General S-Matrix Methods for Calculation of Perturbations on the Strong Interactions*

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Recently, the authors proposed an on-the-mass-shell, S-matrix method for computing the effects of small perturbations on the masses and coupling constants of strongly interacting particles. In the present paper, the method is generalized to the multichannel case. The use of group-theoretical techniques in reducing the complexity of the method is described in detail.

I. INTRODUCTION

RECENTLY, the authors proposed an on-the-mass-shell, S-matrix method[1] for computing the effects of small perturbations on the masses and coupling constants of strongly interacting particles. In this method, particles appear as poles in scattering amplitudes, and weak, electromagnetic, or strong perturbations cause changes in the positions and residues of the poles. Computation of these changes yields the mass and coupling shifts, respectively. The dispersion integrals in the method converge rapidly, and a detailed calculation of the neutron-proton electromagnetic mass difference yielded a result[2] in good agreement with experiment.

In the present paper, we extend and amplify the method preparatory to applying it to a wide range of further problems. Then in the following paper, the method is used to investigate electromagnetic and strong $SU(3)$ symmetry violations in the masses of the $J=\frac{1}{2}^+$ octet and the $J=\frac{3}{2}^+$ decuplet. Some results of the latter calculation, together with a unified discussion of octet enhancement in strong, electromagnetic, and weak violations of $SU(3)$ symmetry, have also been given in a recent letter.[3,4]

The first generalization contained in the present paper is the matrix formalism for obtaining mass and coupling shifts in a multichannel problem. The formalism for the nondegenerate case is presented in Sec. II, and the case of initially degenerate channels is treated in Sec. III.

In our second generalization we discuss how to exploit the fact that, in small violations of a symmetry such as $SU(2)$ or $SU(3)$ invariance, the ratios of many terms follow from group theory independently of the detailed dynamics. This subject is illustrated in Sec. IV by a study of electromagnetic violations of isotopic spin invariance in the ρ meson bootstrap. Section V contains a more general description of the use of group theoretical techniques in reducing the complexity of our matrix formalism.

The paper has been written in such a way that the reader can study the group theoretical techniques of Secs. IV and V without having previously studied in detail the dispersion relations in Secs. II and III.

II. PERTURBATION FORMULA FOR THE MANY-CHANNEL ND^{-1} METHOD

In this section we wish to develop some perturbation techniques based on the partial wave dispersion relations for the scattering amplitude connecting several two-particle channels. Our goal is to derive explicit formulas for the first-order changes in the amplitude, and in particular, changes in the position and residue of bound state poles, in terms of the changes in the left cut and kinematics of the problem.

Let us begin by briefly reviewing our treatment of the one-channel case.[1] The partial wave scattering amplitude T for this case can be written in the form[5]

$$T=e^{i\eta}\sin\eta/\rho\,, \tag{1}$$

where η is the phase shift and ρ^{-1} is a factor which removes the kinematic singularities. We assume that T is an analytic function of the energy variable s with the usual left and right cuts, and that the unperturbed T is known. The first-order effect of a perturbation is

$$\delta T=(\delta\eta/\rho)e^{2i\eta}-(\delta\rho/\rho)(e^{i\eta}\sin\eta/\rho)\,. \tag{2}$$

Recalling that the denominator function D for the unperturbed problem has the phase $e^{-i\eta}$ along the right cut, one finds that the discontinuity across the right cut in the function $J(s)=D^2\delta T(s)$ is simply

$$\mathrm{Im}J(s)=\mathrm{Im}(D^2\delta T)=-\mathrm{Im}((\delta\rho/\rho)DN)=N^2\delta\rho\,, \tag{3}$$

where we have set $T=ND^{-1}$ and used $\mathrm{Im}D=-\rho N$ and $\mathrm{Im}N=0$ along the right cut. Since D has no left cut, we have

$$\mathrm{Im}J=D^2\,\mathrm{Im}\delta T \tag{4}$$

along the left-hand singularities and a simple applica-

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[1] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).
[2] R. Dashen, Phys. Rev. **135**, B1196 (1964).
[3] R. Dashen and S. Frautschi, Phys. Rev. Letters **13**, 499 (1964).
[4] R. Dashen, S. Frautschi, M. Gell-Mann, and Y. Hara, Proceedings of the International Conference on High Energy Physics at Dubna, 1964 (unpublished).
[5] Note that for purposes of making the extension to the many-channel case, we have inverted the meaning of ρ from our previous papers (Refs. 1 and 2).

tion of Cauchy's theorem yields

$$J(s)=\frac{1}{\pi}\int_R \frac{N^2\delta\rho}{s'-s}ds'+\frac{1}{\pi}\int_L \frac{D^2\,\mathrm{Im}\delta T}{s'-s}ds', \quad (5)$$

$$\delta T(s)=J(s)D^{-2}(s),$$

where the integrals L and R run over the left and right cuts. Now let us suppose that the unperturbed problem has a bound state pole at $s=s_B$ so that $T\sim R/(s-s_B)$ and

$$\delta T\sim\frac{R\delta s_B}{(s-s_B)^2}+\frac{\delta R}{s-s_B} \quad (6)$$

near $s=s_B$. Since D^2 has a double zero at $s=s_B$, J has no additional singularities and (5) is still valid. Multiplying both sides of (5) by $(s-s_B)^2$ [note that $(s-s_B)D^{-1}$ is well behaved near $s=s_B$] yields

$$R\delta s_B=J(s_B)\lim_{s\to s_B}\left[\frac{s-s_B}{D(s)}\right]^2=J(s_B)D'^{-2}(s_B) \quad (7)$$

and

$$\delta R=\frac{d}{ds}\left[J(s)\left(\frac{s-s_B}{D(s)}\right)^2\right]\Bigg|_{s=s_B}, \quad (8)$$

where $J(s)$ is given by (5).

In the procedure described above, we multiplied δT by D^2 in order to remove the unitarity part of the right-hand cut and the double pole that will appear if there is a shift in the mass of the bound state. Alternatively, we might have tried multiplying δT by $(T^{-1})^2$, which would also remove the unitarity part of the right cut and the double pole. This alternative procedure, however, has several drawbacks:

(i) Any zeros which T may have produce new double poles in $(T^{-1})^2\delta T=N^{-2}D^2\delta T$, which are not present in $D^2\delta T$.

(ii) The function $N^{-2}D^2\delta T$ has a more complicated left cut than $D^2\delta T$.

(iii) The dispersion relation for $N^{-2}D^2\delta T$ is likely to have worse convergence at large s than does the relation for $D^2\delta T$.

One might also have tried simply multiplying δT by $(s-s_B)^2$, which would remove the double pole and would be free of the first two difficulties we encountered with $(T^{-1})^2\delta T$. However, the dispersion relation for $(s-s_B)^2\delta T$ very likely diverges, whereas the function $D(s)$ responsible for a bound state is likely to grow no faster than powers of $\ln s$ at large s, which makes $D^2\delta T$ much more convergent. In addition, the right cut of $(s-s_B)^2\delta T$ contains "unitarity terms," whereas the use of $D^2\delta T$ provides a calculation of shifts in the dominant "unitarity terms" (i.e., bound states or resonances) from an input which includes kinematic shifts on the right cut and shifts in "force terms" on the left cut but no "unitarity terms."

In the present paper we wish to generalize Eqs. (5)–(8) to the case of n two-body channels where the partial-wave amplitude $\mathbf{T}$ is a symmetric n-by-n matrix. To this end, we note that along the right cut $\mathrm{Im}\mathbf{T}^{-1}=-\boldsymbol{\varrho}$, where $\boldsymbol{\varrho}$ is a matrix which is completely determined by the kinematics of the problem; hence,

$$\mathrm{Im}\delta(\mathbf{T}^{-1})=-\mathrm{Im}\mathbf{T}^{-1}\delta\mathbf{T}\mathbf{T}^{-1}=-\delta\boldsymbol{\varrho} \quad (9)$$

along the right cut. For the same reasons as in the one-channel case, however, $\mathbf{T}^{-1}\delta\mathbf{T}\mathbf{T}^{-1}$ is not the best function to consider. Instead we assume that the unperturbed amplitude has been obtained in the form[6] $\mathbf{T}=\mathbf{N}\mathbf{D}^{-1}=\mathbf{D}^{T^{-1}}\mathbf{N}^T$ ($\mathbf{N}^T$ is the transpose of $\mathbf{N}$) and using the fact that $\mathbf{N}$ has no right cut, we write Eq. (9) as

$$\mathbf{N}^{T^{-1}}\,\mathrm{Im}(\mathbf{D}^T\delta\mathbf{T}\mathbf{D})\mathbf{N}^{-1}=\delta\boldsymbol{\varrho} \quad (10)$$

or

$$\mathrm{Im}(\mathbf{D}^T\delta\mathbf{T}\mathbf{D})=\mathbf{N}^T\delta\boldsymbol{\varrho}\mathbf{N}. \quad (11)$$

Thus the n-channel generalization of the one-channel function J is the matrix function $\mathbf{J}(s)=\mathbf{D}^T(s)\delta\mathbf{T}(s)\mathbf{D}(s)$, and proceeding in the same way as before, we find[7]

$$\mathbf{J}(s)=\frac{1}{\pi}\int_L\frac{\mathbf{D}^T(s')\,\mathrm{Im}\delta\mathbf{T}(s')\mathbf{D}(s')}{s'-s}ds' +\frac{1}{\pi}\int_R\frac{\mathbf{N}^T(s')\delta\boldsymbol{\varrho}(s')\mathbf{N}(s')}{s'-s}ds', \quad (12)$$

$$\delta\mathbf{T}(s)=\mathbf{D}^{T^{-1}}(s)\mathbf{J}(s)\mathbf{D}^{-1}(s).$$

Again, let us assume that the unperturbed problem has a bound-state pole at $s=s_B$ so that $\mathbf{T}\sim\mathbf{R}/(s-s_B)$ near $s=s_B$, where $\mathbf{R}$ is the residue matrix which in terms of the couplings f_i of the bound state to the various channels $i=1\cdots n$ is $R_{ij}=-f_if_j$. The change in the amplitude will then behave like

$$\delta\mathbf{T}\sim\frac{\mathbf{R}}{(s-s_B)^2}\delta s_B+\frac{\delta\mathbf{R}}{s-s_B}, \quad (13)$$

where $\delta\mathbf{R}$ is the change in the residue matrix; $\delta R_{ij}=-f_i\delta f_j+\delta f_if_j$. From (12), it follows that

$$\mathbf{R}\delta s_B=[\lim_{s\to s_B}(s-s_B)\mathbf{D}^{-1}(s)^T] \times\mathbf{J}(s_B)[\lim_{s\to s_B}(s-s_B)\mathbf{D}^{-1}(s)], \quad (14)$$

$$\delta\mathbf{R}=(d/ds)[(s-s_B)\mathbf{D}^{-1}(s)^T \times\mathbf{J}(s)(s-s_B)\mathbf{D}^{-1}(s)]|_{s=s_B}, \quad (15)$$

where $\mathbf{J}$ is given in terms of the left and right cuts of $\delta\mathbf{T}$ by (12). In order to simplify our future formulas, it is

[6] J. Bjorken, Phys. Rev. Letters 4, 473 (1960).

