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Relativistic Equations for Spherical Gravitational Collapse with Escaping Neutrinos*

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The general-relativity equations for the dynamics of a self-gravitating sphere of ideal fluid as given by Misner and Sharp are modified to allow an extremely simplified heat-transfer process in which internal energy is converted (at some rate controlled by an equation of state) into an outward flux of neutrinos which have no subsequent interaction with matter. This outward flux of radiation carries with it an inward non-Newtonian gravitational force field. Thus, if a portion Δm of some central mass is converted into escaping radiation, the corresponding energy increase $\sim G\Delta m/R$ of each unit mass of surrounding matter is seen to be the work done by this non-Newtonian inward impulse accompanying the radiation.

I. INTRODUCTION

IN connection with both supernovae[1] and quasi-stellar radio sources[2] it has been proposed that at a certain stage of collapse, neutrinos could be emitted copiously. This neutrino production occurs only at extreme temperatures attained in regions of intense gravitational fields where general relativity is important. In this paper, the equations of general relativity are written out for a spherically symmetric situation in which one might attempt to study not only the effects of the changing gravitational fields of the collapsing matter on the escaping radiation, but also the forces on the matter due to gravitational fields associated with the escaping radiation. These last could be important if it is possible suddenly to convert a significant fraction of the total mass of the system into radiation.

Except to point out a few qualitative features of the non-Newtonian gravitational field carried by a pulse of radiation, this paper does not continue beyond the formulation of the basic equations to consider any applications.

The idealized situation which we envision is that of a sphere of fluid subject to gravitational and pressure gradient forces. This fluid does not, however, obey a simple adiabatic equation of state but each element of fluid will cool by emission of neutrinos at some rate determined by its temperature and density. To simplify the treatment of the neutrino flux, we assume that all the neutrinos move radially outward when emitted and that they are neither scattered nor absorbed by the surrounding matter.

When the neutrino flux vanishes, the equations derived here reduce to those previously obtained by Misner and Sharp.[3] They also show many similarities to the relativistic hydrodynamic equations with weak heat diffusion.[4] The boundary conditions at the outer surface of these radiating objects are obtained by matching to an exterior solution containing pure outgoing radiation. A convenient form of this solution has been discovered by Vaidya[5] and is discussed in the accompanying paper.[6]

II. THE STRESS-ENERGY TENSOR

The fluid will be described by its local thermodynamic properties such as the matter density or baryon number density n, energy density ϵ, and pressure p. Then matter conservation is expressed by the equation of continuity

$$(nu^\mu)_{;\mu}=0\,, \tag{2.1}$$

where u^μ is the fluid's four-velocity. The stress-energy tensor of the fluid we take to be

$$T^{\mu\nu}=(\epsilon+p)u^\mu u^\nu+pg^{\mu\nu}. \tag{2.2}$$

It fails to satisfy a local conservation law because of the neutrino emission. In fact, if $C(T,n)$ is the cooling rate (rate of decrease of internal energy due to the neutrino emission) for a unit amount of matter, then nC is the rate per unit volume in the rest frame of the fluid and we can write

$$-nC=u^\mu(-T_\mu{}^\nu{}_{;\nu}) \tag{2.3}$$

which reduces to

$$-nC=(\epsilon u^\mu)_{;\mu}+pu^\mu{}_{;\mu}. \tag{2.4}$$

We can further simplify this by writing

$$\epsilon=n(1+e) \tag{2.5}$$

to define a specific internal energy e that does not include rest-mass energy. When this definition and the equation of continuity (2.1) are introduced into Eq. (2.4) there results

$$e_{,\mu}u^\mu=-C-p(1/n)_{,\mu}u^\mu \tag{2.6}$$

which is just the first law of thermodynamics with $-C$ giving the heat input rate.

