Lower Bounds on the Shrinking of Diffraction Peaks*

K. L. KOWALSKI
Department of Physics, Case Institute of Technology, Cleveland, Ohio
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Lower bounds on the width of a diffraction peak are found using unitarity and analyticity in $\cos\theta$ in Lehmann-type ellipses. When the forward elastic-scattering amplitude is predominantly imaginary, the lower bound obtained is proportional to $(\ln s)^{-2}$. In deriving this result no restrictions, except those imposed by unitarity and analyticity, were made concerning the asymptotic behavior of the total cross section.

UPPER bounds on the width w of the diffraction peak at high energies[1] have been deduced on the basis of unitarity and certain other reasonable assumptions.[2–5] On the other hand, it seems necessary to introduce relatively stronger assumptions concerning either the analyticity of the elastic scattering amplitude in $z=\cos\theta$ or the asymptotic behavior of this amplitude in order to derive a lower bound on w.[6] In this note we will exploit these analyticity postulates for the purpose of obtaining a lower bound on w which is useful if the forward elastic amplitude is predominantly imaginary. This last circumstance appears to be realized in the asymptotic limit.[7]

The width of the diffraction peak is defined by[8]

$$w=\tfrac{1}{2}\{\sigma(s,t)[d\sigma(s,t)/dt]^{-1}\}_{t=0},$$

where

$$\sigma(s,t)=(\pi/k^2)(d\sigma/d\Omega),$$

and $(d\sigma/d\Omega)$ is the ordinary differential cross section. The square of the c.m. energy is denoted by s, and $t=-2k^2(1-z)$.

The partial-wave expansion for the elastic amplitude $f(s,z)$ will be written in the form

$$f(s,z)=(s^{1/2}/k)\sum_l (2l+1)f_l(s)P_l(z),$$

where $P_l(z)$ is the Legendre polynomial of degree l and the normalization is such that

$$\sigma(s,t)=(\pi/sk^2)|f(s,z)|^2.$$

Clearly

$$\left|\frac{d\sigma}{dt}\right|\leqslant\left(\frac{2\pi}{k^2s}\right)|f|\left|\frac{df}{dt}\right|,$$

and so

$$|w|\geqslant\frac{1}{4}\left(|f|\left|\frac{df}{dt}\right|^{-1}\right)_{z=1}.$$

We will now proceed to find an upper bound for $|df/dt|$ in the high-energy limit and thereby obtain a lower bound for $|w|$.

We assume that $f(s,z)$ is analytic in a certain domain of the complex z plane which includes an ellipse with foci at $z=\pm1$ and, moreover, that within this domain $|f(s,z)|$ is bounded by a finite polynomial in s for s very large. What is relevant about this domain for the problem at hand is how the intersections z_0 of its boundary with the real line (semimajor axes of the ellipse) approach the physical line as $s\to\infty$. We consider two cases

$$|z_0|\sim1+as^{-2}, \qquad (1a)$$

$$|z_0|\sim1+bs^{-1}, \qquad (1b)$$

where a and b are constants. The first case corresponds to analyticity within a usual Lehmann ellipse[9] and (1b) is in accord with the type of analyticity implied by the Mandelstam representation.[10,11] Now

$$\left|\frac{df}{dt}\right|\leqslant\left(\frac{s^{1/2}}{2k^3}\right)F(1), \qquad (2)$$

where we have introduced the notation

$$F(n)=\sum_l (2l+1)|f_l|x_l^{\,n}, \quad n=0,1,\cdots,$$

and

$$x_l=\tfrac{1}{2}l(l+1).$$

However, using the Schwartz inequality we note that[12]

$$[F(1)]^2\leqslant F(0)F(2).$$

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[1] We consider the usual example of the scattering of two massive, spinless particles. We also set $\hbar=c=1$ and scale all energies in terms of some suitable mass.

[2] For example, it is implicitly assumed that the total cross section does not decrease as fast as k^{-2}, where k is the magnitude of the c.m. momentum. In Refs. 3 and 5 the forward scattering is regarded as being mainly absorptive.

[3] A. Martin, Phys. Rev. **129**, 1432 (1963).

[4] E. Leader, Phys. Letters **5**, 75 (1963).

[5] S. W. MacDowell and A. Martin, Phys. Rev. **135**, B960 (1964).