[7] If some channels are not included in T, an inelasticity term must be added to the right cut, as in Ref. 1.

convenient to introduce the notation

$$\mathbf{\Delta}=\lim_{s\to s_B}(s-s_B)\mathbf{D}^{-1}(s)\quad\text{and}\quad \mathbf{\Delta}'=(d/ds)[(s-s_B)\mathbf{D}^{-1}(s)]|_{s=s_B}.$$

Then, multiplying both sides of (14) by $\mathbf{R}$, setting $R_{ij}=-f_if_j$, and taking the trace, one finds

$$\delta s_B=-(\sum_k f_k^2)^{-2}\sum_{ij} f_if_j(\mathbf{\Delta}^T\mathbf{J}(s_B)\mathbf{\Delta})_{ij}. \tag{16}$$

Using the fact that $\mathbf{R}=\mathbf{N}(s_B)\mathbf{\Delta}$, one can also verify that[8]

$$\delta f_i=-(\sum_k f_k^2)^{-1}\sum_j f_j(\mathbf{\Delta}'^T\mathbf{J}(s_B)\mathbf{\Delta}+\tfrac{1}{2}\mathbf{\Delta}^T\mathbf{J}'(s_B)\mathbf{\Delta})_{ij}, \tag{17}$$

where $\mathbf{J}$ is again given by (12).

Finally, one might be interested in the changes in the position δs_r and couplings of a resonance rather than a bound state. In this case, the poles $\mathbf{R}\delta s_r/(s-s_r)^2+\delta\mathbf{R}/(s-s_r)$ lie on the second Riemann sheet of the function $\delta\mathbf{T}(s)$. Of course, $\mathbf{D}^{-1}$ also has a pole on the second sheet, so we obtain, as before

$$\mathbf{R}\delta s_r=[\lim_{s\to s_r}(s-s_r)\mathbf{D}_{(2)}^{-1\,T}(s)]\times\mathbf{J}_{(2)}(s_r)[\lim_{s\to s_r}(s-s_r)\mathbf{D}_{(2)}^{-1}(s)], \tag{18}$$

$$\delta\mathbf{R}=(d/ds)[(s-s_r)\mathbf{D}_{(2)}^{-1\,T}(s)\times\mathbf{J}_{(2)}(s)(s-s_r)\mathbf{D}_{(2)}^{-1}(s)]|_{s=s_r}, \tag{19}$$

where $\mathbf{D}_{(2)}$ and $\mathbf{J}_{(2)}$ refer to the values of $\mathbf{D}$ and $\mathbf{J}$ on the second sheet. Now $\mathbf{D}_{(2)}$ is, of course, known and to find $\mathbf{J}_{(2)}$ one need only deform the contour of integration in the second integral of Eq. (12), as shown in Fig. 1. Replacing the indented contour by a contour along the real axis plus a loop around the pole at $s=s_r$, one finds

$$\mathbf{J}_{(2)}(s_r)=\frac{1}{\pi}\int_L\frac{\mathbf{D}^T\,\mathrm{Im}\delta\mathbf{T}\mathbf{D}}{s'-s_r}ds'+\frac{1}{\pi}\int_R\frac{\mathbf{N}^T\delta\boldsymbol{\rho}\mathbf{N}}{s'-s_r}ds'+2i\mathbf{N}^T(s_r)\delta\boldsymbol{\rho}(s_r)\mathbf{N}(s_r),$$

where the integral R runs along the real axis and $s_r=\mathrm{Re}s_r+i\,\mathrm{Im}s_r$ with $\mathrm{Im}s_r<0$. Equations (18) and (19) can, of course, be written in the same form as (16) and (17). Note that now δs_r has an imaginary part, just as it should.[9]

Equations (16) and (17) are completely general and, as a result, somewhat cumbersome. In practice, $\mathbf{D}$ will often have some properties which can be used to simplify (16) and (17). For example, consider a two-channel situation where the channels were decoupled before the perturbation was turned on. In this case, N_{ij} and D_{ij} will have the form $N_{ij}=N_i\delta_{ij}$, $D_{ij}=D_i\delta_{ij}$, $i,j=1$, 2 and Eq. (12) can be reduced to

$$\delta T_{ij}=D_i^{-1}D_j^{-1}J_{ij}=D_i^{-1}D_j^{-1}\left[\frac{1}{\pi}\int_L\frac{D_iD_j\,\mathrm{Im}\delta T_{ij}}{s'-s}ds'+\frac{1}{\pi}\int\frac{N_iN_j\delta\rho_{ij}}{s'-s}ds'\right]\quad(i,j=1,2). \tag{20}$$

If the unperturbed problem had a bound state in channel one so that $f_1=f$ and $f_2=0$, Eqs. (16) and (17) become

$$\delta s_B=-f^{-2}[D_1'(s_B)]^{-2}\left[\frac{1}{\pi}\int_L\frac{D_1^2\,\mathrm{Im}\delta T_{11}}{s'-s_B}ds'+\frac{1}{\pi}\int_R\frac{N_1^2\delta\rho_{11}}{s'-s_B}ds'\right], \tag{21}$$

$$\delta f_1=-f^{-1}[D_1'(s_B)]^{-2}\left[\frac{1}{\pi}\int_L\frac{D_1^2\,\mathrm{Im}\delta T_{11}}{(s'-s_B)^2}ds'+\frac{1}{\pi}\int_R\frac{N_1^2\delta\rho_{11}}{(s'-s_B)^2}ds'\right]+f^{-1}D''(s_B)[D'(s_B)]^{-3}\left[\frac{1}{\pi}\int_L\frac{D_1^2\,\mathrm{Im}\delta T_{11}}{s'-s_B}ds'+\frac{1}{\pi}\int_R\frac{N_1^2\delta\rho_{11}}{s'-s_B}ds'\right], \tag{22}$$

$$\delta f_2=-f^{-1}[D_1'(s_B)]^{-1}[D_2(s_B)]^{-1}\left[\frac{1}{\pi}\int_L\frac{D_1D_2\,\mathrm{Im}\delta T_{12}}{s'-s_B}ds'+\frac{1}{\pi}\int_R\frac{N_1N_2\delta\rho_{12}}{s'-s_B}ds'\right]. \tag{23}$$

Equations (21) and (22) are, of course, just our previous one-channel equations.[1,10] Note that here δs_B depends only on the (1,1) elements of the perturbation—just as it does in Schrödinger theory to first order. Some

[8] To derive Eq. (17), consider $f_i\delta f_j+f_j\delta f_i=-(\Delta^{T\prime}J\Delta+\Delta^TJ\Delta'+\Delta^TJ'\Delta)_{ij}$. On the left side write $\delta f_i=\epsilon f_i+\delta\hat{f}_i$, where $\sum_i f_i\delta\hat{f}_i=0$; the left side is now $2\epsilon f_if_j+f_i\delta\hat{f}_j+f_j\delta\hat{f}_i$. Use the relation $\mathbf{\Delta}=\mathbf{N}^{-1}\mathbf{R}$ to convert the right side to $(\Delta^{T\prime}JN^{-1}f)_if_j+f_i(fN^{-1T}J\Delta')_j-f_i(fN^{-1T}J'N^{-1}f)f_j$. The third term on the right contributes only to f_if_j on the left, the first term on the right contributes to f_if_j and $f_j\delta\hat{f}_i$ on the left, and the second term on the right contributes to f_if_j and $f_i\delta\hat{f}_j$. Therefore, the combination $(\Delta^{T\prime}J\Delta+\frac{1}{2}\Delta^TJ'\Delta)_{ij}$ equals $\alpha(\epsilon f_if_j)+\beta(f_j\delta\hat{f}_i)$ with α and β to be determined. Interchanging i and j, one obtains a second relation, $(\Delta^TJ\Delta'+\frac{1}{2}\Delta^TJ'\Delta)_{ij}=\alpha(\epsilon f_if_j)+\beta(f_i\delta\hat{f}_j)$. Adding the two relations and comparing to the original equation, we find $\alpha=\beta=-1$ so that the first relation can be written $f_j\delta f_i=-(\Delta^{T\prime}J\Delta+\frac{1}{2}\Delta^TJ'\Delta)_{ij}$, from which Eq. (17) is obtained by multiplying by f_j and summing over j.

[9] Our formalism is not entirely adequate, however, for resonances just above threshold, or for very lightly bound states. These cases may require special treatment because terms of higher order in the perturbation may become important, particularly for S waves. One source of higher order terms is cusp effects at threshold [see, for example, S. Frautschi, Phys. Letters 8, 141 (1964)]. In electromagnetic interactions there are also the higher order Coulomb effects which must be included near threshold.

[10] In Ref. 1, there is a spurious factor of 2 multiplying the $D''D'^{-3}$ term in the equation for δf_1.

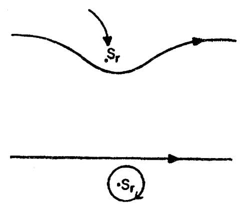

FIG. 1. Deformation of the contour to find the shifts in position and couplings of a resonance.

specific problems where Eqs. (21)–(23) could be applied are:

(i) Suppose we take channel one to be the $J=\frac{1}{2}^+$, $I=\frac{1}{2}$, $I_3=\frac{1}{2}$ πN state, channel two to be the $J=\frac{1}{2}^+$, $I=\frac{3}{2}$, $I_3=\frac{1}{2}$, πN state and take the e^2 electromagnetic corrections to the πN interaction as our perturbation. The proton appears as a bound state in channel one and δs_B will be the proton electromagnetic mass shift and the δf's will be electromagnetic corrections to the πN couplings; in fact, Eq. (21) is essentially that which was used to calculate the proton-neutron mass difference.[2]

(ii) Take the $J=\frac{1}{2}^+$, $I=\frac{1}{2}$, πN state for channel one and the $J=\frac{1}{2}^-$, $I=\frac{1}{2}$, πN state for channel two and let the perturbation be the weak nonparity conserving part of the πN interaction. Again, the nucleon appears as a bound state in channel one, but this time there is no first-order mass shift δs_B because the perturbation does not connect channel one to itself. Here, the interesting quantity is δf_2 which is the parity-violating part of the πN coupling.

(iii) Again let us take channel one to be the $J=\frac{1}{2}^+$, $I=\frac{1}{2}$, πN state but now let channel two be the $J=\frac{1}{2}^+$, γN state. In the unperturbed problem, we neglect all electromagnetic interactions and take the first-order (in e) electromagnetic interactions as our perturbation. Here, there is no scattering in channel two in the unperturbed problem (in fact, there is no scattering in channel two to first order in the perturbation either) so we can take D_2 to be a constant (note that a constant D_2 will cancel out of our formulas). As in example (ii), the perturbation does not connect channel one to itself and the interesting quantity is δf_2, which, apart from kinematic factors, is the nucleon magnetic moment.[10a] One will note, however, that our first-order equations are homogeneous so we can calculate ratios like (nucleon magnetic moment)/(pion charge) but not the absolute scale of electromagnetic interactions. Finally, we note that parameters associated with leptonic decays, e.g., weak magnetic moments and induced pseudoscalar terms, could be treated in a manner similar to the ordinary magnetic moment.