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For the neutrinos we assume the "geometrical optics" form of stress-energy tensor

$$E^{\mu\nu}=qk^{\mu}k^{\nu}, \tag{2.7}$$

where k^{μ} is a null-vector

$$k^{\mu}k_{\mu}=0 \tag{2.8}$$

and q will be the energy flux density in some frame which depends on the normalization of k^{μ}. The total stress-energy tensor satisfies local energy and momentum conservation laws

$$(T^{\mu\nu}+E^{\mu\nu})_{;\nu}=0. \tag{2.9}$$

The energy balance part of these equations gives, from Eq. (2.3), the equation

$$u^{\mu}(-E_{\mu}{}^{\nu}{}_{;\nu})=nC, \tag{2.10}$$

which will govern the behavior of the neutrino flux.

III. COORDINATES AND METRIC

The metric will be chosen as in previous work[3] to have the form

$$ds^2=-e^{2\phi}dt^2+e^{\lambda}dr^2+R^2d\Omega^2, \tag{3.1}$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2 \tag{3.2}$$

and r is a comoving coordinate so that

$$D_t\equiv u^{\mu}(\partial/\partial x^{\mu})=e^{-\phi}(\partial/\partial t). \tag{3.3}$$

(That is, $u^i=0$ for $i=r, \theta, \varphi$.) Then, the derivative

$$U\equiv D_tR \tag{3.4}$$

of the metric component $R(r,t)$ describes the velocity of the fluid. Differentiation along an outward radial unit vector orthogonal to u^{μ} is given by

$$D_r\equiv e^{-\lambda/2}(\partial/\partial r). \tag{3.5}$$

The metric component e^{λ} which appears here can be written in terms of a function $m(r,t)$ according to the definition[3]

$$e^{\lambda}=(1+U^2-2mR^{-1})^{-1}(\partial R/\partial r)^2. \tag{3.6}$$

We choose the null vector k^{μ} which defines the direction of propagation of the neutrinos so that

$$k^{\mu}\frac{\partial}{\partial x^{\mu}}=e^{-\phi}\frac{\partial}{\partial t}+e^{-\lambda/2}\frac{\partial}{\partial r}=D_t+D_r. \tag{3.7}$$

Consequently, the quantity q in Eq. (2.7) is the energy density of the neutrinos in the rest frame of the fluid, i.e., $q=u_{\mu}u_{\nu}E^{\mu\nu}$.

IV. THE FIELD EQUATIONS

The field equation

$$R_{tr}=8\pi(T_{tr}+E_{tr}) \tag{4.1}$$

in the present case gives

$$D_t(\tfrac{1}{2}\lambda)=(\partial U/\partial R)-4\pi Rq\alpha(1+U^2-2mR^{-1})^{-1/2} \tag{4.2}$$

and will be used to eliminate $D_t\lambda$ from all other field equations. We are using the symbol $(\partial/\partial R)$ to mean $(R')^{-1}(\partial/\partial R)$ where $R'=\partial R/\partial r$, and we have also written $\alpha=R'/|R'|$ so that our equations will continue to be correct in case R' is negative.[7] When we substitute from Eq. (3.6) also in the left-hand side of Eq. (4.2) there results

$$D_t(1+U^2-2mR^{-1})^{1/2}=\alpha[UD_{r\phi}+(L/R)], \tag{4.3}$$

where

$$L\equiv 4\pi R^2q \tag{4.4}$$

is the total neutrino flux or luminosity.

When Eqs. (4.2) and (3.6) are used to simplify the equation

$$R_t{}^t-\tfrac{1}{2}R=8\pi(T_t{}^t+E_t{}^t) \tag{4.4'}$$

there results

$$\frac{\partial m}{\partial R}=4\pi R^2\left\{\epsilon+q\left[1+\frac{\alpha U}{(1+U^2-2mR^{-1})^{1/2}}\right]\right\}. \tag{4.5}$$

This is an ordinary differential equation which defines m in each $t=$constant hypersurface. But, in contrast to the $q=0$ case, the appearance of m on the right-hand side prevents its solution being given as a simple integral.