[6] A. C. Finn, Phys. Rev. **132**, 836 (1963).

[7] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).

[8] This is of physical interest, of course, only if $(d\sigma/dt)>0$ at $t=0$, that is, if a "diffraction peak" exists. On the basis of the general assumptions made about the scattering amplitude in this paper there is no reason to expect this to be true. However, if the forward-scattering amplitude is primarily absorptive, then $w>0$.

[9] H. Lehmann, Nuovo Cimento **10**, 579 (1958).

[10] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[11] However, as emphasized by Martin (Ref. 3), this is by no means equivalent to assuming the validity of the Mandelstam representation.

[12] This is permissible since all series involved are composed of positive terms and are convergent since for large l, f_l decreases faster than any power of l as a consequence of our assumed analyticity in z.

After n repetitions of this procedure one finds

$$[F(1)]^2 \leqslant [F(0)]^{p(n)} [F(2^n)]^{q(n)}. \quad (3)$$

Here

$$p(n) = \sum_{m=0}^{n-1} (\tfrac{1}{2})^m,$$

and

$$q(n) = (\tfrac{1}{2})^{n-1}.$$

Unitarity coupled with our previous analyticity assumptions enables one to establish[6] using standard techniques[3,10,13] that for s sufficiently large

$$F(1) < C_1 (\ln s/\alpha)^4.$$

where $\alpha = a' k^{-2}$ or $\alpha = b' k^{-1}$ corresponding to (1a) or (1b), respectively, and where a', b', and C_1 are constants. These arguments can easily be extended to $F(n)$ and one finds

$$F(n) < C_n (\ln s/\alpha)^{2n+2}, \quad (4)$$

with C_n essentially given by

$$C_n = (n+1)^{-1} N^{2n+2};$$

N is the degree of the polynomial in s required to bound the amplitude in the relevant domain of the z plane.

Combining Eqs. (2)–(4) we see that

$$\left|\frac{df}{dt}\right| < \left(\frac{s^{1/2}}{2k^3}\right) [F(0)]^{\frac{1}{2}p(n)} (C_{2^n})^{\frac{1}{2}q(n)} \left(\frac{\ln s}{\alpha}\right)^{2+q(n)}. \quad (5)$$

The right-hand side of (5) is evidently convergent as $n \to \infty$; in fact, it follows that in this limit we have

$$\left|\frac{df}{dt}\right| < C\left(\frac{s^{1/2}}{k^3}\right) F(0) \left(\frac{\ln s}{\alpha}\right)^2.$$

[13] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

Thus, for large s

$$|w| > C' [|f(s,1)|/F(0)] (\alpha k/\ln s)^2. \quad (6)$$

The bound (6) is not very useful unless one can determine a nontrivial lower bound for $[|f(s,1)|/F(0)]$. One way of doing this is to employ the upper bound (4) for $F(0)$. Then

$$|w| > C'' [\sigma(s,0)]^{1/2} (k\alpha/\ln s)^4. \quad (7)$$

Despite the apparent difference in form, the bound (7) is essentially that already found by Finn.[6]

However, if the forward-scattering amplitude is predominantly imaginary then the ratio $[|f(s,1)|/F(0)]$ is of order unity. Then, since $w > 0$ under these conditions,

$$w > C' (k \ln s)^{-2}, \quad (8a)$$

$$w > C' (\ln s)^{-2}, \quad (8b)$$

corresponding to the behaviors (1a) and (1b), respectively.

The result (8b) is very close to the form for w which follows from a Regge-type asymptotic behavior for the elastic amplitude, namely, (const)$\times(\ln s)^{-1}$. Finally, we emphasize that we have made no assumptions[14] about the total cross section other than the bounds which are implicit in unitarity and our analyticity postulates.[3,10,13]

Note added in proof. After the submission of this paper for publication Professor T. Kinoshita kindly informed me that he had already obtained the result (8b) under identical assumptions using a technique introduced by Sugawara (Ref. 14). [T. Kinoshita, Lectures Presented at the Conference on Particles and High Energy Physics, University of Colorado, 1964 (unpublished).] However, the analysis and the principal result [Eq. (6)] of the present paper are somewhat more general than Kinoshita's in that the restriction to purely absorptive scattering is not required except to obtain the specific bounds (8).

[14] H. Sugawara, Progr. Theoret. Phys. (Kyoto) **30**, 404 (1963).