The preceding examples were simple because the different channels decoupled in the unperturbed problem. Generally speaking, this will happen only when some conservation law [e.g., parity in example (ii)] prevents the channels from mixing, and there are many cases where this simplification is not present. For example, the octet amplitudes for baryon-pseudoscalar-meson scattering in $SU(3)$, with 8_S and 8_A mixing, present a true two-channel problem. The basic difference between the above examples where one can make an energy-independent diagonalization of the unperturbed amplitude, and intrinsically more complicated problems where one cannot, is illustrated in the following simple example. Consider a situation where $\delta\boldsymbol{\varrho}=0$ and $\mathrm{Im}\,\delta\mathbf{T}=\pi\mathbf{C}\delta(s-s_0)$. Let us also suppose that the unperturbed problem has a bound state which couples only to channel one, i.e., $f_k=f\delta_{1k}$. Then from (16), we have

$$\delta s_B=-f^{-2}\sum_{ij}(\mathbf{\Delta D}(s_0))_{j1}(\mathbf{\Delta D}(s_0))_{i1}C_{ij}\,, \tag{24}$$

and in the particular case where the different channels are decoupled before the perturbation is applied, $\mathbf{\Delta D}$ is diagonal and

$$\delta s_B=-f^{-2}(\mathbf{\Delta D}(s_0))_{11}{}^2C_{11}\,. \tag{25}$$

Now the point of this example is that if the unperturbed channels decouple, δs_B depends only on C_{11} for any s_0, but in the general case the particular combination of the C_{ij} that contributes to δs_B will depend on s_0. This is, of course, in agreement with Schrödinger equation theory, in which the first-order change in energy of a bound state is

$$\delta E=\sum_{ij}\int_1 \psi_i^*(r)V_{ij}(r)\psi_j(r)dr\,, \tag{26}$$

where ψ_i is the wave function ($\sum_i\int|\psi_i|^2dr=1$) and V_{ij} is the perturbing potential. Here, if the channels decouple in the unperturbed problem, then $\psi_i(r)=0$ for $i\neq 1$ and δE depends only on V_{11}; but in general, δE will depend on a different combination of the V_{ij} for each value of r in the integral.

III. DEGENERATE PERTURBATIONS: THE MASS MATRIX

In the previous section we dealt only with problems in which there is a single bound state at a given energy. Our future applications of the formalism will be mostly concerned with violations of $SU(2)$ and $SU(3)$ symmetries, where one has to deal with problems in which the unperturbed solution has several degenerate poles.

To see what we should expect in this situation, let us review the analogous problem in Schrödinger equation theory. Suppose we start with a Hamiltonian which has two bound states ψ_1 and ψ_2 with energies E_1 and E_2 and then add a perturbing potential V. The first-order changes in the energies and wave functions are $\delta E_1=V_{11}$, $(E_2=V_{22}$, $\delta\psi_1=V_{12}\psi_2/(E_1-E_2)+\cdots$, and $\delta\psi_2=V_{21}\psi_1/(E_2-E_1)+\cdots$, where

$$V_{ij}=\int\psi_i^*V\psi_jdr\,.$$

Now if $E_2\approx E_1$, the first-order corrections to the wave

[10a] R. Dashen, Phys. Letters **11**, 89 (1964).

functions are large and lowest order perturbation theory cannot be expected to give good results. However, if E_2 is *exactly* equal to E_1, one can choose for the unperturbed wave functions any two linear combinations ψ_1' and ψ_2' of the original ψ's. In particular, one can choose ψ_1' and ψ_2' so that $V_{1'2'}=V_{2'1'}=0$, which makes the first-order correction to the wave function finite. The first-order energy changes are then $V_{1'1'}$ and $V_{2'2'}$, which are, of course, just the eigenvalues of V_{ij}, since V_{ij} is diagonal in the $1'$, $2'$ representation.

Now in our dispersion theoretic approach, the analog of $\delta\psi$ is δf_k so we would expect our equations for δf_k to blow up when the unperturbed problem has two bound states at the same energy. That this is, in fact, the case can be seen from Eq. (23) for δf_2 which contains a factor $D_2^{-1}(s_B)$ that becomes large if there is a bound state in channel two with a mass close to s_B. In the next paragraph we will show how this difficulty can be avoided by diagonalizing the mass perturbation, just as one does in the Schrödinger theory.

To see how the present formalism works when there are degenerate poles in the unperturbed problem, let us consider an n-channel problem, where the unperturbed solution contains n degenerate bound-state poles all at $s=s_B$ and all with the same residue f^2, i.e., $T_{ij}\sim -f^2\delta_{ij}/(s-s_B)$ near $s=s_B$. Since the poles are degenerate, we have some freedom in what we choose for our unperturbed states or "particles." More precisely, given any set of numbers $e_i^\alpha(\alpha,i=1\cdots n)$ which satisfy $\sum_\alpha e_i^\alpha e_j^\alpha=\delta_{ij}$ and $\sum_j e_j^\alpha e_j^\beta=\delta_{\alpha\beta}$, we can define "particle" or pole α to be the pole $-f^2 e_i^\alpha e_j^\alpha/(s-s_B)$ whose coupling to channel i is fe_i^α; summing the n poles in $\mathbf{T}$, we recover

$$T_{ij}\sim -\sum_\alpha \frac{f^2 e_i^\alpha e_j^\alpha}{s-s_B}=\frac{-f^2\delta_{ij}}{s-s_B}.$$

Choosing a set of poles defined by a particular set of e_i^α is analogous to choosing a particular set of unperturbed wave functions in the Schrödinger theory. Now after the perturbation has been turned on, the amplitude will have n bound state poles like $f_i^\alpha f_i^\alpha/(s-s_B^\alpha)$, $\alpha=1\cdots n$, where s_B^α is the position of the αth pole and f_i^α is the coupling of particle α to channel i. If the perturbation is to be small, f_i^α must be of the form $f_i^\alpha=fe_i^\alpha+\delta f_i^\alpha$, where δf_i^α is small and the e_i^α are some, as yet unspecified, set of couplings for the unperturbed problem. Then the first-order charge in the amplitude will behave like

$$\delta T_{ij}\sim (f^2/(s-s_B)^2)\sum_\alpha e_i^\alpha e_j^\alpha \delta s_B^\alpha + (f/(s-s_B))\sum_\alpha e_i^\alpha \delta f_j^\alpha + \delta f_i^\alpha e_j^\alpha \quad (27)$$

near $s=s_B$, which looks like the nondegenerate case of the previous section with δs_B replaced by the real symmetric matrix $(\delta s_B)_{ij}=\sum_\alpha e_i^\alpha e_j^\alpha \delta s_B^\alpha$. Evidently, $(\delta s_B)_{ij}$ is given by

$$f^2(\delta s_B)_{ij}=-(\mathbf{\Delta}^T\mathbf{J}(s_B)\mathbf{\Delta})_{ij} \quad (28)$$

in the notation of the previous section. Since the δs_B^α and e_i^α, $\alpha=1\cdots n$, are just the eigenvalues and eigenvectors of $(\delta s_B)_{ij}$; they are completely determined by Eq. (28). Once e_i^α has been determined by diagonalizing $(\delta s_B)_{ij}$, then δf_i^α, as can be easily verified, is equal to

$$\delta f_i^\alpha=-f^{-1}\sum_j(\mathbf{\Delta}^{T\prime}\mathbf{J}(s_B)\mathbf{\Delta}+\tfrac{1}{2}\mathbf{\Delta}^T\mathbf{J}'(s_B)\mathbf{\Delta})_{ij}e_j^\alpha. \quad (29)$$

Note that the δf's are now perfectly finite quantities. Thus we have a situation completely analogous to that in Schrödinger theory; if there are degenerate bound states, one has to diagonalize a matrix whose eigenvalues turn out to be the mass shifts and whose eigenvectors determine, apart from the small corrections δf_i^α, the couplings (wave functions) of the "physical" particles.

In the previous paragraph we saw how the concept of a mass shift matrix $\boldsymbol{\delta}\mathbf{s}_B$ arises naturally out of a study of perturbations on a set of degenerate poles. Evidently, it is not the particular mass shifts δs_B^α, but the matrix $\boldsymbol{\delta}\mathbf{s}_B$ which is the fundamental object. For one thing, $\boldsymbol{\delta}\mathbf{s}_B$ contains more information; remember $\boldsymbol{\delta}\mathbf{s}_B$ also determines the couplings e_i^α. Also, as we will see in the next section, if one is studying the violations of a symmetry group such as $SU(2)$ or $SU(3)$, the group-theoretic properties of the problem become apparent only when one works with the matrix $\boldsymbol{\delta}\mathbf{s}_B$. In many cases involving degenerate poles, the perturbation obeys some conservation law which determines the representation in which $\boldsymbol{\delta}\mathbf{s}_B$ is diagonal and if one uses this representation from the beginning, the problem can be worked out without explicit reference to a mass matrix. One such problem is example (i) of the previous section. There, the unperturbed $J=\frac{1}{2}^+$, $T=\frac{1}{2}$, πN scattering amplitude had two degenerate poles which were taken to correspond to the two isospin states of the nucleon. Since the electromagnetic perturbation conserves T_3, our choice of states implicitly ensured a diagonal mass shift matrix, leaving us free to concentrate on the $I_3=\frac{1}{2}$ state.

IV. THE USE OF GROUP THEORY TO SIMPLIFY THE PERTURBATION FORMALISM; AN EXAMPLE

In Secs. II and III, we have presented a formalism for making dynamical calculations of the effect of small perturbations on masses and couplings. Most of the perturbations one wants to study in practice involve violations of symmetries such as $SU(2)$ or $SU(3)$ invariance. In such cases, as Glashow[11] and Cutkosky and Tarjanne[12,13] have pointed out, the ratios of many terms in the mass shift and coupling shift matrices can be obtained from group theory alone, thus permitting a simplification of the dynamical equations. The simplifications are of two types: first, many terms vanish on account of group theoretical considerations, and secondly, the ratios of many of the nonvanishing terms are fixed.