The continuity equation (2.1) can be written either in an integrated form

$$4\pi R^2n|\partial R/\partial r|(1+U^2-2mR^{-1})^{-1/2}=(dA/dr)_{t=0}, \tag{4.6}$$

where the right-hand side is time-independent, or as a differential equation

$$\frac{1}{n}D_tn+\frac{1}{R^2}\frac{\partial}{\partial R}(R^2U)=4\pi Rq\alpha\left(1+U^2-\frac{2m}{R}\right)^{-1/2}, \tag{4.7}$$

where the term in q was introduced through the use of Eq. (4.2).

The local energy balance for the fluid is governed by Eq. (2.6) which reads

$$D_te=-C-pD_t(1/n). \tag{4.8}$$

Using the thermodynamic relation $de=Tds-pdv$, where the specific volume is $v=1/n$, gives an equation for the specific entropy

$$TD_ts=-C. \tag{4.9}$$

The momentum balance for the fluid from the radial component of Eq. (2.9) can be simplified somewhat

[7] *Note added in proof*. This modification involving $\alpha=\pm1$ was introduced into the manuscript after numerical computations by M. May and R. H. White done at the Lawrence Radiation Laboratory (Livermore) showed in some examples of adiabatic collapse that negative values of R' could evolve from nonrelativistic initial conditions. The limiting value of $\partial U/\partial R=U'/R'$ when $R'=0$ [which is needed in Eq. (4.7)] is obtained by differentiating the square of Eq. (4.6) with respect to r.

using Eq. (2.4) to read

$$(\epsilon+p)u^{\mu}{}_{;\nu}u^{\nu}=-(g^{\mu\nu}+u^{\mu}u^{\nu})p_{,\nu}-E^{\mu\nu}{}_{;\nu}+nCu^{\mu}. \quad (4.10)$$

The reaction forces from the neutrino emission can be handled most simply by using the identity

$$k_{\mu}E^{\mu\nu}{}_{;\nu}=0\,, \quad (4.11)$$

which is obtained by differentiating the identity $k_{\mu}E^{\mu\nu}=0$ and noticing that $k_{\mu;\nu}E^{\mu\nu}$ vanishes since $k_{\mu;\nu}k^{\mu}=(\frac{1}{2}k_{\mu}k^{\mu})_{,\nu}=0$. From Eqs. (4.11) and (2.10) then we find a simple expression for the radial component of $E_{\mu}{}^{\nu}{}_{;\nu}$, namely,

$$e^{-\lambda/2}E_{r}{}^{\mu}{}_{;\mu}=nC. \quad (4.12)$$

With this substitution, the radial component of Eq. (4.10) reads

$$(\epsilon+p)\frac{\partial\phi}{\partial R}=-\frac{\partial p}{\partial R}-\frac{\alpha nC}{(1+U^2-2mR^{-1})^{1/2}}. \quad (4.13)$$

Our choice of co-moving coordinates makes this equation of hydrodynamics look like a simple hydrostatic balance of forces. It is this equation which one uses to determine the metric component $(-g_{00})^{1/2}\equiv e^{\phi}$ on each $t=$ constant surface. A convenient boundary condition is to make $\phi=0$ at the outer surface of the body, so that coordinate time, t, becomes proper time there.

We turn now to Eq. (2.10) for the neutrino flux. In flat space the total power L would be independent of R on each null cone, so we choose to write the equation in terms of L instead of q. It reads

$$4\pi R^2nC=e^{-\lambda}D_t(e^{\lambda}L)+e^{-2\phi}D_r(e^{2\phi}L)\,. \quad (4.14)$$

Since the above equation is unfamiliar, it is appropriate to study a simple case. None of the Einstein equations has been used to simplify Eq. (2.10) to the form (4.14), so it applies also to the case where L is weak enough not to influence the metric. For a static metric $(D_t\lambda=0=D_t\phi)$ and outside the region of neutrino production $(C=0)$, Eq. (4.14) reduces to

$$(D_t+D_r)(Le^{2\phi})=0. \quad (4.15)$$

One understands this equation by considering a pulse of neutrinos; each neutrino's energy gets redshifted according to the factor $e^{\phi}=(-g_{00})^{1/2}$ as it moves out, and the energy flux is further reduced by another factor e^{ϕ} because of the time-dilation of the duration of the pulse.

