CONCLUSION

Results of the two analyses are consistent with each other and with previous experiments of Baglin *et al.*[5] and Lind *et al.*[6]

A value of $|C_A/C_V|=0.94$ predicted by Sakurai,[7] who assumed a Λ_β-decay branching ratio of 0.82×10^{-3}, is just compatible with our results. However, the value $|C_A/C_V|=0.72$ given by Cabbibo[8] is not in good agreement with our result from P_t which favors a predominately axial-vector interaction.

ACKNOWLEDGMENT

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Inverse Compton Scattering of Cosmic-Ray Electrons

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It has long been known that an important mode of energy loss for cosmic-ray electrons is inverse Compton scattering with photons of starlight. Previous calculations of $\langle d\mathcal{E}/dt\rangle_{\text{av}}$ due to this process have involved nonsystematic approximations involving the form of the Klein-Nishina formula and the angular distribution of the radiation as seen in the electron's rest frame. The present paper considers an electron of arbitrary energy in an isotropic thermal radiation field of temperature T. A formally correct expression for $\langle d\mathcal{E}/dt\rangle_{\text{av}}$ is obtained as an asymptotic expansion in the quantity $\mathcal{E}kT/(m_ec^2)^2$ considered as a small parameter. The often quoted result $\langle d\mathcal{E}/dt\rangle_{\text{av}}\propto\mathcal{E}^2$ is seen to be the zero-order term in this expansion. It is also seen that the energy-loss rate changes sign at an energy $\mathcal{E}\approx\frac{3}{2}kT$ as would be expected from thermodynamics. A derivation of the zero-order term is given from classical radiation theory, and from this it is seen that this term also describes the energy-loss rate due to synchrotron radiation as well as from inverse Compton scattering.

I. INTRODUCTION

THE scattering of energetic electrons by low-energy photons, called "inverse" Compton scattering, has been of astrophysical interest for many years. It was first investigated by Feenberg and Primakoff[1] as a process by which cosmic-ray electrons (and protons) would lose energy during their passage through the galaxy. Later Donahue[2] applied the general method of Feenberg and Primakoff to the case of electrons trapped in orbits about the sun.

The result of these two papers that the mean energy loss of an energetic electron of energy $\mathcal{E}$ is proportional to both the photon energy density and to $\mathcal{E}^2$ was applied by Hayakawa and Kobayashi[3] and by Hayakawa and Okuda[4] to the problem of the equilibrium of cosmic-ray electrons in the galaxy. More recently, Felten and Morrison[5] have considered this process as a possible source of galactic x rays[6–8] and gamma rays,[9–11] and Shklovsky[12] has proposed it as a source of x rays in solar flares.

In the calculations of Feenberg and Primakoff and of Donahue the relevant cross-section formula is the Klein-Nishina formula $\sigma(\epsilon',\chi')$ for the scattering of a photon of energy ϵ' by a *stationary* electron through an angle χ'. In essence the scattering probability is expressed in the electron's rest frame and then transformed to the laboratory frame to determine the mean energy transferred from the electron to the photon. In the previous calculations the full Klein-Nishina formula was not used but rather the asymptotic forms for $\epsilon'\ll m_ec^2$ (Thompson scattering) and for $\epsilon'\gg m_ec^2$. In Feenberg and Primakoff the two forms are used in the

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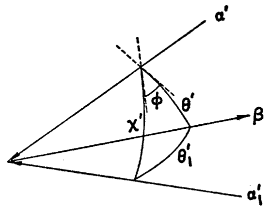

FIG. 1. Angles involved in the scattering process viewed from the electron center-of-mass system.

two regions $\epsilon' < m_e c^2$ and $\epsilon' > m_e c^2$, respectively, as an approximation to the correct formula. Donahue, on the other hand, uses the two formula in the regions $\epsilon' < m_e c^2/4$ and $\epsilon' > 4m_e c^2$ respectively and connects the two regions of his results with an "eyeball" curve. Both authors assume that the electron is energetic enough so that in its rest frame all of the incident radiation has $\theta' \approx 0$ where $\pi - \theta'$ is the angle between the photon momentum in the electron rest frame and the original direction of the electron momentum. This assumption obviously limits the validity of the results to high-energy electrons.

