$ is graphically represented in Fig. 1 too. The numerical differentiation in (4.14) was performed with the help of Milne's formulas.[14]

The graph $y(x)$ in Fig. 1 shows that the Newtonian force of the central body (i.e., of the local agglomeration of matter above the cosmic average) progressively retards the motion of the test body until it reaches rest ($y_{\max}=145.05$ at $x=3000.6$), and then causes its free fall towards the central body which it reaches at the moment $x=6860$. Since the Newtonian field within a local system is in reality created by a group of celestial bodies rotating around their center of gravity, we may expect that the free fall changes over to a rotation too. This plausible transition can be computed with sufficient accuracy by the methods of classical analytical mechanics as a restrained problem of three bodies.[15]

As the graph of the function $w(x)$ in Fig. 1 exhibits, the change of the total energy of the test body is of considerable magnitude. As soon as the function $p(x)$ amounts to 250, the function $w(x)$ reaches practically its asymptotic value 0.0130. In this region the law of conservation of energy holds again.

The differential equation (4.8) was numerically calculated supposing the initial values (4.9). Because the Newtonian force influences the motion of the test body even before $x=1$, the intial velocity $(dy/dx)_{x=1}$ lies in fact slightly lower than at $\frac{2}{3}y_1$. This difference affects the process quantitatively, but by no means qualitatively.

Note added in proof. Since $\Lambda<0$ and $dw/dx<0$, the energy of the test body increases during its capture by the field of a local system. This agrees, of course, also with Eq. (2.16), if we consider the process from the standpoint of an observer situated at the center of gravity of the local system. However, if we compute $d\Lambda/dt$ by Eq. (2.16) from the standpoint of another observer very distant from the local system and participating on the cosmic expansion, we find that it depends on the position of this observer (i.e., on the chosen standard reference frame), whether the energy of the test body increases ($\dot{\mathbf{r}}-H\mathbf{r}<0$) or decreases ($\dot{\mathbf{r}}-H\mathbf{r}>0$). It is therefore very doubtful what use the introduction of the concept of energy into the cosmological considerations can bring to us.

With regard to these results[16] and taking also into consideration a new model of an oscillating isotropic universe[17] in which the absence of a singularity with the infinite density of matter depends essentially on the existence of a negative stress indirectly proportional to the fourth power of the curvature of space and causing the continuous creation of matter we arrive at the conclusion that the fundamental laws of physics cannot have in an expanding universe the form of conservation laws.

[13] W. E. Milne, *Numerical Solution of Differential Equations* (John Wiley & Sons, Inc., New York, 1953), p. 88.

[14] W. E. Milne, *Numerical Calculus* (Princeton University Press, Princeton, New Jersey, 1949), p. 96.

[15] See, for instance: E. T. Whittaker, *Analytische Dynamik der Punkte und starren Körper* (Springer-Verlag, Berlin, 1924), p. 376.

[16] From another point of view D. Layzer, Astrophys. J. **138**, 174 (1963), shows that the conservation law of energy is not generally valid in an expanding universe.

[17] J. Pachner (to be published).