might be emitted by the fluid particle just after the pulse of intense radiation went by. Owing to the weaker gravitational binding just after the pulse, one expects this additional radiation to reach infinity with a smaller redshift than if it were emitted ahead of the strong pulse. However, we see that this effect is just canceled by the greater Doppler shift from the moving fluid after the impulse. The total conversion factor from local radiant energy to its value at infinity is, as we have seen in Eqs. (4.18), (5.7), and (6.1), just $U+\gamma$, and from Eqs. (6.6) and (6.8) we compute

$$\Delta(U+\gamma)=0. \tag{6.9}$$

ACKNOWLEDGMENTS

The author wishes to thank Robert Schwartz, Dr. David Sharp, Dr. Michael May, and Dr. Richard White for helpful discussions.

PHYSICAL REVIEW VOLUME 137, NUMBER 5B 8 MARCH 1965

Vaidya's Radiating Schwarzschild Metric*

R. W. LINDQUIST† AND R. A. SCHWARTZ‡
Lawrence Radiation Laboratory, University of California, Livermore, California

AND

C. W. MISNER
Department of Physics and Astronomy,§ University of Maryland, College Park, Maryland
and
Lawrence Radiation Laboratory, University of California, Livermore, California
(Received 9 September 1964)

In Vaidya's metric for a radiating sphere,

$$ds^2=-(1-2mr^{-1})du^2-2dudr+r^2d\Omega^2,$$

where $m(u)$ is a nonincreasing function of the retarded time $u=t-r$, we verify that $-dm/du$ is the total power output as given by the Landau-Lifshitz stress-energy pseudotensor, and relate it through red-shift and Doppler-shift factors to the apparent luminosity L for an observer moving radially in this gravitational field. We argue that the hypersurface $r=2m(u)$ cannot be realized physically, but see that a hypersurface $r=2m(\infty)$ at $u=\infty$ (which is not adequately represented in presently available coordinate systems) shows the total red-shift characteristic of the Schwarzschild "singularity." The geodesic equations are written out to display a gravitational "induction field" $-GL/c^3r$ associated with a changing mass in the Newtonian $-Gm/r^2$ field.

I. INTRODUCTION

THE metric field surrounding a star, idealized as a *radiating* sphere, cannot be the Schwarzschild solution,[1] except in the excellent approximation in which one neglects the energy density of the emitted radiation. In this paper we investigate the metric outside a spherically symmetric body when radiation is included. For a normal star, the influence of radiation on the metric is negligible when compared with the effects of deviations from spherical symmetry, caused by rotation, magnetic fields, etc.[2] Nevertheless, this metric may have some relevance to the study of a collapsing supernova core,[3] if one allows for the production of a copious supply of neutrinos, but neglects their subsequent absorption in the outer envelope. A realistic treatment must, of course, analyze the problem of neutrino transport in detail. The solution described here is thus chiefly useful as an extreme limiting case, in which the neutrino optical depth of the envelope is negligible.

We therefore seek a spherically symmetric solution of the Einstein equations[4]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu} \tag{1.1}$$

with the "geometrical optics" stress-energy tensor of

* Work supported in part by NASA Grant NsG 436 and the U. S. Atomic Energy Commission.

† Present address: Department of Physics, The University of Texas, Austin, Texas.

‡ Permanent address: Goddard Institute of Space Studies, New York, New York and Department of Physics, Columbia University, New York, New York.

§ Permanent address.

[1] K. Schwarzschild, Sitzber. Preuss. Akad. Wiss. Physik-Math. Kl. 189 (1916).

[2] A metric for empty space surrounding a rotating object has been given by R. Kerr, Phys. Rev. Letters **11**, 237 (1963). The asymptotic form had been obtained previously by A. Papapetrou, Proc. Roy. Irish Acad. **A52**, 11 (1948). Static metrics for empty space surrounding objects with axial symmetry were given by H. Weyl, Ann. Physik **54**, 117 (1917); **59**, 185 (1919), and have been further studied by M. Misra, Proc. Natl. Inst. India **A26**, 673 (1960); **A27**, 373 (1961); G. Erez and N. Rosen, Bull. Res. Council Israel **8F**, 47 (1959); D. Zipoy (unpublished).