The primary problem with these approximations is that they are nonsystematic; they do not suggest how to apply a higher order correction. In the present calculation the mean rate of energy loss $-\langle d\mathcal{E}/dt \rangle$ of an electron in an isotropic radiation field in thermal equilibrium is determined. The only approximation used for a wide range of $\mathcal{E}$ is a systematic one in that the result is obtained as a true asymptotic expansion in powers of a small parameter $\iota \equiv \mathcal{E}kT/(m_e c^2)^2$ where $\mathcal{E}$ is the electron energy and T is the temperature characteristic of the radiation field. When $\mathcal{E}$ gets so large that $\iota \sim 1$ the expansion is no longer useful and the result must be obtained by methods which are less systematic but which are, nevertheless, quite accurate. We will also discover an interesting relationship between inverse Compton scattering, as described by the zero-order term of the expansion, and synchrotron radiation.

II. FORMULATION OF THE PROBLEM

Consider an electron of energy γ (we shall express the electron energy $\mathcal{E} = \gamma mc^2$, the photon energy $\epsilon = \alpha mc^2$, and $kT = \Theta mc^2$ in terms of the dimensionless parameters γ, α, and Θ) moving in a region of space in which the photon density $n(\alpha,\theta)$ is given as a function of energy α and angle θ where $\pi - \theta$ is the angle between the photon and electron velocity vectors. Letting a prime indicate quantities expressed in the electron's rest frame, we may write for the number of Compton collisions per unit time

$$\left\langle \frac{dN}{dt} \right\rangle_{\rm av} = \left\langle \frac{dN}{\gamma dt'} \right\rangle_{\rm av} = \int d\Omega'(\theta') \times \int_0^\infty d\alpha' \frac{c}{\gamma} n'(\alpha',\theta')\sigma'(\alpha'), \quad (1)$$

where $\sigma'(\alpha')$ is the Klein-Nishina total cross section. Since $n'(\alpha',\theta)d\Omega'(\theta')d\alpha'$ is a number density, it transforms under the Lorentz transformation like an energy α' so that $n'(\alpha',\theta)d\Omega(\theta')d\alpha'/\alpha'$ is an invariant.

We may then set

$$n'(\alpha',\theta)d\Omega'(\theta')d\alpha' = (\alpha'/\alpha)n(\alpha,\theta)d\Omega(\theta)d\alpha \quad (2)$$

and obtain

$$\left\langle \frac{dN}{dt} \right\rangle_{\rm av} = c\int d\Omega(\theta) \int d\alpha \frac{\alpha'}{\gamma\alpha} n(\alpha,\theta)\sigma'(\alpha'). \quad (3)$$

If we denote the lab frame energy of the scattered photon by α_1, the energy transfer in the scattering process is $(\alpha_1 - \alpha)$ and the mean energy loss of the electron is given by

$$\left\langle \frac{-d\gamma}{dt} \right\rangle_{\rm av} = c\int d\Omega(\theta) \int d\alpha \frac{\alpha'}{\gamma\alpha} n(\alpha,\theta) \times \int d\Omega'(\chi')\sigma'(\alpha',\chi')(\alpha_1 - \alpha), \quad (4)$$

where $\sigma'(\alpha',\chi')$ is the differential Klein-Nishina formula for scattering a photon of energy α' through an angle χ'. We have the usual angle-energy relationship for Compton scattering

$$\alpha_1' = \alpha'/[1 + \alpha'(1 - \cos\chi')]. \quad (5)$$

Employing the well-known Doppler shift formula

$$\alpha_1 = \alpha_1'\gamma(1 - \beta\cos\theta_1') = \frac{\alpha'\gamma(1 - \beta\cos\theta_1')}{1 + \alpha'(1 - \cos\chi')}, \quad (6)$$

$$[\beta = (\gamma^2 - 1)^{1/2}/\gamma = v/c],$$

we have

$$\alpha_1 - \alpha = \alpha'\left\{\frac{\gamma(1 - \beta\cos\theta_1')}{1 + \alpha'(1 - \cos\chi')} - \frac{\alpha}{\alpha'}\right\}. \quad (7)$$

Consulting Fig. 1, we obtain the following formula from spherical trigonometry:

$$\cos\theta_1' = \cos\theta'\cos\chi' + \sin\theta'\sin\chi'\cos\phi. \quad (8)$$