To get a qualitative picture of why such simplifica-

[11] S. Glashow, Phys. Rev. **130**, 2132 (1963).
[12] R. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963).
[13] R. Cutkosky and P. Tarjanne, Phys. Rev. **133**, B1292 (1964).

tions are possible, consider $\pi\pi$ scattering. Although we have not used the concept of a potential in Secs. II and III, it is helpful for the moment to think of ρ exchange as providing a potential for the $\pi\pi$ system. Electromagnetic shifts in the ρ masses modify the range of the ρ exchange potential, which in turn will modify the mass of any bound state or resonance, such as the ρ, appearing in the $\pi\pi$ channel. Now electromagnetic shifts in the ρ masses transform like 1 or $T_3{}^2$ in isotopic spin space [T_3 is absent because $m(\rho^+)=m(\rho^-)$]. In terms of irreducible representations, they transform like a linear combination of ($T=0$, $T_3=0$) and ($T=2$, $T_3=0$) states. If we restrict ourselves to lowest order effects, the shift in exchanged mass transforming like $T=0$ can only lead to potential shifts transforming like $T=0$ and thence to shifts in the resonant ρ mass transforming like $T=0$. Similarly, $T=2$, $T_3=0$ shifts in exchanged mass cause only $T=2$, $T_3=0$ shifts in the resonant mass. We can express this mathematically by

$$\begin{aligned} \delta m_{T=0}{}^{\text{direct}\,\rho} &= A_0 \delta m_{T=0}{}^{\text{exch}\,\rho} + \cdots, \\ \delta m_{T=2,\,T_3=0}{}^{\text{direct}\,\rho} &= A_2 \delta m_{T=2,\,T_3=0}{}^{\text{exch}\,\rho} + \cdots. \end{aligned} \tag{30}$$

In order to make a dynamical calculation[13a] of δm_0 and δm_2, A_0 must be calculated dynamically, but the ratio A_2/A_0 follows from group theory alone. This is because the dispersion integrals representing the effect of $\delta m_0{}^{\text{exch}}$ on $\delta m_0{}^{\text{dir}}$, and $\delta m_2{}^{\text{exch}}$ on $\delta m_2{}^{\text{dir}}$, are just the same except for crossing coefficients giving quantities such as the ratio of ρ^+ to ρ^0 exchange in the $T=1$, $T_3=1$ direct channel, etc. Once this point is recognized, the crossing coefficients can be calculated without further reference to the detailed dispersion relation (or to the potential concept which we introduced as an intermediate step in the above reasoning).

Techniques for calculating those factors that depend only on group theory have been developed by Glashow[11] and by Cutkosky and Tarjanne.[12,13] In the present section, we take the particular case of perturbations on the ρ bootstrap, classify the various terms that appear, and show, following Glashow, Cutkosky and Tarjanne, how one actually uses group theory to greatly reduce the number of dispersion integrals that have to be evaluated.

Specifically, we consider the e^2 electromagnetic corrections to ρ meson masses and couplings. Only the quantities obtainable by group theory will be calculated. We begin by assuming that there exists an $SU(2)$ symmetric bootstrap model of the ρ meson as a resonance in the $\pi\pi$ system. For simplicity, we suppose that all inelastic channels can be neglected and that the left cut is completely dominated by ρ exchange.

Now the singularities which will appear in our dispersion integrals for the ρ mass and coupling shifts can be divided into three general classes, each of which has a rather different status in a bootstrap theory of the ρ meson; we list the classes as follows:

(i) First, the dispersion integrals will include changes in the ρ exchange cut created by shifts in the ρ masses and couplings. Since these are the same shifts we are calculating, we treat them self-consistently.

(ii) The pion mass shifts will give rise to both right-hand singularities (through the term $N^2\delta\rho$) and left-hand singularities (the π masses affect the position of the ρ exchange cut) in our dispersion integrals. In a complete calculation we would, of course, also be calculating the pion mass shifts self-consistently, but here we shall take the pion masses as given.

(iii) Finally, there will be cuts due to intermediate states which contain photons, e.g., the γ and $\pi+\gamma$ exchange cuts. The discontinuities across these cuts are given by the squares of the amplitudes of order e for processes like $\pi\pi\to\pi\gamma$ and are therefore independent of the order e^2 shifts in the strong interaction parameters; thus we can take the discontinuity across the $\pi\gamma$ cuts, for example, as a completely predetermined quantity in our calculation. Singularities of this type will be called "driving terms." Of course, in practical calculations, other terms which are not strictly speaking driving terms will be treated as though they were, in the sense that they are taken as given and not calculated self-consistently, e.g., the π masses in the present example.

The above separation of singularities into driving terms and singularities to be treated self-consistently would be a general feature of any calculation of the e^2 corrections to strong interactions. Note that the requirements of self-consistency may have an important effect on the nature of the solution, but the scale of electromagnetic corrections will always be determined by the driving terms.

To calculate the changes in the ρ masses and couplings, we must study all the $J=1^-$, $\pi\pi$ scattering amplitudes. Since Bose statistics requires the pions to be in an $I=1$ state, we have three channels which we label $i=+1, 0, -1$ according to the third component of isospin. In the absence of electromagnetic corrections, the scattering is the same in all three channels.

Now charge conservation tells us that, even when electromagnetic effects are included, the channels do not mix, but the group theoretic properties of the problem will become more transparent if we temporarily put aside this fact and use the multichannel degenerate perturbation theory outlined in the previous section. Thus we take the ρ mass shifts to be a matrix $\delta m_{ij}, i,j=-1$, 0, 1, which will, of course, turn out to be diagonal with $\delta m_{-1-1}=\delta m_{\rho-}$, $\delta m_{11}=\delta m_{\rho+}$, and $\delta m_{00}=\delta m_{\rho^0}$. In the same spirit, we take the pion mass shifts to be a matrix $\delta\mu_{ij}$, $i,j=-1, 0, 1$.

Finally, it is best to characterize the $\rho\pi\pi$ coupling shifts by dimensionless quantities which are independent of the scale of mass. Thus, taking the dimensions of the

[13a] To simplify the typography we use δm instead of δm^2. Actually it makes no difference if δm is small as in electromagnetic corrections.

residue matrix R_{ij} of the ρ pole to be (mass)^n, we define $\delta\gamma_{ij}=\delta(R_{ij}/(\bar{\mu})^n)$ where $\bar{\mu}$ is the *average* pion mass, including the perturbation $\delta\mu$.[14] Again, $\delta\gamma_{ij}$ will turn out to be diagonal with $\delta\gamma_{11}$, for example, related to the shift in the residue of the ρ^+ pole in the $I_3=1$ $\pi\pi$ amplitude.

Now let us suppose that we have worked out all the singularities listed above and performed the dispersion integrals for δm_{ij} and $\delta\gamma_{ij}$. Clearly, we will have relations like

$$\begin{aligned}\frac{\delta m_{ij}}{m}&=\sum_{kl}\left(A_{ij,kl}{}^{mm}\frac{\delta m_{kl}}{m}+A_{ij,kl}{}^{m\mu}\frac{\delta\mu_{kl}}{\mu}+A_{ij,kl}{}^{m\gamma}\delta\gamma_{kl}\right)+D_{ij}{}^{m},\\ \delta\gamma_{ij}&=\sum_{kl}\left(A_{ij,kl}{}^{\gamma\gamma}\delta\gamma_{kl}+A_{ij,kl}{}^{\gamma m}\frac{\delta m_{kl}}{m}+A_{ij,kl}{}^{\gamma\mu}\frac{\delta\mu_{kl}}{\mu}\right)+D_{ij}{}^{\gamma},\end{aligned}\tag{31}$$

where the A's are numbers which depend only on the strong interactions, the D's are the driving terms defined in (iii) and we have introduced the unperturbed ρ and π masses, m and μ, to make the A's dimensionless.

The quantities which are most amenable to a group theoretical analysis, and which we shall study in the remainder of this paper, are the A coefficients in Eq. (31).[15] Because (i) the strong interactions conserve isospin, and (ii) bootstrap equations do not determine a unit of mass, it will turn out that we can relate all the $A_{ij,kl}{}^{mm}$, $A_{ij,kl}{}^{\gamma\gamma},\cdots(i,j,k,l=-1\cdots1)$ to four numbers which can be obtained by simple $SU(2)$ symmetric calculations.

Our first step is to note that $A_{ij,kl}{}^{mm}$, for example, must be invariant under simultaneous isospin rotations of the four indices i, j, k, l.[11] Physically, this follows from the fact that $A_{ij,kl}{}^{mm}$ depends only on the strong interactions which do not pick out any particular direction in isospin space. Thus, $A_{ij,kl}{}^{mm}$ has no direction associated with it and must be a scalar. Later, we will show how one can derive explicit formulas for quantities like $A_{ij,kl}{}^{mm}$, which do, in fact, turn out to be invariant.

The easiest way to exploit the invariance of A is to expand δm_{ij}, $\delta\mu_{ij}$, $\delta\gamma_{ij}$, and the D_{ij}'s in irreducible tensors[16] in isospin space. To this end, we introduce the nine matrices $e_{ij}{}^0$, $e_{ij}{}^{1,n}(n=-1\cdots1)$, and $e_{ij}{}^{2,n}$ $(n=-2\cdots2)$, where $e_{ij}{}^{I,n}$ transforms under I-spin rotations like the nth component of an object with total isospin I and we assume that the e_{ij}'s are normalized such that $\sum_{ij}e_{ij}{}^{I,n}e_{ij}{}^{I',n'}=\delta_{II'}\delta_{nn'}$. Specifically, $e_{ij}{}^0$ is equal to $\delta_{ij}/\sqrt{3}$ and $e_{ij}{}^{1,0}$ and $e_{ij}{}^{2,0}$ are diagonal matrices whose nonzero elements are

$$e_{11}{}^{1,0}=-e_{-1-1}{}^{1,0}=1/\sqrt{2}\,,\quad e_{11}{}^{2,0}=e_{-1-1}{}^{2,0}=-1/\sqrt{6}\,,\quad e_{00}{}^{2,0}=2/\sqrt{6}\,.\tag{32}$$

The remaining e's may be obtained by rotations in isospin space.