[3] S. A. Colgate and R. W. White, Rev. Mod. Phys. (to be published).

[4] Throughout this paper we choose units such that $G=1$, $c=1$.

radiation

$$T^{\mu\nu}=qk^{\mu}k^{\nu} \tag{1.2}$$

in which k^{μ} is a null vector directed radially outward.

This problem has been considered by Vaidya,[5] by Raychaudhuri,[6] and by Israel,[7] and the most convenient form for the solution is that given by Vaidya[8] which reads

$$ds^2=-[1-2m(u)/r]du^2-2dudr+r^2d\Omega^2, \tag{1.3}$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2. \tag{1.4}$$

Here $m(u)$ is an arbitrary nonincreasing function of the retarded time coordinate u. In Sec. II below we discuss this and other closely related metrics. In Sec. III we verify from the Landau-Lifshitz pseudotensor that m and $L_\infty\equiv-(dm/du)$ are, respectively, the mass and total energy output at infinity, and compare this to the energy flux measured by local observers. Section IV analyzes the special features of the "Schwarzschild surface" $r=2m(u)$. Finally, in Sec. V, we study the geodesics in this metric, and exhibit the main features of the non-Newtonian gravitational field tied to a pulse of radiation.

II. RELATED METRICS

The geometry of spherically symmetric space time is usually described by[9]

$$ds^2=-e^{2\phi}dT^2+e^{\lambda}dr^2+r^2d\Omega^2. \tag{2.1}$$

The retarded time coordinate u used by Vaidya in the metric (1.3) can be introduced through the equation

$$du=f^{-1}(e^{\phi}dT-e^{\lambda/2}dr), \tag{2.2}$$

where f is an integrating factor to make du an exact differential. The metric then takes the form

$$ds^2=-f^2du^2-2fe^{\lambda/2}dudr+r^2d\Omega^2. \tag{2.3}$$

From Eqs. (2.1) and (2.2) it follows that $u^{;\mu}$ is a null vector. This can also be seen from Eq. (2.3) directly; the metric induced on a 3-dimensional hypersurface $u=$constant has the signature $(0,+,+)$ of a null hypersurface. For our choice of signs in Eq. (2.2) these hypersurfaces of constant u contain *outgoing* null rays, since r increases with increasing T when $du=0$.

A generalization of Vaidya's radiation coordinates of Eq. (2.3) to the case with axial symmetry has been given by Bondi and van der Burg,[10] and to the general case without symmetry by Sachs.[11]

The coordinates introduced by Finkelstein[12] for the Schwarzschild metric have their counterpart also in the radiating case. Setting

$$u=t-r \tag{2.4}$$

in Vaidya's metric (1.3) one finds

$$ds^2=-(1-2mr^{-1})dt^2-2(2mr^{-1})dtdr +(1+2mr^{-1})dr^2+r^2d\Omega^2. \tag{2.5}$$

If m is a constant then this is just the Schwarzschild metric, as Finkelstein showed by introducing Schwarzschild's time coordinate T through the formula

$$T=t+2m\ln(r-2m). \tag{2.6}$$

A corresponding transformation to explicitly diagonalize the metric (1.3) or (2.5) when $dm/du\neq0$ is not known.