Since the cross-section formula cannot depend on ϕ, we may choose χ' and ϕ as our coordinate angles for $d\Omega'(\chi',\phi)$ and immediately integrate over ϕ. This has the effect of multiplying Eq. (4) by 2π and replacing $\cos\theta_1'$ with $\langle \cos\theta_1' \rangle_{\rm av}$, where

$$\langle \cos\theta_1' \rangle_{\rm av} = \cos\theta'\cos\chi'. \quad (9)$$

The Klein-Nishina cross-section formula is

$$\sigma'(\alpha',x') = \frac{r_0^2}{2} \frac{(1 + \cos^2\chi')}{[1 + \alpha'(1 - \cos\chi')]^2} \times \left\{1 + \frac{\alpha'^2(1 - \cos\chi')^2}{(1 + \cos^2\chi')[1 + \alpha'(1 - \cos\chi')]}\right\}, \quad (10)$$

$$(r_0 = e^2/mc^2).$$

Making the following substitutions:

$$\beta\cos\theta' = 1-(\alpha/\gamma\alpha'),\quad 1-\cos\chi' = f,$$
$$d\Omega'(\chi',\phi) = 2\pi df,$$

and inserting Eq. (10) in Eq. (4), we obtain

$$\left\langle\frac{-d\gamma}{dt}\right\rangle_{\rm av} = \pi r_0^2 c\int d\Omega(\theta)\int_0^\infty d\alpha\, n(\alpha,\theta)\frac{\alpha'}{\gamma\alpha}$$
$$\times\int_0^2 df\frac{(f^2-2f+2)}{(1+\alpha'f)^2}\left[1+\frac{(\alpha'f)^2}{(f^2-2f+2)(1+\alpha'f)}\right]$$
$$\times\frac{(\gamma-\alpha/\alpha'-\alpha)f\alpha'}{(1+\alpha'f)}. \quad (11)$$

It should be noted that the variables in this expression are not independent since α, α', γ, and θ are related by the Doppler shift formula

$$\alpha' = \gamma\alpha(1+\beta\cos\theta).$$

We now specialize to the case where $n(\alpha,\theta)$ represents an isotropic radiation field, i.e., $n(\alpha,\theta) = n(\alpha)/4\pi$:

$$d\Omega(\theta) = 2\pi d(\cos\theta) = 2\pi d\alpha'/(\beta\gamma\alpha),$$

and obtain

$$\left\langle\frac{-d\gamma}{dt}\right\rangle_{\rm av} = \tfrac{1}{2}\pi r_0^2 c\int_0^\infty d\alpha\frac{n(\alpha)}{\beta\gamma^2\alpha^2}$$
$$\times\int_0^2 df\int_{\gamma\alpha(1-\beta)}^{\gamma\alpha(1+\beta)} d\alpha'[\alpha'^2(\gamma-\alpha)-\alpha'\alpha]$$
$$\times\left[\frac{(f^3-2f^2+2f)}{(1+\alpha'f)^3}+\frac{\alpha'^2f^3}{(1+\alpha'f)^4}\right]. \quad (12)$$

The integration over f and α' may be done by a straightforward and repeated application of a good set of integral tables. After some time we arrive at the result

$$\left\langle\frac{-d\gamma}{dt}\right\rangle_{\rm av} = \tfrac{1}{2}\pi r_0^2 c\int_0^\infty d\alpha\frac{n(\alpha)}{\beta\gamma^2\alpha^2}F(\alpha,\gamma),$$
$$F(\alpha,\gamma) = \gamma[f_1(\alpha\bar\gamma)-f_1(\alpha/\bar\gamma)]$$
$$-\alpha[f_2(\alpha\bar\gamma)-f_2(\alpha/\bar\gamma)], \quad (13)$$

where $\bar\gamma = \gamma(1+\beta) = \gamma+(\gamma^2-1)^{1/2}$ and

$$f_1(z) = (z+6+3/z)\ln(1+2z)-(22z^3/3+24z^2$$
$$+18z+4)(1+2z)^{-2}-2+2\,{\rm Li}_2(-2z), \quad (14)$$

$$f_2(z) = (z+31/6+5/z+3/2z^2)\ln(1+2z)$$
$$-(22z^3/3+28z^2+103z/3+17+3/z)$$
$$\times(1+2z)^{-2}-2+{\rm Li}_2(-2z). \quad (15)$$