The matrices $e_{ij}{}^0$ and $e_{ij}{}^{2,n}$ are invariant under charge conjugation, but charge conjugation changes the sign of $e_{ij}{}^{1,n}$. Therefore, none of the matrices δm_{ij}, $\delta\mu_{ij}$, $\delta\gamma_{ij}$, or the D_{ij}'s can have a component along $e_{ij}{}^{1,n}$, and we can write

$$\delta m_{ij}=\delta m_0e_{ij}{}^0+\textstyle\sum_n\delta m_{2,n}e_{ij}{}^{2,n}\tag{33}$$

and similar expressions for $\delta\mu_{ij}$, $\delta\gamma_{ij}$, and the D_{ij}'s. Note that charge conservation actually implies that only δm_0 and $\delta m_{2,0}$ can be nonzero. Now the reason for writing δm_{ij} in the form (33) is as follows. Since $A_{ij,kl}{}^{mm}$ is invariant under isospin rotations, we must have[17]

$$\begin{aligned}&\sum_{kl}A_{ij,kl}{}^{mm}e_{kl}{}^0=A_0{}^{mm}e_{ij}{}^0\,,\\ &\sum_{kl}A_{ij,kl}{}^{mm}e_{kl}{}^{2,n}=A_2{}^{mm}e^{2,n}\quad n=-2\cdots2\,.\end{aligned}\tag{34}$$

Then, substituting (33) and similar expansions for $\delta\mu_{ij}$, $\delta\gamma_{ij}$, and the D_{ij}'s into (31), and identifying the coefficients of $e_{ij}{}^0$ and $e_{ij}{}^{2,n}$ on both sides of the equations, we obtain

$$\begin{aligned}\delta m_0/m&=A_0{}^{mm}(\delta m_0/m)+A_0{}^{m\mu}(\delta\mu_0/\mu)+A_0{}^{m\gamma}\delta\gamma_0+D_0{}^m\,,\\ \delta\gamma_0&=A_0{}^{\gamma\gamma}\delta\gamma_0+A_0{}^{\gamma m}(\delta m_0/m)+A_0{}^{\gamma\mu}(\delta\mu_0/\mu)+D_0{}^\gamma\,,\\ \delta m_{2,n}/m&=A_2{}^{mm}(\delta m_{2,n}/m)+A_2{}^{m\mu}(\delta\mu_{2,n}/\mu)+A_2{}^{m\gamma}\delta\gamma_{2,n}+D_{2,n}{}^m\quad n=-2,\cdots2\,,\\ \delta\gamma_{2,n}&=A_2{}^{\gamma\gamma}\delta\gamma_{2,n}+A_2{}^{\gamma m}(\delta m_{2,n}/m)+A_2{}^{\gamma\mu}(\delta\mu_{2,n}/\mu)+D_{2,n}{}^\gamma\quad n=-2,\cdots2\,.\end{aligned}\tag{35}$$

Thus by simply using the fact that the strong interactions conserve isospin, we have reduced the matrix equa-

[14] The dimensionless coupling is defined in this way so that $\delta\gamma$ will vanish for a perturbation on the masses that amounts to changing the over-all scale of mass. This definition will lead to simplifications in future formulas.

[15] Some readers, on the basis of past experience with electromagnetic corrections, may have been surprised that some terms follow from group theory alone. Now, in any theory of electromagnetism, one has driving terms, and we have to use dynamics to calculate them. But the self-consistent terms are less familiar, and it is the A factors connecting the self-consistent terms which simplify owing to group theory.

[16] See, for example, A. E. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[17] To see why the e_{ij}'s are "eigenvectors" of A^{mm}, consider $A_{ij,kl}{}^{mm}$ as a matrix which carries the nine-dimensional i, j space to the nine-dimensional k, l space. Next, make a change of basis from i, j to I, n, where the I, n basis vector is defined by $e_{ij}{}^{I,n}$ and write A^{mm} as $A_{I'n',In}$. Now the "matrix" A^{mm} is invariant with respect to $SU(2)$ rotations so it commutes with all the isospin operators in i, j space and must therefore be diagonal and independent of n in the I,n representation; hence, $A_{I'n',In}{}^{mm}=A_I{}^{mm}\delta_{I'I}\delta_{n'n}$ or $\sum_{ij}A_{kl,ij}{}^{mm}e_{ij}{}^{I,n}=A_I{}^{mm}e_{kl}{}^{I,n}$.

tion (31) to one set of numerical equations which determines the mass and coupling shifts that transform like $I=0$ and another, decoupled set which determines the shifts which transform like $I=2$. It is very important to note that the equations for the $I=2$ shifts are independent of n so that the particular direction in isospin space along which the $\delta\gamma$'s and δm's point is entirely determined by the nature of the driving terms and pion mass shifts (in a complete calculation we would also treat the pion mass shifts self-consistently so that only the driving terms would define a direction in isospin space). We know, of course, that only the 0 and 2,0 components of $\delta\mu_{ij}$ and the D_{ij}'s are nonvanishing.

We have not yet exhausted the implications of group theory for the A matrix. Actually, as suggested at the beginning of this section, it is possible to explicitly determine the $ijkl$ dependence of, say, $A_{ij,kl}{}^{m\mu}$, and find ratios like $A_2{}^{m\mu}/A_0{}^{m\mu}$ from group theory alone. To see how this goes, let us examine the isospin dependence of $A_{ij,kl}{}^{m\mu}$. A change in the mass of the pions affects the singularities of a partial wave $\pi\pi$ scattering amplitude in two ways. First, variation of the pion mass changes the kinematics of the right-hand unitarity cut [i.e., the second integral in Eq. (12)]. The effect on δm_{ij} of these singularities is expressed graphically in Fig. 2(a),[12,13] where the blobs represent arbitrary isospin-conserving $\pi\pi$ scattering processes and the wiggly lines are schematic ρ mesons which we use to express the fact that we are projecting out the $J=1^-$, $I=1$, $I_3=i\rightarrow I_3=j$ part of the $\pi\pi$ amplitude. Variation of the pion mass also changes the position of the left-hand cuts [i.e., the first

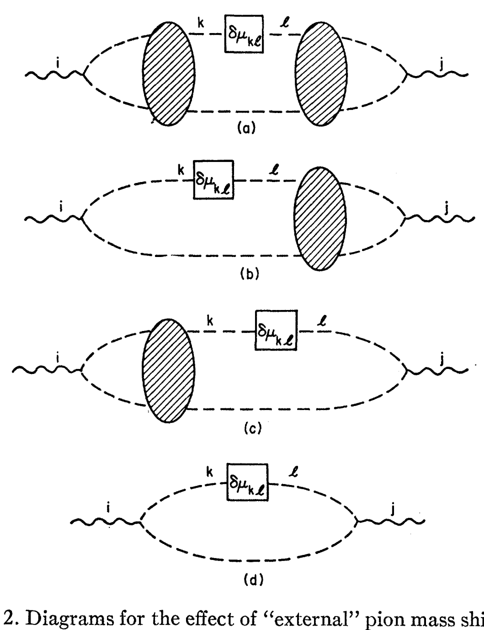

FIG. 2. Diagrams for the effect of "external" pion mass shifts on the ρ mass. The dashed lines are pions and the wiggly lines denote ρ mesons. Diagram (a) illustrates the effect of $\delta\mu$ on the right-hand (unitarity) cut and diagrams (b) and (c) represent kinematic charges on the left cut. Diagram (d) represents the essential isospin properties of (a)–(c).

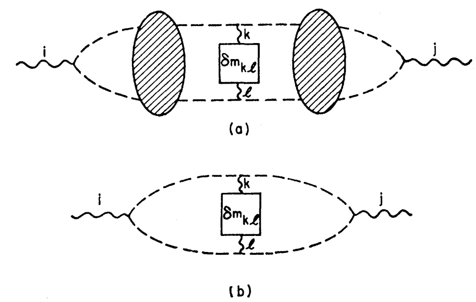

FIG. 3. Diagrams showing the change in the ρ mass due to a ρ mass change in a cross channel. Diagram (b) represents the essential isospin properties of (a).

integral in Eq. (12)]. This effect is represented in Figs. 2(b) and 2(c).

Now the point of all this is that, since the blobs conserve isospin, the $ijkl$ dependence of the diagrams in Figs. 2(a)–2(c) is the same as that of the simple "bubble diagram" in Fig. 2(d). The latter diagram should, for our purposes, be interpreted simply as the sum of products of Clebsch-Gordan coefficients. That is, the isospin properties of $A^{m\mu}$ follow directly from the diagram but the over-all normalization of $A^{m\mu}$ must be determined from some dynamical scheme.

From the insight just gained into the isospin dependence of $A_{ij,kl}{}^{m\mu}$, we can find $A_2{}^{m\mu}/A_0{}^{m\mu}$ in the following manner. Let us call the Clebsch-Gordan coefficient at the $\rho^i\pi^j\pi^k$ vertex g^{ijk}, where g^{ijk} is normalized such that $\sum_{jk} g^{ijk}g^{i'jk}=\delta_{ii'}$. Then, according to Fig. 2, we have

$$A_{ij,kl}{}^{m\mu}=k^{m\mu}(\textstyle\sum_x g^{ikx}g^{jlx})\,, \qquad (36)$$

where $k^{m\mu}$ is a number independent of i, j, k, and l.

Notice here that: (i) Since the $\rho\pi\pi$ coupling $\sum_{ijk} g^{ijk} \times\rho^i\pi^j\pi^k$ is invariant under simultaneous isospin rotations of ρ and π, g^{ijk} must be invariant under simultaneous rotations if i, j, and k. Hence, $A_{ij,kl}{}^{m\mu}$ as given by Eq. (36) is an invariant. (ii) The "diagonal" elements $C_{ii,kk}{}^{m\mu}$ of $C^{m\mu}$, which refer to the physical particles, are simply $k^{m\mu}\sum_x(g^{ikx})^2$ or $k^{m\mu}$ times the probability that π^k appears in the ρ^i wave function. Physically, this means that the *relative* effect on the ρ^+, ρ^0, and ρ^- masses of changing, say, the π^0 mass is given by the probabilities that ρ^+, ρ^0, and ρ^- contain a π^0, a point which has previously been noted by Capps.[18] Now to find $A_2{}^{m\mu}/A_0{}^{m\mu}$ we need only remember that, according to Eq. (34), $e_{ij}{}^0$ and $e_{ij}{}^{2,0}$ are eigenvectors of $A_{ij,kl}{}^{m\mu}$, which, using the fact that $e_{ij}{}^0$ and $e_{ij}{}^{2,0}$ are diagonal, implies that

$$\begin{aligned} A_0{}^{m\mu}e_{00}{}^0&=\textstyle\sum_k A_{00,kk}{}^{m\mu}e_{kk}{}^0\,,\\ A_2{}^{m\mu}e_{00}{}^{2,0}&=\textstyle\sum_k A_{00,kk}{}^{m\mu}e_{kk}{}^{2,0}\,. \end{aligned} \qquad (37)$$

Since the $A_{00,kk}{}^{m\mu}$ $k=-1\cdots1$ are, as pointed out above, $k^{m\mu}$ times the probabilities that π^k appears in the

[18] R. Capps, Phys. Rev. **134**, B1396 (1964).