In the form

$$ds^2=ds_F{}^2+2mr^{-1}du^2, \tag{2.7}$$

where $ds_F{}^2$ is the flat space metric, Vaidya's metric (1.3) is of the type recently studied for empty space by Kerr and Schild.[13]

III. THE OBSERVED ENERGY FLUX

The value of q in Eq. (1.2) is not well defined because there is no natural normalization for k^{μ}. Let us then define q to be the energy density of the radiation as measured locally by an observer with 4-velocity v^{μ}, so

$$q\equiv v^{\mu}v^{\nu}T_{\mu\nu}. \tag{3.1}$$

Therefore, in this observer's local Lorentz frame, $k^{\mu}=(1;1,0,0)$; it follows that q is the energy flux as well as the energy density measured in this frame. We shall only consider radially moving observers, and define

$$U\equiv v^r=dr/d\tau. \tag{3.2}$$

Then, from $v^{\mu}v_{\mu}=-1$, with $v^{\theta}=0$, $v^{\varphi}=0$, there follows

$$du/d\tau=v^u=(1-2mr^{-1})^{-1}(\gamma-U)=(\gamma+U)^{-1}, \tag{3.3}$$

where

$$\gamma\equiv(1+U^2-2mr^{-1})^{1/2}. \tag{3.4}$$

The covariant Ricci tensor for the metric (1.3) has only one nonvanishing component: $R_{uu}=-2r^{-2}(dm/du)$. From Eqs. (3.1) and (1.1) one computes

$$q=(8\pi)^{-1}v^{\mu}v^{\nu}R_{\mu\nu}, \tag{3.5a}$$

$$q=-[(dm/du)/4\pi r^2][1/(\gamma+U)^2]. \tag{3.5b}$$

[5] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951); Curr. Science **21**, 96 (1952).

[6] A. K. Raychaudhuri, Z. Physik **135**, 225 (1953).

[7] W. Israel, Proc. Roy. Soc. (London) **A248**, 404 (1958).

[8] P. C. Vaidya, Nature **171**, 260 (1953). We thank Professor D. Zipoy for pointing out this work to us.

[9] R. C. Tolman, Proc. Natl. Acad. Sci. Wash. **20**, 3 (1934).

[10] H. Bondi, Nature, **186**, 535 (1960); H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962).

[11] R. Sachs, Proc. Roy. Soc. (London) **A270**, 103 (1962).

[12] D. Finkelstein, Phys. Rev. **110**, 965 (1958).

[13] R. P. Kerr and A. Schild, invited paper presented to the International Meeting on General Relativity, Florence, Italy, September 1964 (unpublished).