The function ${\rm Li}_2(z)$ is the Eulerian dilogarithm[13] defined by

$${\rm Li}_2(z) = -\int_0^z\frac{\ln(1-z')}{z'}dz' \quad\text{for complex } z,$$

$${\rm Li}_2(z) = \sum_{n=1}^\infty\frac{z^n}{n^2} \quad\text{for}\quad |z|<1.$$

III. EVALUATION OF THE INTEGRAL

We will now consider that the photon density is described reasonably well by the Planck radiation formula

$$n(\alpha) = \frac{\langle\alpha\rangle}{3!\zeta(4)\Theta^4}\frac{\alpha^2}{\exp(\alpha/\Theta)-1},$$

where $\langle\alpha\rangle = \int_0^\infty\alpha n(\alpha)d\alpha$ = "energy" density and $\zeta(p)$ is the Riemann zeta function defined by

$$\zeta(p) = 1+1/2^p+1/3^p+\cdots$$

and

$$\zeta(4) = \pi^4/90.$$

Expression (13) may now be written

$$-\langle d\gamma/dt\rangle_{\rm av} = \pi r_0^2 c\langle\alpha\rangle N(\gamma), \quad (16)$$

$$N(\gamma) = \frac{(15/\pi^4)}{\Theta^4 2(\gamma^2-1)^{1/2}}$$
$$\times\left[\frac{1}{\bar\gamma}\int_0^\infty\frac{f_1(z)dz}{\exp(z/\bar\gamma\Theta)-1}-\bar\gamma\int_0^\infty\frac{f_1(z)dz}{\exp(z\bar\gamma/\Theta)-1}\right]$$
$$-\frac{(15/\pi^4)}{\Theta^4 2\gamma(\gamma^2-1)^{1/2}}\left[\frac{1}{\bar\gamma^2}\int_0^\infty\frac{zf_2(z)dz}{\exp(z/\bar\gamma\Theta)-1}-\bar\gamma^2\right.$$
$$\left.\times\int_0^\infty\frac{zf_2(z)dz}{\exp(z\bar\gamma/\Theta)-1}\right], \quad (17)$$

where we have transformed variables in such a way as to make the arguments of f_1 and f_2 the variable of integration.

Up to this point our calculations have been exact within the framework of the physical situation that we have considered. We now ask whether there is some systematic approximation scheme that will allow us to evaluate the integrals in Eq. (17) in some simple manner.

To this end we notice that f_1 and f_2 have a second-order pole and a branch point at $z=-\frac{1}{2}$. This means that the power expansions of f_1 and f_2,

$$f_1 = \sum_{n=1}^\infty A_n z^n,\quad f_2 = \sum_{n=1}^\infty B_n z^n,$$

are convergent only for $|z|<\frac{1}{2}$ and our integrals are

[13] L. Lewin, *Dilogarithms and Associated Functions* (MacDonald and Company, Ltd., London, 1958).

TABLE I. Expansion coefficients for f_1 and f_2.

n	A_n	B_n
0	0.0	0.0
1	0.0	0.0
2	0.0	0.13333320×10^1
3	0.88888827×10^0	-0.28444412×10^1
4	-0.27999992×10^1	0.69999923×10^1
5	0.78400015×10^1	-0.16784715×10^2
6	-0.20520631×10^2	0.39161810×10^2
7	0.51156461×10^2	-0.89469178×10^2
8	-0.12304761×10^3	0.20112988×10^3
9	0.28815807×10^3	-0.44645477×10^3
10	-0.66115232×10^3	0.98096851×10^3

over the range $0\geq z<\infty$. However, we notice that the term $[\exp(kz)-1]^{-1}$ is a function that peaks at $1/k$ and drops off as $\exp(-kz)$ for values of z significantly greater than $1/k$. Therefore, if $\Theta\bar{\gamma}\ll\frac{1}{2}$, only the portion of f_1 and f_2 for $z<\frac{1}{2}$ will contribute significantly to the integral. Since $\Theta=kT/m_ec^2\approx10^{-6}$ for $T=6000°$K, we may consider $\bar{\gamma}\Theta$ as a small parameter of order ι (since $\bar{\gamma}\geq1$, $\Theta/\bar{\gamma}\leq\bar{\gamma}\Theta$). The expansion coefficients A_n and B_n for f_1 and f_2 respectively are obtained in a straightforward manner from the known expansions of $\ln(1+2z)$, $(1+2z)^{-2}$, and $\mathrm{Li}_2(-2z)$. These coefficients for n up to ten are given in Table I.