wave function $\rho^0=(1/\sqrt{2})[\pi^+(1)\pi^-(2)-\pi^-(1)\pi^+(2)]$, they must be given by $A_{00,11}{}^{m\mu}=A_{00,-1-1}{}^{m\mu}=\frac{1}{2}k^{m\mu}$ and $A_{00,00}{}^{m\mu}=0$. Then, using $e_{kk}{}^0=\delta_{kk}/\sqrt{3}=1/\sqrt{3}$ and the values for $e_{kk}{}^{2,0}$ given in Eq. (32), one finds $A_0{}^{m\mu}=k^{m\mu}$ and $A_2{}^{m\mu}=-\frac{1}{2}k^{m\mu}$ so that

$$A_2{}^{m\mu}/A_0{}^{m\mu}=-\tfrac{1}{2}. \qquad (38)$$

Next let us turn from the isospin structure of $A^{m\mu}$ to that of A^{mm}. Graphically, the effect on δm_{ij} of changing the mass of an exchanged ρ is illustrated by Fig. 3(a). Again the blobs conserve isospin so the isospin content of Fig. 3(a) is the same as that of the bubble in Fig. 3(b). Thus, $A_{ij,kl}{}^{mm}$ turns out to be proportional to a sum over products of four Clebsch-Gordan coefficients; specifically,

$$A_{ij,kl}{}^{mm}=k^{mm}\sum_{xyzu}g^{ixy}g^{kxz}g^{jzu}g^{lyu}. \qquad (39)$$

According to the discussion of the last paragraph, however, we can find $A_2{}^{mm}/A_0{}^{mm}$ from a knowledge of just the three numbers $A_{00,kk}{}^{mm}$, $k=-1, 0, 1$. To find these numbers, consider the graph in Fig. 3(b) with $i=j=0$, which is like placing a ρ^0 on each end. A neutral ρ couples only to $\pi^+\pi^-$ so the intermediate pions are all π^+'s or π^-'s. Then the pions in the crossed channels are again all π^+'s or π^-'s. Since the only ρ meson exchange which can come from two charged pions is ρ^0 exchange, $A_{00,kk}{}^{mm}$ must be zero unless $k=0$, and proceeding in the same manner as before, we find $A_2{}^{mm}=A_{00,00}{}^{mm}$ and $A_0{}^{mm}=A_{00,00}{}^{mm}$ or

$$A_2{}^{mm}/A_0{}^{mm}=1. \qquad (40)$$

To find the remaining ratios of A_0 to A_2's, we need only observe that $\delta\gamma_{ij}$ will appear in our graphs in exactly the same way as δm_{ij}, and one finds

$$\frac{A_2{}^{\gamma m}}{A_0{}^{\gamma m}}=\frac{A_2{}^{m\gamma}}{A_0{}^{m\gamma}}=\frac{A_2{}^{\gamma\gamma}}{A_0{}^{\gamma\gamma}}=\frac{A_2{}^{mm}}{A_0{}^{mm}}=1,$$
$$\frac{A_2{}^{\gamma\mu}}{A_0{}^{\gamma\mu}}=\frac{A_2{}^{m\mu}}{A_0{}^{m\mu}}=-\tfrac{1}{2}. \qquad (41)$$

Substituting these ratios into Eqs. (35) for $\delta m_{2,n}$ and $\delta\gamma_{2,n}$, we find

$$\delta m_{2,n}/m=A_0{}^{mm}(\delta m_{2,n}/m)-\tfrac{1}{2}A_0{}^{m\mu}(\delta\mu_{2,n}/\mu)+A_0{}^{m\gamma}\delta\gamma_{2,n}+D_{2,n}{}^{m}, \qquad (42)$$
$$\delta\gamma_{2,n}=A_0{}^{\gamma\gamma}\delta\gamma_{2,n}-\tfrac{1}{2}A_0{}^{\gamma\mu}(\delta\mu_{2,n}/\mu)+A_0{}^{\gamma m}(\delta m_{2,n}/m)+D_{2,n}{}^{\gamma}.$$

We have now gone as far as is possible using group theory alone. Let us review what progress has been achieved. First, we found that the problem of calculating the electromagnetic corrections to the ρ masses and $\rho\pi\pi$ couplings splits into two completely independent problems, one for the corrections which transform like $I=0$ and one for the corrections which transform like $I=2$. Aside from its intrinsic interest, this result would lead to a considerable saving of labor in a practical calculation of the δm's and $\delta\gamma$'s. Secondly, we showed how the parameters appearing in the $I=2$ problem can be related to those in the $I=0$ problem and obtained, as our end result, Eq. (42). The value of Eq. (42) lies, of course, in the fact that it is much easier to calculate the A_0's dynamically than the A_2's; the A_0's can be obtained from a simple $SU(2)$ symmetric calculation. For example, the dispersion integral for $A_0{}^{mm}$ is simply

$$A_0{}^{mm}=\frac{1}{\pi}\frac{1}{[\mathrm{Re}D'(m_\rho{}^2)]^2}\frac{1}{R}\times\int_L\frac{D^2(s')(d/dm_\rho{}^2)[\mathrm{Im}B(m_\rho{}^2,s')]}{s'-m_\rho{}^2}ds', \qquad (43)$$

where D is the denominator function for $\pi\pi$ scattering in the $J=1^-$, $I=1$ state, R is the residue of the ρ pole and $B(m_\rho{}^2,s')$ is the ρ exchange amplitude. In general, the effects of $I=0$ shifts in exchanged masses and couplings are readily computed in terms of integrals, such as Eq. (43), over the left cut associated with the exchange.

The effects of $I=0$ shifts in "external" masses (i.e., the pions in the present example) are generally more complicated because external mass shifts affect the entire right and left cuts instead of only a single piece of the left cut. One might hope that the over-all effect of external mass shifts would be simple in view of the example of low-energy nuclear physics where, for instance, electromagnetic mass shifts in the neutron and proton components of the dueteron simply shift the deuteron mass by $\delta m_n+\delta m_p$. Unfortunately, the situation is more complicated in the relativistic case, as can easily be seen by considering two particles, both of mass M, which interact to produce a bound state with mass $M_B=2M-E_B$ where E_B is the binding energy. If we change M by δM, then M_B will change by $\delta M_B=2\delta M-(\partial E_B/\partial M)\delta M$. Now in low-energy physics, one always has a situation where $E_B\ll M$ so $\delta M_B\approx 2\delta M$, but in relativistic problems $\partial E_B/\partial M$ can easily be of order unity.

Although external mass terms like $A^{m\mu}$ and $A^{\gamma\mu}$ are hard to evaluate directly, there is a general property of bootstrap equations which we have not yet made use of and which reduces the number of independent terms. In fact, in the simple model we are presently considering, this property actually enables us to eliminate $A^{m\mu}$ and $A^{\gamma\mu}$ from the equations, leaving only the more readily evaluated terms associated with exchanges. The property in question is the invariance of the $SU(2)$ symmetric ρ bootstrap equations under the transformation $m\to\lambda m$, $\mu\to\lambda\mu$, and $\gamma\to\gamma$, where λ is any positive number. In the present context, this implies that Eqs. (35) for δm_0 and $\delta\gamma_0$ must have a solution with

$D_0{}^\gamma = D_0{}^m = 0$ and this solution must have the form $\delta m_0/m = \delta\mu_0/\mu$, $\delta\gamma_0 = 0$, which can be the case only if[12,13]

$$A_0{}^{m\mu} + A_0{}^{mm} = 1\,, \quad A_0{}^{\gamma\mu} + A_0{}^{\gamma m} = 0\,. \tag{43'}$$

Finally, substituting (41) and (43) into (35) and performing a few algebraic manipulations, we find

$$\begin{aligned}
\delta m_0/m &= (\delta\mu_0/\mu) + (1 - A_0{}^{mm})^{-1}(A_0{}^{\gamma m}\delta\gamma_0 + D_0{}^m)\,,\\
\delta m_{2,n}/m &= -\tfrac{1}{2}(\delta\mu_{2,n}/\mu) + (1 - A_0{}^{mm})^{-1}\\
&\qquad \times (A_0{}^{\gamma m}\delta\gamma_{2,n} + D_{2,n}{}^m)\,,\\
\delta\gamma_0 &= \{(1 - A_0{}^{\gamma\gamma})(1 - A_0{}^{mm}) - A_0{}^{m\gamma}A_0{}^{\gamma m}\}^{-1}\\
&\qquad \times [(1 - A_0{}^{mm})D_0{}^\gamma + A_0{}^{\gamma m}D_0{}^m]\,,\\
\delta\gamma_{2,n} &= \{(1 - A_0{}^{\gamma\gamma})(1 - A_0{}^{mm}) - A_0{}^{m\gamma}A_0{}^{\gamma m}\}^{-1}\\
&\qquad \times [(1 - A_0{}^{mm})D_{2,n}{}^\gamma + A_0{}^{\gamma m}D_{2,n}{}^m]\,.
\end{aligned} \tag{44}$$

These equations have a number of amusing properties: (i) The dependence of δm and $\delta\gamma$ on the pion mass differences is completely determined. Should it turn out that the driving terms $D_{2,0}{}^m$ and $D_{2,0}{}^\gamma$ are small compared to $\delta\mu_{2,0}/\mu$, we would have $\delta\gamma_{2,0} \approx 0$ and $\delta m_{2,0} \approx -(m/2\mu)\delta\mu_{2,0}$, or $m_\rho{}^+ - m_\rho{}^0 \approx -(m/2\mu)(\mu_\pi{}^+ - \mu_\pi{}^0)$. (ii) Suppose we set the pion mass shifts $\delta\mu$ and the driving terms equal to zero and consider the possibility of a nonzero solution for $\delta m_{2,n}$ and $\delta\gamma_{2,n}$. Since, without the pion mass and driving terms, the equations for $\delta\gamma_0$ and δm_0 are the same as those for $\delta\gamma_{2,n}$ and $\delta m_{2,n}$, it is not possible to find such a "spontaneous" violation of $SU(2)$ in the ρ bootstrap unless the bootstrap equations have more than one $SU(2)$-symmetric solution.

In the previous paragraphs, we showed how, using only group theory and the scaling properties of bootstrap equations, one can reduce the rather complicated Eq. (31) and the remarkably simple set (44). Obviously, it will be advantageous to use the same sort of procedure in any problem involving violations of $SU(2)$. However, the degree of simplification which can be achieved by these general arguments alone, will not in general, be as great as it was in this particular example. To see why, let us consider what would happen if we tried to improve our calculation of the ρ mass shifts by including the $\pi\omega$ channel as well as the $\pi\pi$ channel. If we again simplify the left cut in the scattering amplitude to just ρ exchange and the driving terms, the equation for δm_{ij} becomes

$$\frac{\delta m_{ij}}{m} = \sum_{kl}\left(A_{ij,kl}{}^{mm}\frac{\delta m_{kl}}{m} + A_{ij,kl}{}^{m\mu}\frac{\delta\mu_{kl}}{\mu}\right) + A_{ij,kl}{}^{mM}\delta M + \cdots\,, \tag{45}$$

where we have suppressed the terms involving coupling shifts and driving terms and M is the ω mass (since ω is an isosinglet, there is no need to use a matrix for δM). Again we expand δm_{ij}, $\delta\mu_{ij}\cdots$ in irreducible tensors, which yields

$$\begin{aligned}
\frac{\delta m_0}{m} &= A_0{}^{mm}\frac{\delta m_0}{m} + A_0{}^{m\mu}\frac{\delta\mu_0}{\mu} + A_0{}^{mM}\frac{\delta M}{M} + \cdots\,,\\
\frac{\delta m_{2,n}}{m} &= A_2{}^{mm}\frac{\delta m_{2,n}}{m} + A_2{}^{m\mu}\frac{\delta\mu_{2,n}}{\mu} + \cdots\,.
\end{aligned} \tag{46}$$