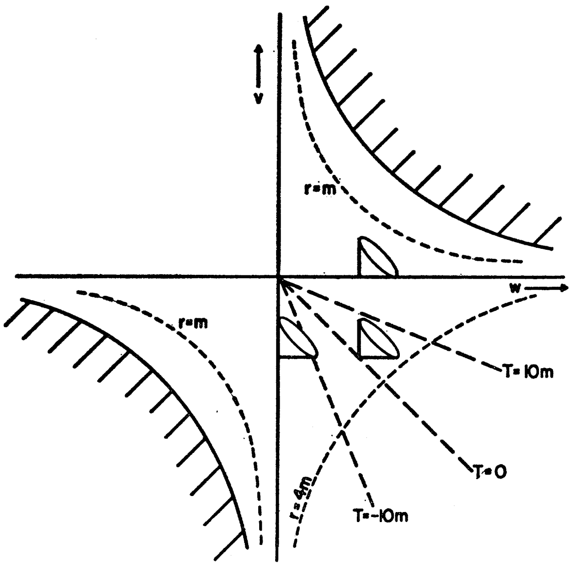

FIG. 1. Kruskal's coordinate system for the Schwarzschild (constant m) metric shows that there are two different $r=2m$ hypersurfaces, corresponding to $T=\pm\infty$. The Schwarzschild metric in the above $vw\theta\varphi$ coordinates is $ds^2=-16m^2e^{-x}dvdw+4m^2x^2d\Omega^2$, where $x=r/2m$ is defined in the region of regularity $vw\lessgtr 1$ by $-vw=(x-1)e^x$. Thus hypersurfaces of constant r are represented by hyperbolas of constant vw, and the Schwarzschild "singularity" $x=1$ consists of the two quite regular null hypersurfaces $v=0$ and $w=0$. Schwarzschild's time coordinate T is defined by $-w/v=e^{T/2m}$ and is therefore constant along lines through the origin of the vw plane. Schwarzschild's coordinate system covers only the fourth quadrant of the diagram above. Vaidya's coordinate u of Eq. (1.3) is defined here $(dm/du=0)$ by $v=-e^{-u/4m}(2m)^{-1/2}$ and covers the lower half-plane. The curves $r=$constant $(r<2m)$ are represented by a sample curve, $r=m$. One may see that this curve always has a negative slope. Thus, since light cones are lines parallel to the axes, $r=m$ must be a *space-like* hypersurface and no particle can have $r=m$ for a world line. Similiarly, the lines $T=$constant are time-like in this region. However, there is no singularity in the coordinates system for $0<r<\infty$. This is the maximal extension of the exterior Schwarzschild solution; the singularity at $r=0$ cannot be eliminated. If one demands that the geometry be regular there, and satisfy spherical symmetry, then flat space is the only solution.

Since q, being an energy density, must be positive, it follows from (3.5b) that $dm/du\leqslant 0$.

For an observer at rest at infinity we find a total luminosity of

$$L_\infty(u)=\lim_{r\to\infty,\ U=0}(4\pi r^2q)=-(dm/du)\,. \tag{3.6}$$

Note also that if we define

$$L=4\pi r^2q \tag{3.7}$$

then Eq. (3.5) can be rewritten as

$$L_\infty=L(\gamma+U)^2 \tag{3.8}$$

showing that the locally observed luminosity L is reduced (or increased if $U>m/r$) by one factor of $(\gamma+U)$ to red shift the energy involved as it moves out, and a second $(\gamma+U)$ factor for the dilation of the time interval over which this energy is emitted.

Another way to compute the total-energy output of the system described by the metric of Eq. (1.3) is by means of the various stress-energy pseudotensors.[14] The one defined by Landau and Lifshitz[15] is usually best because one can remember the formulas. Landau and Lifshitz have rewritten the Einstein equations in the form

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}=T_{\text{tot}}{}^{\mu\nu}\,, \tag{3.9}$$

where $T_{\text{tot}}{}^{\mu\nu}$ involves both the stress-energy tensor of matter and an expression quadratic in first derivatives of the metric. The left-hand side contains an expression with the symmetries of the Riemann tensor; it is defined by

$$H^{\mu\alpha\nu\beta}=(16\pi)^{-1}(\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta}-\mathfrak{g}^{\mu\beta}\mathfrak{g}^{\nu\alpha}) \tag{3.10}$$

with

$$\mathfrak{g}^{\mu\nu}=(-g)^{1/2}g^{\mu\nu}. \tag{3.11}$$

The total energy-momentum vector is

$$P_{\text{tot}}{}^{\mu}=\int T_{\text{tot}}{}^{\mu0}d^3x=\oint H^{\mu\alpha0k}{}_{,\alpha}d^2S_k. \tag{3.12}$$

Similarly the power output can be computed from the right-hand side of

$$L_{\text{tot}}=\oint T_{\text{tot}}{}^{0i}d^2S_i=\oint H^{0ki\beta}{}_{,k\beta}d^2S_i. \tag{3.13}$$

These integrals must be evaluated in asymptotically rectangular coordinate systems. (This is a reasonable restriction since energy and momentum have meaning only with reference to Lorentz transformations, and therefore some means must be provided to introduce the reference flat space at infinity into the calculations.)