In the general case, where Eq. (4.15) is not valid, the additional terms in $D_t\lambda$ are therefore gravitational fields produced by moving sources and acting on the neutrinos. Equation (4.14) is equally valid, of course, for an energy flux due to photons or gravitons moving through a transparent medium. By using Eq. (4.2) we can rewrite Eq. (4.14) as

$$D_tL=4\pi R^2nC-e^{-2\phi}D_r(Le^{2\phi})-2L\left[\frac{\partial U}{\partial R}-\frac{L}{R}\alpha\left(1+U^2-\frac{2m}{R}\right)^{-1/2}\right]. \quad (4.16)$$

The last of the field equations is the Einstein equation for $R_{\theta}{}^{\theta}$ or $R_{\varphi}{}^{\varphi}$. Simplifying it as before,[3] we find

$$D_tU=\left(1+U^2-\frac{2m}{R}\right)\frac{\partial\phi}{\partial R}-\frac{m+4\pi R^3(p+q)}{R^2}. \quad (4.17)$$

The first term on the right-hand side here represents the mechanical forces through Eq. (4.13); the second term gives the gravitational forces.

By using Eq. (4.17) to carry out some differentiations in Eq. (4.3) we obtain

$$D_tm=-4\pi R^2pU-L[U+\alpha(1+U^2-2mR^{-1})^{1/2}], \quad (4.18)$$

which can be interpreted in terms of energy conservation.

A complete set of equations for this problem could consist of Eqs. (4.7), (4.8), (4.16), (3.4), and (4.17) giving the time derivatives of the basic independent variables n, e, L (or q), R, and U, together with Eqs. (4.5) and (4.13) to define the auxiliary quantities m and ϕ, and equations of state to give $p(n,e)$, $C(n,e)$, and if desired the temperature $T(n,e)$ or other thermodynamic variables. Equations (4.18) and (4.3) are identities resulting from the above system of equations. Equation (4.6) is a restriction (constraint) on the initial conditions which will be preserved thereafter as an identity from the system of equations just described. When general relativity is initially unimportant so that $2mR^{-1}\ll 1$, Eq. (4.6) can be satisfied trivially; it then merely defines $A(r)$ in terms of the initial values of n, R, and U.

V. BOUNDARY CONDITIONS

Outside the region occupied by the fluid sphere, the metric must deviate from the Schwarzschild form owing to the neutrino flux which also permeates the exterior region. A convenient form of this "radiating Schwarzschild" metric has been found by Vaidya[5] and is discussed in more detail in the accompanying paper.[6] This solution reads

$$ds^2=-[1-2M(u)R^{-1}]du^2-2dudR+R^2d\Omega^2, \quad (5.1)$$

where $M(u)$ is an arbitrary nonincreasing function, and $-dM/du$ is the radiated power reaching infinity at retarded time $u=T-R$. If the boundary surface is described in these external coordinates by

$$R=R_s(u) \quad (5.2)$$

then the induced metric on the surface is

$$(ds^2)_s=-[1+2(dR_s/du)-2MR^{-1}]du^2+R_s{}^2d\Omega^2. \quad (5.3)$$

In comparison, the interior metric (3.1) on the boundary $r=r_s=$constant, is

$$(ds^2)_s=-dt^2+R^2(r_s,t)d\Omega^2 \quad (5.4)$$

when we impose the boundary condition

$$\phi(r_s,r)=0. \quad (5.5)$$

The equality of these two boundary-hypersurface metrics for arbitrary angular displacements $d\Omega^2$ gives the equation of the boundary surface in the exterior coordinates