Note that since the ω mass shift transforms like $I=0$, it cannot appear in an equation for $\delta m_{2,n}$. Now let us see to what extent Eqs. (46) can be simplified. First, the analog of Eq. (43) is $A_0{}^{mm} + A_0{}^{m\mu} + (\sqrt{3})^{-1}A_0{}^{mM} = 1$ [the factor $(\sqrt{3})^{-1}$ enters here because we defined the $I=0$ π and ρ mass shifts as $\delta m^0 e_{ij}{}^0 = \delta m^0\delta_{ij}/\sqrt{3}$] which can be used to eliminate only one of the external mass parameters $A_0{}^{m\mu}$ or $A_0{}^{mM}$; the other must be computed explicitly. Secondly, consider the determination of the ratio $A_2{}^{m\mu}/A_0{}^{m\mu}$, which is now complicated by the fact that π's appear as external particles in two different channels. Proceeding graphically, one finds that the analog of the single graph in Fig. 2(a) is the set of four graphs shown in Fig. 4. Again using the fact that the blobs conserve isospin, we observe that the isospin dependence of Figs. 4(a) and 4(b) is the same as that of the 2π bubble in Fig. 4(e) and that the isospin dependence of Figs. 4(c) and 4(d) is given by the isospin properties of the $\pi\omega$ bubble in Fig. 4(f). Then, recalling that the $ijkl$ dependence of Fig. 4(e) is given by $\sum_x g^{ikx}g^{jlx}$ and noting that, since ω is an isosinglet, the $ijkl$ dependence of Fig. 4(f) is simply $\delta_{ik}\delta_{lj}$, one can convince himself that

$$A_{ij,kl}{}^{m\mu} = k_a{}^{m\mu}(\textstyle\sum_x g^{ikx}g^{jlx}) + k_b{}^{m\mu}\delta_{ik}\delta_{lj}\,, \tag{47}$$

where $k_a{}^{m\mu}$ and $k_b{}^{m\mu}$ are numbers independent of i, j, k, and l. Application of the same group theory techniques as before gives the relation

$$\frac{A_2{}^{m\mu}}{A_0{}^{m\mu}} = (-\tfrac{1}{2}k_a{}^{m\mu} + k_b{}^{m\mu})/(k_a{}^{m\mu} + k_b{}^{m\mu})\,. \tag{48}$$

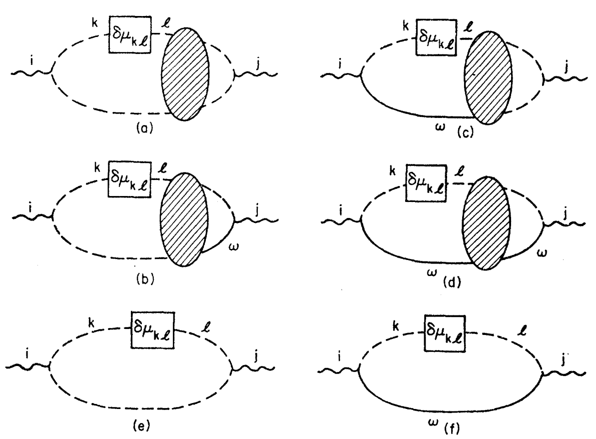

FIG. 4. Some "external mass" diagrams analogous to those of Fig. 2 for the ρ bootstrap including both the $\pi\pi$ and $\pi\omega$ channels. The solid line is an ω meson.

But in this case, since the ratio $k_a^{m\mu}/k_b^{m\mu}$ must be obtained from some dynamical model, we no longer have a purely group theoretic prediction for $A_2^{m\mu}/A_0^{m\mu}$. Of course, if the $\pi\omega$ channel has little effect on the ρ mass, then $k_a^{m\mu}\gg k_b^{m\mu}$ and (48) reduces to the single-channel result $A_2^{m\mu}/A_0^{m\mu}=-\frac{1}{2}$.

V. THE USE OF GROUP THEORY TO SIMPLIFY THE PERTURBATION FORMALISM; GENERAL PROPERTIES

In the previous section we illustrated, by means of a specific example, some group theoretic methods which will often be useful in studying violations of symmetry groups. The present section will be devoted to a more general discussion of these methods. Although the techniques in question can be used to study violations of any symmetry group, we shall continue to concentrate on $SU(2)$, with a few concluding remarks about $SU(3)$.

Let us consider, then, the general problem of calculating the e^2 electromagnetic corrections to masses and couplings of the strongly interacting particles. As in the example of the previous section, we assume a bootstrap theory of strongly interacting particles, treating stable and unstable particles on the same footing. Presumably, the strong interaction bootstrap equations have an $SU(2)$ symmetric solution, in which the isospin multiplets of particles appear as degenerate sets of poles in scattering amplitudes with the proper quantum numbers. Adding electromagnetism to the strong interactions then breaks the $SU(2)$ symmetry and causes shifts in the positions and residues of these poles. In a bootstrap theory, the mass shift matrix $\delta m_{ij}{}^{\alpha}$ for the particles in multiplet α will depend on: (i) the mass shifts $\delta m_{ij}{}^{\alpha}$ of all the multiplets of strongly interacting particles; (ii) the coupling shifts; (iii) a "driving term" $D_{ij}{}^{\alpha}$ which, as discussed in the previous section, is associated with the explicit appearance of photons in the dispersion relations. Thus we have

$$\delta m_{ij}{}^{\alpha}/m^{\alpha}=\sum_{k,l} A_{ij,kl}{}^{\alpha\alpha'}(\delta m_{kl}{}^{\alpha'}/m^{\alpha'})+D_{ij}{}^{\alpha} +(\text{terms involving coupling shifts}). \quad (49)$$

In the last section we saw that equations like (49) became simpler when one expands the δm's, D's and coupling shifts in terms of irreducible tensors in isospin space; hence, we write

$$\delta m_{ij}{}^{\alpha}=\sum_{I=0}^{2I_\alpha}\sum_{n=-I}^{I}\delta m_{I,n}{}^{\alpha}e_{ij}{}^{I,n},$$

$$D_{ij}{}^{\alpha}=\sum_{I=0}^{2I_\alpha}\sum_{n=-I}^{I}D_{I,n}{}^{\alpha}e_{ij}{}^{I,n},$$

and so forth, where as before, $e_{ij}{}^{I,n}$ is the tensor with total isospin I and third component of isospin n and I_α is the isospin of multiplet α. Again, since $A_{ij,kl}{}^{\alpha\alpha'}$ is invariant under simultaneous isospin rotations of i, j, k, and l, we have $\sum_{kl}A_{ij,kl}{}^{\alpha\alpha'}e_{kl}{}^{I,n}=A_I{}^{\alpha\alpha'}e_{ij}{}^{I,n}$ and Eq. (49) can be written in the form

$$\delta m_{I,n}{}^{\alpha}/m=\sum_{\alpha'}A_I{}^{\alpha\alpha'}(\delta m_{I,n}{}^{\alpha'}/m^{\alpha'})+D_{I,n}{}^{\alpha} +(\text{terms involving coupling shifts}). \quad (50)$$

In the example of the previous section, we were able to set our coupling shifts equal to a matrix which had the same group theoretic properties as a mass shift matrix. The reader will recall that this was possible because in the coupling of a ρ to two π's, Bose statistics requires that the two pions always be in an $I=1$ state. In general, the parametrization of coupling shifts is more difficult. Consider, for example, the coupling constant shifts $\delta\Gamma_{ijk}$ for a vertex connecting one particle with isospin one to two particles with isospin $\frac{1}{2}$. Here, the subscripts i, j, k run over the I_3 values; i.e., $i=-1$, 0, 1 and j and $k=-\frac{1}{2},\frac{1}{2}$. Just as for the mass matrix, it will be most convenient to expand $\delta\Gamma_{ijk}$ in a set of irreducible tensors in i, j, k space.

Now i, j, k space contains one independent irreducible tensor $e_{ijk}{}^{I,n}$, $n=-I\cdots I$, for each time the representation I appears in the reduction of the direct product $1\otimes\frac{1}{2}\otimes\frac{1}{2}$ into irreducible representations. Thus, decomposing $1\otimes\frac{1}{2}\otimes\frac{1}{2}$ according to $1\otimes\frac{1}{2}\otimes\frac{1}{2}=1\otimes(0\oplus1)=0\oplus1\oplus1\oplus2$, we can write

$$\delta\Gamma_{ijk}=\sum_{I=0}^{2}\sum_{n=-I}^{I}\sum_{\beta}\delta\Gamma_{I,n}{}^{\beta}e_{ijk}{}^{I,n;\,\beta}, \quad (51)$$

where $e_{ijk}{}^{I,n;\,\beta}$ is the $I_3=n$ component of an irreducible tensor which transforms with total isospin I and β is an index that distinguishes between the two $I=1$ representations that appear in the triple product $1\otimes\frac{1}{2}\otimes\frac{1}{2}$, e.g., one can take β as the total isospin, 0 or 1, associated with the indices j and k.

Next we turn to the most general case: perturbations on the coupling of three or more multiplets with isospins I_α, $I_{\alpha'}$, $I_{\alpha''}\cdots$. Denoting the coupling shift by $\delta\Gamma_{ijk\cdots}$, $i=-I_\alpha\cdots I_\alpha$, $j=-I_{\alpha'}\cdots I_{\alpha'}\cdots$, we can always expand $\delta\Gamma$ in irreducible tensors according to

$$\delta\Gamma_{ijk\cdots}=\sum_{I,n,\beta}e_{ijk\cdots}{}^{I,n;\,\beta}\delta\Gamma_{I,n}{}^{\beta}, \quad (52)$$

where I now runs over all the distinct values of total isotopic spin which appear in the direct product $I_\alpha\otimes I_{\alpha'}\otimes I_{\alpha''}\otimes\cdots$ and β is again a parameter which distinguishes between representations with the same total isospin which occur in this particular direct product. We can also use the index β to distinguish between the different vertices (e.g., $\rho\pi\pi,\cdots\rho\omega\pi$, etc.) at which the coupling corrections appear; thus we list all charges in couplings by $\delta\gamma_{I,n}{}^{\beta}$ where β now runs over all independent corrections to the strong interaction couplings which transform like (I,n).