Introducing Cartesian coordinates $x^i (i=1,2,3)$ in place of (r,θ,φ), one computes

$$\begin{aligned} g^{00}&=-(1+2mr^{-1})\,,\\ g^{0i}&=-2mx^ir^{-2}\,,\\ g^{ij}&=\delta_{ij}-2mr^{-3}x^ix^j\,,\\ (-g)^{1/2}&=1\,, \end{aligned} \tag{3.14}$$

and then, from (3.12) and (3.13),

$$P^\mu=(m;0,0,0)\,, \tag{3.15}$$

$$L_{\text{tot}}=-(dm/du)=L_\infty. \tag{3.16}$$

[14] J. N. Goldberg, Phys. Rev. **111**, 315 (1958); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961), *Proceedings of The International Conference on The Theory of Gravitation* (PWN, Warsaw and Gauthier-Villars, Paris, 1964), p. 189; A. Trautman in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 169.

[15] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1951), Sec. 11-9.

Equivalent results requiring more extensive computation with other stress-energy pseudotensors have been obtained by Møller.[16]

IV. THE SCHWARZSCHILD SURFACE

In Kruskal's form[17] of the Schwarzschild metric one sees that there are really two distinct Schwarzschild surfaces $r=2m$; one coincides with $T=-\infty$, the other with $T=+\infty$ (see Fig. 1). In Vaidya's metric the hypersurface $r=2m(u)$ is analogous to the Schwarzschild hypersurface $T=-\infty$, $r=2m$. Its character can be seen most easily in Fig. 2 where we have sketched light cones bounded by the radial null vectors

$$\mathbf{k}=k^\mu(\partial/\partial x^\mu)=\partial/\partial r \tag{4.1}$$

and

$$\mathbf{l}=l^\mu(\partial/\partial x^\mu)=-(1-2mr^{-1})\frac{\partial}{\partial r}+2\frac{\partial}{\partial u} \tag{4.2}$$

in the ur plane.

From this sketch one sees that, for $dm/du\neq 0$, the surface $r=2m(u)$ lies outside the light cone; i.e., it is a space-like hypersurface. This is also obvious from the form of the induced metric on the hypersurface

$$(ds^2)_{r=2m(u)}=2(-dm/du)du^2+r^2d\Omega^2\,, \tag{4.3}$$

which has signature $(+,+,+)$ whenever $dm/du<0$. We may properly consider it to be a space-like hypersurface lying in the *past* of the region $r>2m(u)$, since no light ray starting in this region will intersect the $r=2m(u)$ hypersurface. It is also evident from the figure that no material particle following a time-like path can reach the $r=2m(u)$ hypersurface starting from the outside. For this reason we consider the region $r\leqslant 2m(u)$ to be unphysical; the sources of the strong gravitational fields there cannot be objects which once existed in an $r>2m(u)$ region and were then assembled into something in the region $r<2m(u)$. As these regions cannot, in principle, be produced experimentally, we ignore them. Every physical situation must consequently contain a boundary hypersurface $r=f(u)>2m(u)$ with the interior metric in the region $r\leqslant f(u)$ differing from Vaidya's metric because of the presence of matter or other fields.

The hypersurface $r=2m(\infty)$ at $u=\infty$ in Vaidya's metric is analogous to the Schwarzschild hypersurface $r=2m$ at $T=+\infty$ in Kruskal's metric. It is reasonable to suppose that such surfaces are sometimes formed in the gravitational collapse of the cores of stars at the supernova stage. In any case they can, in principle, be produced.[18]

The most characteristic property of the hypersurface $u=+\infty$ is an infinite time dilation, which we now

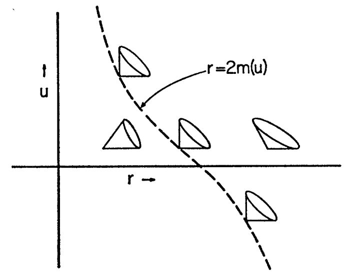

FIG. 2. The light cones for Vaidya's metric are shown projected on to the u-r plane. Note that time-like vectors in the forward light cone on the hypersurface $r=2m(u)$ all have $dr/du>0$, so no time-like path crosses this hypersurface starting from the outside.