If we now insert the series form of f_1 and f_2 into the integrals in Eq. (17) and ignore the fact that they are not convergent for $|z|>\frac{1}{2}$ we will obtain a formal series for $N(\gamma)$ which we hope will not be in error by very much. Making use of the formula

$$\int_0^\infty \frac{x^n dx}{\exp(x/\epsilon)-1}=n!\zeta(n+1)\epsilon^{n+1}, \tag{18}$$

we have

$$\int_0^\infty \frac{f_1(z)dz}{\exp(z/\epsilon)-1}=\sum_n A_n n!\zeta(n+1)\epsilon^{n+1},$$
$$\int_0 \frac{zf_2(z)dz}{\exp(z/\epsilon)-1}=\sum_n B_n(n+1)!\zeta(n+2)\epsilon^{n+2}. \tag{19}$$

We may see at once that these series are not convergent, since from the known circle of convergence of the series for f_1 and f_2 we have $\lim_{n\to\infty}(A_n/A_{n-1})=\lim_{n\to\infty}(B_n/B_{n-1})=2$.

Therefore, since $\lim_{z\to\infty}\zeta(z)=1$, we have

$$\lim_{n\to\infty}\left(\frac{A_n n'\zeta(n+1)\epsilon^{n+1}}{A_{n-1}(n-1)!\zeta(n)\epsilon}\right)=2n\epsilon.$$

So for any finite ϵ there is a value of $n\approx1/2\epsilon$ beyond which the terms of the series grow without limit. It is fairly easy to see that this is the same series that would be generated by repeated partial integrations of the integrals in (19) since $A_n n!=d^nf_1/dz^n$. Furthermore, it is demonstrated in the Appendix that this is in fact a correct asymptotic expansion of the integrals in the sense that

$$F(\epsilon)=\sum_{n=1}^{N} C_n\epsilon^n+R_N(\epsilon)$$

and

$$R_N(\epsilon)=O(\epsilon^{N+1}).$$

Noting from Table I that

$$A_0=A_1=A_2=B_0=B_1=0,$$

we may, after some rearrangement of terms, write for $N(\gamma)$

$$N(\gamma)=(15/\pi^4)\sum_{m=0}[\gamma^2A_{m+3}C_{m+3}(\gamma)$$
$$-B_{m+2}C_{m+2}(\gamma)](m+3)!\zeta(m+4)(\gamma\Theta)^m, \tag{20}$$

where

$$C_n(\gamma)=\sum_{i=\mathrm{odd}}^{n}\binom{n}{i}(1-\gamma^{-2})^{(i-1)/2}=\frac{\bar{\gamma}^n-1/\bar{\gamma}^n}{2\gamma^{n-1}(\gamma^2-1)^{1/2}}.$$

We see that for $\gamma\Theta\ll1$ (in our case $\gamma\ll10^6$) this series gives an excellent approximation provided you do not sum beyond $n\approx1/2\gamma\Theta$.

The zero-order term in this series is

$$[\gamma^2A_3C_3(\gamma)-B_2C_2(\gamma)]3!\zeta(4)(15/\pi^4)=A_3(4\gamma^2-1)-2B_2$$
$$=4A_3(\gamma^2-1)$$

since $2B_2=3A_3$.

Inserting this approximation to $N(\gamma)$ in expression (16), we have to zero-order

$$-\langle d\mathcal{E}/dt\rangle_{\mathrm{av}}=3.555\pi r_0^2c\rho(p/m_ec)^2, \tag{21}$$

where ρ is the photon energy density in conventional units. We see that for electrons of sufficiently low energy such that $pc\neq\mathcal{E}$, the energy-loss rate is proportional to p^2 rather than $\mathcal{E}^2$.