$$R_s(u)=R(r_s,t)\ . \tag{5.6}$$

The relationship between the distant observer's proper time u and the interval of proper time on the boundary surface dt is then

$$dt=du[U_s+(1+U_s^2-2MR_s^{-1})^{1/2}]\ , \tag{5.7}$$

where $U_s=U(r_s,t)$. The second fundamental forms for this surface can also be compuated in each metric; their equality, equivalent to the equality of first derivatives of the metric, yields the boundary conditions

$$M(u)=m(r_s,t) \tag{5.8}$$

and

$$p(r_s,t)=0\ . \tag{5.9}$$

(We assume that $\alpha=\pm 1$ at the boundary.) From these equations and Eq. (4.19) we find an expression for the luminosity as observed at $R=\infty$,

$$L_\infty=-dM/du=L_s[U_s+(1+U_s^2-2m_sR_s^{-1})^{1/2}]^2\ . \tag{5.10}$$

All the quantities on the right-hand side of this equation are obtained from the interior solution and evaluated at the surface. Note that for a collapsing surface, $U_s<0$, the power output L_∞ vanishes as the fluid surface sinks to the characteristic Schwarzschild limit

$$R_s=2m_s\ . \tag{5.11}$$

The $uR\theta\varphi$ coordinates of Eq. (5.1) do not extend into a region which would match onto the collapsing surface when $R_s<2m_s$.

VI. PHYSICAL INTERPRETATION

By integrating Eq. (4.5) we obtain a formula for $m(r,t)$ which we interpret as being the total energy contained inside a sphere of coordinate radius r, namely,

$$m(r,t)=\int_0^r d^3V\{\epsilon\alpha(1+U^2-2mR^{-1})^{1/2} +q[\alpha U+(1+U^2-2mR^{-1})^{1/2}]\}\ . \tag{6.1}$$

The element of proper volume which appears here is

$$d^3V=4\pi R^2e^{\lambda/2}dr\ . \tag{6.2}$$

In particular, for $r=r_s$ the integral in Eq. (6.1) correctly gives the proper total energy[8] of the system $M(u)$. We find it helpful then to think of $\epsilon\alpha(1+U^2-2mR^{-1})^{1/2}$ as an energy density associated with matter moving in this (spherical) gravitational field, and

$$q[\alpha U+(1+U^2-2mR^{-1})^{1/2}]$$

as an energy density associated with the outgoing radiation. At the surface of the body we had even found in Eq. (5.7) that the factor which multiplies the locally measured energy density q here, is in fact the proper Doppler and red-shift factor which would come in in converting a quantum of local energy at the surface to its value at infinity. Correspondingly, in the exterior Schwarzschild metric,

$$\gamma\equiv(1+U^2-2mR^{-1})^{1/2}\ , \tag{6.3}$$

is the (conserved) total energy per unit mass for a test particle on a radial geodesic, so we are pleased that the same quantity can through Eq. (6.1), be used as the total energy, per gram of local internal energy, inside the fluid sphere. [But note that in regions where R' is negative ($\alpha=-1$), the main ϵ term contributes negatively to m. This is essential in order that this formula yield $m=0$ for spherically symmetric *closed* universes.]

Let us now study Eq. (4.3) which gives the rate of change of this energy per unit local mass γ:

$$D_t\gamma=U\gamma(\partial\phi/\partial R)+\alpha(L/R)\ . \tag{6.4}$$

The terms on the right-hand side here represent the rate at which work is being done on a mass element by (a) mechanical forces per unit mass $\partial\phi/\partial R$ as given by Eq. (4.13) and (b) gravitational forces associated with the neutrino flux L. According to Eq. (6.4) we may consider that the gravitational forces produced by the fluid itself, which survive in the $L=0$ limit, do only that work which is accounted for by the gravitational potential term m/R in γ.