examine. Let a particle move radially inward along a time-like path with a finite nonzero "velocity" $U=dr/d\tau$. Then, Eq. (3.3) gives $du/d\tau=(\gamma+U)^{-1}$. Since light signals travel outward along rays of constant u, two signals emitted by the particle at an interval du are received at infinity with the same separation du. For an observer at rest at infinity, du is just his proper time $d\tau_{\rm obs}$. Consequently we may write Eq. (3.3) in the form

$$d\tau_{\rm obs}=(\gamma+U)^{-1}d\tau\,. \tag{4.4}$$

Since γ is necessarily positive, infinite time dilations occur when $U=-\gamma\leqslant 0$; from Eq. (3.4) it follows that $r=2m(u)$. If u is finite, a time-like vector at $r=2m(u)$, which lies in the future light cone, necessarily has $U\geqslant 0$; hence infinite time dilation only occurs at $r=2m(\infty)$. Because this time dilation also affects the frequency of a photon, there will correspondingly be a total red shift of light emitted by a particle crossing the $r=2m(\infty)$, $u=+\infty$ hypersurface.

It would be desirable, of course, to introduce a coordinate system which included the $r=2m(\infty)$, $u=+\infty$ hypersurface in its interior, so that one could study this interesting hypersurface in detail and follow the world lines of particles into the region $r\leqslant 2m(\infty)$. Because $dm/du\leqslant 0$ and $m>0$, there are two possibilities to consider: Either (1) $m(u)$ is constant for all sufficiently large u, say $u>0$, or (2) $m(u)\to m_0\geqslant 0$ as $u\to\infty$. In case (1) the metric in the $u>0$ region is just the Schwarzschild metric, whose continuation through $u=+\infty$ has been given by Kruskal.[17] We have been unable to find explicitly the transformation to Kruskal-like coordinates for solutions of type (2).

V. GEODESICS

The geodesics for the metric (1.3) will all lie in "planes" because of the spherical symmetry. When we orient the coordinate system to make the "plane" in question $\theta=\pi/2$, then the geodesic equations result from varying the action integral

$$I=\int \mathcal{L}d\tau=\frac{1}{2}\int[-(1-2mr^{-1})\dot{u}^2-2\dot{u}\dot{r}+r^2\dot{\varphi}^2]d\tau\,, \tag{5.1}$$

[16] C. Møller, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **34**, No. 3 (1964).
[17] M. Kruskal, Phys. Rev. **119**, 1743 (1960): This form of the Schwarzschild metric is analyzed in detail by R. W. Fuller and J. A. Wheeler, Phys. Rev. **128**, 919 (1962).
[18] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

where a dot is used for proper time derivatives. Two of the generalized momenta associated with this variation principle are useful to us: The angular momentum per unit mass

$$l\equiv(\partial\mathfrak{L}/\partial\dot{\varphi})=r^2\dot{\varphi}\,, \tag{5.2}$$

and the energy per unit mass

$$\gamma\equiv-(\partial\mathfrak{L}/\partial\dot{u})=(1-2mr^{-1})\dot{u}+\dot{r}\,. \tag{5.3}$$

Although γ will be constant only when m is, l will of course always be constant. The normalization condition $v_\mu v^\mu=-1$ can be written in terms of l, γ, and the radial momentum per unit mass $U=\dot{r}$ in the form

$$\gamma^2=(1-2mr^{-1})(1+l^2r^{-2})+U^2\,. \tag{5.4}$$

The geodesic equations will be further simplified by introducing the apparent flux L defined by Eqs. (3.7) and (3.5a)

$$L=L_\infty\dot{u}^2=-\dot{u}^2(dm/du)\,. \tag{5.5}$$

The equations resulting from the variation of Eq. (5.1) are then

$$dl/d\tau=0\,, \tag{5.6a}$$

$$d\gamma/d\tau=L/r\,, \tag{5.6b}$$

$$\frac{dU}{d\tau}=-\frac{L}{r}-\frac{m}{r^2}+\frac{l^2}{r^3}-\frac{3ml^2}{r^4}\,. \tag{5.6c}$$