We note further that expression 20 is negative for γ sufficiently close to 1. This means that a very low-energy electron can, on the average, gain energy from the radiation field. Setting expression (20) to zero, we have to first order

$$A_34!\zeta(4)(\gamma^2-1)=-[4A_4-3B_3+(\gamma^2-1)$$
$$\times(8A_4-B_3/\gamma^2)]4!\zeta(5)(\gamma\Theta). \tag{22}$$

Since Eq. (22) states that $\gamma^2-1=O(\epsilon)$, we may neglect the term in γ^2-1 on the right-hand side and obtain

$$\gamma^2-1=[(3B_3-4A_4)/A_3][\zeta(5)/\zeta(4)]\gamma\Theta$$
$$=3\gamma\Theta[\zeta(5)/\zeta(4)]$$
$$=0.958(3\gamma\Theta). \tag{23}$$

We see that this is a few percent below the equipartition energy for a relativistic gas,[14] $\gamma^2-1=3\gamma\Theta$. A similar calculation for the energy loss of a test particle

[14] R. C. Tolman, *The Principles of Statistical Mechanics* (Oxford University Press, London, 1938), p. 97.

in a Maxwellian, hard-sphere gas gives $\langle d\mathcal{E}/dt\rangle=0$ for $=0.981(3kT/2)$ which is in qualitative agreement with Eq. (23).

For $\gamma\gtrsim\Theta^{-1}$ the asymptotic series (20) is no longer useful. In this region we must resort to less systematic but nevertheless quite accurate methods of approximation. First of all, note that if $\gamma\Theta\gtrsim1$ then $\gamma/\Theta\gg1$. This means that the second and fourth integrals in expression (17) may be completely ignored. We may, in fact, rewrite expression (17), noting that $\bar{\gamma}\approx2\gamma$, as

$$N(\gamma)=(15/\pi^4)\left[(4\gamma^2)^{-1}\int_0^\infty\frac{f_1(z)dz}{\exp(z/2\gamma\Theta)-1}-(8\gamma^4)^{-1}\int_0^\infty\frac{zf_2(z)dz}{\exp(z/2\gamma\Theta)-1}\right]. \quad (24)$$

From expressions (14) and (15) we also see that both f_1 and f_2 tend to $z\ln(az)$ for z large, where $a=2e^{-11/6}$. In fact, for $z\geq z'=7.75\times10^4$ this approximation is good to within one part in 10^4. We may, therefore, perform the integrations indicated in expression (24) in two parts; from zero to z' we evaluate the integrals numerically and from z' to ∞ we use $z\ln(az)$ for f_1 and f_2 and obtain analytic expressions.

Carrying out this procedure we obtain the following, rather opaque expression, where $\Gamma=2\Theta\gamma$:

$$\begin{aligned}N(\gamma)=(60/\pi^4)\Bigg\{\gamma^2\Bigg[&\Gamma^{-4}\left(\int_0^{z'}\frac{f_1(z)dz}{\exp(z/\Gamma)-1}\right)_{\text{numerical}}\\&-\Gamma^{-3}z'\ln(az')\ln(1-e^{-z}/\Gamma)\\&+\Gamma^{-2}[\ln(az')+1]\,\mathrm{Li}_2(e^{-z/\Gamma})\\&+\Gamma^{-2}\sum_{n=1}^{\infty}\frac{-\mathrm{Ei}(-nz'/\Gamma)}{n^2}\Bigg]\\&-\frac{1}{2}\Bigg[\Gamma^{-4}\left(\int_0^{z'}\frac{zf_2(z)dz}{\exp(z/\Gamma)-1}\right)_{\text{numerical}}\\&-\Gamma^{3}z'^2\ln(az')\ln[1-\exp(-z'/\Gamma)]\\&+\Gamma^{-2}z'[2\ln(az')+1]\,\mathrm{Li}_2[\exp(-z'/\Gamma)]\\&+\Gamma^{-1}[2\ln(az')+3]\,\mathrm{Li}_3[\exp(-z'/\Gamma)]\\&+\Gamma^{-1}2\sum_{n=1}^{\infty}\frac{-\mathrm{Ei}(-nz'/\Gamma)}{n^3}\Bigg]\Bigg\}. \quad (25)\end{aligned}$$