From Eq. (4.18) we see that the conversion of a portion Δm of the mass in a region into radiation, if it takes place in a short proper time interval $\Delta\tau$, corresponds to a flux (we henceforth assume $\alpha=+1$)

$$L\approx(\Delta m/\Delta\tau)(U+\gamma)^{-1}\ . \tag{6.5}$$

As this pulse of radiation passes a fluid particle, the gravitational potential will rise from $-m/R$ to $-(m-\Delta m)/R$ and Eq. (6.4) gives the net-energy change

$$\Delta\gamma=(\Delta m/R)(U+\gamma)^{-1} \tag{6.6}$$

which is always positive. To see the forces which account for this energy increase, we must turn to Eq. (4.17) where only the term

$$-4\pi qR=-(L/R)=-(\Delta m/R\Delta\tau)(U+\gamma)^{-1} \tag{6.7}$$

can be important if we consider impulse where $\Delta\tau\to 0$ for finite Δm. Since $\Delta m>0$, this force is directed toward the center. The corresponding change in U is

$$\Delta U=-(\Delta m/R)(U+\gamma)^{-1}\ . \tag{6.8}$$

Consider now a small amount of radiation which

[8] In the general, nonsymmetrical, asymptotically flat system, no compelling arguments are known to show that there should be a basis for preferring any of several distinct definitions of energy *density*, although total energy should be unambiguous.

might be emitted by the fluid particle just after the pulse of intense radiation went by. Owing to the weaker gravitational binding just after the pulse, one expects this additional radiation to reach infinity with a smaller redshift than if it were emitted ahead of the strong pulse. However, we see that this effect is just canceled by the greater Doppler shift from the moving fluid after the impulse. The total conversion factor from local radiant energy to its value at infinity is, as we have seen in Eqs. (4.18), (5.7), and (6.1), just $U+\gamma$, and from Eqs. (6.6) and (6.8) we compute

$$\Delta(U+\gamma)=0. \tag{6.9}$$

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Vaidya's Radiating Schwarzschild Metric*

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In Vaidya's metric for a radiating sphere,

$$ds^2=-(1-2mr^{-1})du^2-2dudr+r^2d\Omega^2,$$

where $m(u)$ is a nonincreasing function of the retarded time $u=t-r$, we verify that $-dm/du$ is the total power output as given by the Landau-Lifshitz stress-energy pseudotensor, and relate it through red-shift and Doppler-shift factors to the apparent luminosity L for an observer moving radially in this gravitational field. We argue that the hypersurface $r=2m(u)$ cannot be realized physically, but see that a hypersurface $r=2m(\infty)$ at $u=\infty$ (which is not adequately represented in presently available coordinate systems) shows the total red-shift characteristic of the Schwarzschild "singularity." The geodesic equations are written out to display a gravitational "induction field" $-GL/c^3r$ associated with a changing mass in the Newtonian $-Gm/r^2$ field.

I. INTRODUCTION

The metric field surrounding a star, idealized as a *radiating* sphere, cannot be the Schwarzschild solution,[1] except in the excellent approximation in which one neglects the energy density of the emitted radiation. In this paper we investigate the metric outside a spherically symmetric body when radiation is included. For a normal star, the influence of radiation on the metric is negligible when compared with the effects of deviations from spherical symmetry, caused by rotation, magnetic fields, etc.[2] Nevertheless, this metric may have some relevance to the study of a collapsing supernova core,[3] if one allows for the production of a copious supply of neutrinos, but neglects their subsequent absorption in the outer envelope. A realistic treatment must, of course, analyze the problem of neutrino transport in detail. The solution described here is thus chiefly useful as an extreme limiting case, in which the neutrino optical depth of the envelope is negligible.

We therefore seek a spherically symmetric solution of the Einstein equations[4]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu} \tag{1.1}$$

with the "geometrical optics" stress-energy tensor of

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[4] Throughout this paper we choose units such that $G=1$, $c=1$.