Just as happened with the mass shifts, a coupling shift $\delta\Gamma_{I,n}{}^{\beta}$ can only affect a shift in coupling or mass of the same I and n, and the effect of $\delta\Gamma_{I,n}{}^{\beta}$ on $\delta m_{I,n}{}^{\alpha}$ or $\delta\Gamma_{I,n}{}^{\beta'}$ must be independent of n. Thus, explicitly writing out the coupling terms, Eq. (49) must have the form

$$\delta m_{I,n}{}^{\alpha}/m^{\alpha}=\sum_{\alpha'} A_I{}^{\alpha\alpha'}(\delta m_{I,n}{}^{\alpha'}/m^{\alpha'}) +\sum_{\beta'} A_I{}^{\alpha\beta'}\delta\Gamma_{I,n}{}^{\beta'}+D_{I,n}{}^{\alpha}. \quad (53)$$

Similarly, the coupling shifts are given by equations like

$$\delta\Gamma_{I,n}{}^{\beta}=\sum_{\beta'} A_I{}^{\beta\beta'}\delta\Gamma_{I,n}{}^{\beta'} +\sum_{\alpha'} A_I{}^{\beta\alpha'}(\delta m_{I,n}{}^{\alpha'}/m^{\alpha'})+D_{I,n}{}^{\beta}. \quad (54)$$

Equations (53) and (54) are completely general—note that in setting up these equations we made no reference to approximations such as two-particle unitarity or single-particle exchange. A few general properties of (53) and (54) are worth noting: (i) the over-all problem of determining the electromagnetic corrections to strong interactions splits up into a set of completely independent problems, one for each (I,n) type of $SU(2)$ violation. Once again, ratios between some of the nonzero A coefficients are given by group theory. (ii) If we parametrize all our couplings in terms of dimensionless numbers,[14] the strong interaction, $SU(2)$ symmetric bootstrap equations must be invariant under the transformation $m^{\alpha}\rightarrow\lambda m^{\alpha}$, with no changes in the coupling constants. In the present context, this implies that Eqs. (53) and (54) for $I=0$ have a solution with $D_0{}^{\alpha}=D_0{}^{\beta}=\delta\Gamma_0{}^{\beta}=0$ and $\delta m_0{}^{\alpha}=\epsilon m^{\alpha}(2I_{\alpha}+1)^{1/2}$ [the factor $(2I^{\alpha}+1)^{1/2}$ is required here because we have normalized our tensors such that $\sum_{ij}(e_{ij}{}^{I,n})^2=1$; hence, $e_{ij}{}^0=\delta_{ij}(2I_{\alpha}+1)^{-1/2}$], which requires that

$$\sum_{\alpha'} A_0{}^{\alpha\alpha'}(2I_{\alpha'}+1)^{1/2}=(2I_{\alpha}+1)^{1/2},$$
$$\sum_{\alpha'} A_0{}^{\beta\alpha'}(2I_{\alpha'}+1)^{1/2}=0. \quad (55)$$

This is, of course, the generalization of Eq. (43) of the previous section. (iii) If Eqs. (53) and (54) for $I\neq0$ have a nonzero solution with $D_{I,n}{}^{\alpha}=D_{I,n}{}^{\beta}=0$, there would be an instability in the strong interaction bootstrap equations which could lead to a "spontaneous" breakdown of $SU(2)$. Apparently this situation does not occur in nature, however, since $SU(2)$ is conserved except for small electromagnetic and weak corrections.[19] (iv) Finally, we recall that since the electric current transforms isotopically like a scalar plus the third component of a vector, the e^2 driving terms actually contain only $(I,n)=(0,0)$, $(1,0)$, and $(2,0)$ pieces.

[19] On the calculational side, estimates such as the one in Sec. IV of the present paper also indicate that $SU(2)$ does not undergo spontaneous breakdown. This point has been particularly emphasized by E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963).

Having discussed the general properties of electromagnetic corrections, let us return to the approximation of two-particle unitarity and the N/D method. In the two-particle unitarity approximation, one can calculate all the A's appearing in (53) and (54) with the N/D perturbation techniques developed in Sec. I. (If desired, one could also include some multiparticle channels in a phenomenological way.) Furthermore, as we saw in the last section, one can often use group theoretical methods to find relations between parameters like $A_0{}^{\alpha\alpha}$ and $A_2{}^{\alpha\alpha}$, thus simplifying the calculations. We conclude our discussion of electromagnetic corrections with an example which illustrates both the possibilities and limitations of the group theoretical methods for finding such ratios. Suppose that particle c is an $I=1$ bound state of two $I=\frac{1}{2}$ particles a and b and that c exchange is the principal binding force. Let us parametrize the abc coupling shifts according to $\delta\gamma_{I,n}{}^{\beta}$ where as usual I and n give the isospin transformation properties of the coupling shift and we choose $\beta=0$, 1 as the total isospin of particles a and b. Since a mass shift $\delta m_{I,n}$ of particle c will affect $\delta\Gamma_{I,n}{}^{\beta}$, we have a relation like

$$\delta\Gamma_{I,n}{}^{\beta}=A_I{}^{\beta}\frac{\delta m_{I,n}}{m}+\cdots, \quad (56)$$

where $\beta=1$ for $I=0$, 2 and $\beta=0$, 1 for $I=1$.

For the $\beta=1$ coupling shifts $\delta\Gamma_{I,n}{}^1$ the particles a and b are always in an $I=1$ state, so just as in the case of corrections to the $\rho\pi\pi$ couplings, $\delta\Gamma_{I,n}{}^1$ has the same group theoretical behavior as the mass matrix $\delta m_{I,n}$. Thus by drawing diagrams similar to those shown in Fig. 3, one can determine the ratios $A_1{}^1/A_0{}^1$ and $A_2{}^1/A_0{}^1$. On the other hand, group theory alone cannot give any relations between $A_1{}^0$ and $A_0{}^1$. To see why this is so, recall that in the perturbation formulas given in Eqs. (21) to (23), $\delta\Gamma_{I,n}{}^1$ depends only on the denominator function D_1 for a, b scattering in the $I=1$ state whereas $\delta\Gamma_{I,n}{}^0$ depends on both D_1 and the denominator function D_0 for the $I=0$ state. Since group theory by itself does not give a relation between D_1 and D_0, it cannot determine a ratio like $A_1{}^0/A_0{}^1$. Finally, let us suppose that particles a and b have isospin 1 instead of $\frac{1}{2}$. We can still use the labels β, I, and n, and write $\delta\gamma_{I,n}{}^{\beta}=A_I{}^{\beta}\times\delta m_{I,n}/m+\cdots$ but now we have $\beta=1$ for $I=0$, $\beta=0$, 1, 2 for $I=1$, $\beta=1$, 2 for $I=2$, and $\beta=2$ for $I=3$. We leave it to the reader to convince himself that, in this case, group theory can provide the ratios $A_0{}^1{:}A_1{}^1{:}A_2{}^1$ and $A_1{}^2{:}A_2{}^2{:}A_3{}^2$ but cannot give any relations between the A's for different values of β.

In conclusion, we briefly discuss the application of these methods to violations of $SU(3)$. To change Eqs. (53) and (54) from $SU(2)$ to $SU(3)$ one need only: (i) interpret the label I as the dimension of an $SU(3)$ representation and the label n as a component of representation I. (ii) Let the indices α and β on $\delta m_{I,n}{}^{\alpha}$ and $\delta\Gamma_{I,n}{}^{\beta}$ run over all independent mass and coupling

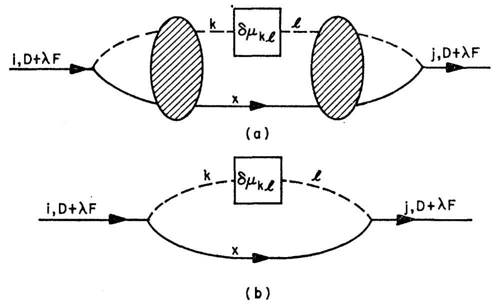

FIG. 5. Diagrams representing the effect of "external" psuedo-scalar meson mass shifts on the mass of the $J=\frac{1}{2}^+$ baryon octet. The dashed lines are pseudoscalar mesons and the solid, directed lines represent baryons. As explained in the text, diagram (b) has the same $SU(3)$ structure as (a), provided that the D/F ratios for $B\Pi$ scattering in the $J=\frac{1}{2}^+$ octet states do not vary rapidly with energy.

shifts which transform like (I,n). In counting indices, one must keep in mind that whereas in $SU(2)$ the product of two representations contains a given representation only once, in $SU(3)$ a representation can occur more than once in the decomposition of a product, e.g., $\mathbf{8}\otimes\mathbf{8}$ contains $\mathbf{8}$ twice.

The group theoretic techniques which we used to simplify calculation of the A matrix in our $SU(2)$ examples can, of course, be generalized to $SU(3)$. In most cases the generalization is perfectly straightforward, but in a few situations some additional complexities appear. Consider, for example, a problem where it is assumed that the octet of baryons B is a bound state of B and the octet of pseudoscalar mesons Π, and one wants to determine the effect on the B masses of different types of Π mass splittings. As usual, we write

$$\delta m_{ij}/m=\sum_{kl} A_{ij,kl}(\delta\mu_{kl}/\mu)+\cdots, \qquad (57)$$

where δm_{ij} $(i,j=1\cdots 8)$ is the B mass matrix and $\delta\mu_{ij}$ $(i,j=1\cdots 8)$ is the Π mass matrix. We ask, to what extent we can use group theory to determine the $ijkl$ dependence of $A_{ij,kl}$. First note that since the direct product $\mathbf{8}\otimes\mathbf{8}$ contains two octets, $\mathbf{8}_S$ and $\mathbf{8}_A$ in the usual symmetric-antisymmetric notation, the unperturbed $J=\frac{1}{2}^+$, ΠB octet amplitudes in which the degenerate B poles appear form a coupled two-channel problem. Now the reader will recall that in our $SU(2)$ example, group theory was sufficient to determine the $ijkl$ dependence of $A_{ij,kl}{}^{m\mu}$ as long as we kept only the $\pi\pi$ channel, but was no longer sufficient when we added the $\pi\omega$ channels. Thus, from our experience in $SU(2)$, we suspect that since there are two channels in the present $SU(3)$ example, group theory by itself will not provide us with complete information on the $ijkl$ dependence of A. To see what happens, consider the graph in Fig. 5(a) which represents the effect on δm_{ij} of changing the Π masses in the unitarity (right-hand) cut of the ΠB scattering amplitude.

In this graph, the external baryon lines labeled i, $D+\lambda F$ and j, $D+\lambda F$ represent Clebsch-Gordan coefficients which couple B and Π to the i and j components of that combination of $\mathbf{8}_S$ and $\mathbf{8}_A$ which is observed for the $BB\Pi$ couplings at the B pole. The blobs preserve $SU(3)$, which implies that the pairs (kx) and (lx) are in octet states. But the blobs can mix 8_S and 8_A, so the F/D ratio of a pair such as (kx) need not be the same as λ, but will generally vary with energy in the dispersion relations, and is not given by group theory alone. Therefore, without some dynamical model for the unperturbed $B\Pi$ scattering amplitudes [i.e., the blobs in Fig. 5(a)] we can only partially determine the $ijkl$ dependence of $A_{ij,kl}$. However, one suspects that if we choose our representation for the two by two matrix, which represents the unperturbed $J=\frac{1}{2}^+$, ΠB octet scattering amplitudes, such that the amplitude is diagonal at the baryon pole, then the amplitude will be roughly diagonal over a reasonable range of energies around the pole. If this is the case, and if our dispersion integrals are dominated by low-mass singularities, it is clear that, for our purposes, the graph in Fig. 5(a) will have the same $ijkl$ dependence as the simple bubble in Fig. 5(b), where $(D+\lambda F)$ again indicates a ΠBB coupling with the F/D ratio λ which appears at the B pole. We will take this point of view in the following paper on $SU(3)$ violations in the baryon octet and the decuplet of $\frac{3}{2}^+$ resonances.