$\mathrm{Li}_3(z)$ is the trilogarithm where, in general,

$$\mathrm{Li}_n(z)=\int_0^z\frac{\mathrm{Li}_{n-1}(z')}{z'}dz'=\sum_{m=1}^{\infty}z^m/m^n,\quad |z|<1$$

and $\mathrm{Ei}(z)$ is the exponential integral defined by

$$-\mathrm{Ei}(z)=\int_z^\infty\frac{e^{-t}}{t}dt.$$

The values of this expression may be calculated quite easily on a computer; however, it is instructive to consider the situation when γ (hence Γ) becomes very large. A brief examination shows that every term in the expression tends towards zero with the exception of the first series $\sum[-\mathrm{Ei}(-nz'/\Gamma)/n^2]$. From the known properties of $\mathrm{Ei}(-x)$ we have

$$\mathrm{Ei}(-x)=C+\ln x+f(x),$$

where $C=0.577215665\cdots$ is Euler's constant and $f(0)=0$ and $|f(x)|\to|C+\ln(x)|$ as $x\to\infty$.

Therefore $\sum f(nx)/n^2$ converges uniformly for all finite x and converges to zero for x equal to zero. We are then left with

$$-\sum_n\frac{C+\ln(z'/\Gamma)}{n^2}-\sum_n\frac{\ln n}{n^2}=\zeta(2)\ln(2\Theta\gamma/z)+\text{const}.$$

We then have

$$N(\gamma)\to(15/\pi^4\Theta^2)[\zeta(2)\ln\gamma+\text{const}] \quad (26)$$

as γ becomes very large.

IV. RESULTS AND CONCLUSIONS

In Fig. 2 we present curves of $N(\gamma)=-\langle d\mathcal{E}/dt\rangle\times(\pi r_0^2c\rho)^{-1}$ as a function of $\gamma=\mathcal{E}/m_ec^2$ for a range of T from 5000 to 10 000°K. The necessary computation was done on an IBM 7094. For values of $2\gamma\Theta\leq10^{-2}$ the asymptotic series 20 was used including the seventh-order term. For values of $2\gamma\Theta>10^{-2}$ expression (25) was evaluated.

It is immediately seen that the approximation $-\langle d\mathcal{E}/dt\rangle\propto\mathcal{E}^2$ is very good for $4<\gamma<4000$. Below 4 the dependence on p^2 rather than $\mathcal{E}^2$ is manifest. Between about 10^8 and 10^9 the curves achieve the form

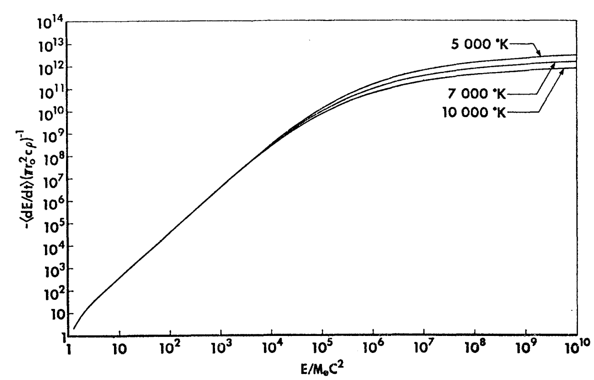

FIG. 1. $N(\gamma)$ versus γ for values of Θ corresponding to $T=5000$, 7000, and 10 000°K.

of expression (26), namely, proportional to $\ln \mathcal{E}$ and T^{-2}. The values of $N(\gamma)$ for $\gamma \to 1$ cannot be shown in Fig. 2 due to the logarithmic scale; however, the zero- and first-order terms of expression (20) give quite accurate values in the region $\gamma \sim 1$.

It is of interest to compare inverse Compton scattering and synchrotron radiation as an energy-loss mechanism for cosmic-ray electrons. To this end we first note that for small values of $\gamma\Theta$ the inverse Compton scattering process is a classical radiation process; in this limit the Klein-Nishina cross section is just the Thompson scattering cross section.