From our previous definitions we can add the equations

$$d\varphi/d\tau=l/r^2\,, \tag{5.7a}$$

$$dr/d\tau=U\,, \tag{5.7b}$$

$$\frac{du}{d\tau}=\frac{1+l^2r^{-2}}{\gamma+U}\,, \tag{5.7c}$$

to form with Eqs. (5.6) a system of first-order equations for which Eq. (5.4) gives a first integral. These equations differ from the Schwarzschild set only in the terms containing L.

The acceleration

$$a_L=-GL/c^3r \tag{5.8}$$

in Eq. (5.6c) is a non-Newtonian gravitational field associated with radiated power L emitted by the central source, or one may call it an *induction field* associated with the changing Newtonian Gm/r^2 field. The terms $-Gm/r^2$ and l^2/r^3 in this equation are the Newtonian gravitational force, and the centrifugal force, while the term $-3Gml^2/c^2r^4$ is a non-Newtonian gravitational force which [together with the distinction (5.7c) between coordinate time and proper time] accounts for the Einstein perihelion precession in Mercury and other planets.

We see in Eq. (5.6c) that the gravitational induction field L/r is directed *toward* the center of force, and from Eq. (5.6b) it always acts to *increase* the energy of the test particle on which it acts. (This feature was first discovered in the interior metric discussed in the accompanying paper.[19]) In normal situations it is negligible, as we can see by comparing it with acceleration a_{43} which gives the Einstein 43″/century perihelion precession in Mercury's orbit

$$a_L/a_{43}=(GL/c^3r)(3Gml^2/c^2r^4)^{-1}=5\times10^{-11}\,.$$

In order to find a situation where the induction field a_L is comparatively large, we may try to take advantage of the fact that $a_L=-L/r$ decreases more slowly with r than the other terms in Eq. (5.6c). Thus we can define a distance R where a_L is comparable to the Newtonian acceleration $a_m=-m/r^2$. From

$$a_L/a_m=(GL/c^3r)(Gm/r^2)^{-1}=Lr/c^3m \tag{5.9}$$

we find

$$R=mc^3/L\,. \tag{5.10}$$

While this radius R for the sun exceeds the "radius of the universe," for the most luminous known objects it is small enough to have some meaning without reference to cosmology, and can even be made small with prejudiced data: A low minimum mass for the quasistellar source[20] 3C273 is $m\sim10^6M_0$, while a generous value of L would be 10^{47} erg/sec. These figures give $R\simeq5\times10^5$ light years which is not much larger than the complex radio source itself. More probable mass estimates increase this value of R by several orders of magnitude.

Although the induction field L/r may be comparable to the Newtonian m/r^2 field at sufficiently large distances from a strongly radiating source, it will not have important effects there because of its limited duration in time. The maximum time a constant luminosity L can be maintained by a mass m is $\Delta t=mc^2/L$, so the maximum velocity change a test particle can have due to the GL/c^3r acceleration during this interval is $\Delta U=\Delta(v/c)=a_L\Delta t/c=Gm/c^2r$. But if the test particle were originally in a Newtonian orbit with $v^2\sim Gm/r$, this gives

$$\Delta(v/c)\sim v^2/c^2\ll v/c\,.$$

Thus the induction field L/r is likely to be of importance, if at all, only in catastrophic phases of gravitational collapse where one might find $v\sim c$ and $Gm/c^2r\sim1$.

[19] C. W. Misner, preceding paper, Phys. Rev. **137**, B1360 (1965).

[20] J. L. Greenstein and M. Schmidt, Astrophys. J. **140**, 1 (1964).