For an electron in arbitrary electromagnetic field the instantaneous radiated power is given by[15]

$$P=\frac{2e^4}{3m^2c^3}\left\{\frac{(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{H})^2-(\boldsymbol{\beta}\cdot\mathbf{E})^2}{1-\beta^2}\right\}$$

$$=\frac{2e^4}{3m^2c^3}\left\{E^2+\frac{(\boldsymbol{\beta}\times\mathbf{E})^2+(\boldsymbol{\beta}\times\mathbf{H})^2-2\boldsymbol{\beta}\cdot(\mathbf{E}\times\mathbf{H})}{1-\beta^2}\right\}. \quad (27)$$

If we now assume that any energy flow is isotropic $[\langle\boldsymbol{\beta}\cdot(\mathbf{E}\times\mathbf{H})\rangle_{\text{av}}=0]$ and that the fields are unpolarized $[\langle(\boldsymbol{\beta}\times\mathbf{E})^2\rangle_{\text{av}}=\frac{2}{3}\beta^2E^2;\ \langle(\boldsymbol{\beta}\times\mathbf{H})^2\rangle_{\text{av}}=\frac{2}{3}\beta^2H^2]$, we now have

$$\langle P\rangle_{\text{av}}=(8/3)\pi r_0^2(cE^2/4\pi)+3.555\cdots \pi r_0^2c[(E^2+H^2)/8\pi](p/mc)^2. \quad (28)$$

If we consider the situation that the only $\mathbf{E}$ fields present are radiation fields, the radiation energy flux incident on the electron is just $(8/3)\pi r_0^2(cE^2/4\pi)$ so that the loss of mechanical energy is just radiation out minus radiation in, or

$$-\langle d\mathcal{E}/dt\rangle_{\text{av}}=3.555\cdots\pi r_0^2c[(E^2+H^2)/8\pi](p/mc)^2, \quad (29)$$

where the energy density includes radiation *and* static magnetic fields.

This is, of course, identical to Eq. (21) and we see that the relative importance of inverse Compton scattering compared to synchrotron radiation depends only on the energy density of the radiation field versus the energy density of the magnetic fields.[16] We see, therefore, that in the galaxy where the energy density of both starlight and magnetic fields is of the order of 1 eV/cc, the two processes will be on a roughly equal footing.

[15] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 212.

[16] Note that the above arguments in no way imply that the frequency spectrum of the radiation emitted by the particle will be similar in the two cases; this depends on the frequency spectrum of the thermal radiation in the first case and the structure of the magnetic field over distances of the order of $\sim mc^2/eH$ in the second case.

APPENDIX: GENERATION OF AN ASYMPTOTIC EXPANSION BY REPEATED PARTIAL INTEGRATIONS

Consider the integral

$$I=\int_0^\infty \frac{f(z)dz}{\exp(z/\epsilon)-1},$$

where $f(z)$ is analytic on the positive real line including zero, and it and all of its derivatives increase no faster than a polynomial as $z\to\infty$. We may expand $[\exp(z/\epsilon)-1]$ as

$$\sum_{n=1}^{\infty} e^{-nz/\epsilon},$$

which series converges uniformly in z for $z>0$. Since the series does not converge for $z=0$ we must also demand that $f(z)$ go to zero at least as fast as z for $z\to 0$. Since the integral now has contributions only in the region of uniform convergence, we may integrate term by term. Integrating a particular term by parts N times, we obtain

$$\int_0^\infty f(z)e^{-nz/\epsilon}dz=\sum_{m=0}^{N} f^{(m)}(0)\left(\frac{\epsilon}{n}\right)^{m+1}+R_N,$$

where

$$R_N=\int_0^\infty \frac{f^{(N+1)}(z)e^{-nz/\epsilon}dz}{(n/\epsilon)^{N+1}}.$$

Due to the analyticity of $f(z)$ on the positive real axis and the limitation on its growth as $z\to\infty$, $f^{(N+1)}(z)$ may be bounded by $|f^{(N+1)}(z)|\le A_N+B_Nz^q$ so that

$$|R_N|\le A_N(\epsilon/n)^{N+2}+B_N\Gamma(q+1)(\epsilon/N)^{N+2+q}.$$

If we now sum over n, remembering that $f(0)=0$, we have

$$I=\sum_{m=1}^{N} S_m+R'_N,$$

where

$$S_m=\zeta(m+1)f^{(m)}(0)\epsilon^{m+1}$$

and

$$|R'_N|\le A_N\zeta(N+2)\epsilon^{N+2}+B_N\Gamma(q+1)\zeta(N+2+q)\epsilon^{N+2+q}.$$

This expansion, therefore, satisfies the definition[17] of an asymptotic expansion of I as $\epsilon\to 0$.

[17] A. Erdelyi, *Asymptotic Expansions* (Dover Publications, Inc., New York, 1956), p. 11.