Self-Consistent, Nondegenerate Multiplets*

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The restrictions of self-consistency are investigated for two sets of interacting particles—vector and scalar (or pseudoscalar)—with unequal masses. Self-consistency is studied within a field-theoretic framework and within a bootstrap framework. It is assumed that solutions exist when the particles of a given set have equal masses and the coupling constants are proportional to the structure constants of SU_3. The unequal-mass case is studied by perturbing the equal-mass solutions and retaining only first-order terms in the mass and coupling-constant shifts. It is found that fully self-consistent solutions do not exist in either case, but it it is seen how such solutions can come about in the field-theoretic case. In the bootstrap analysis it is very difficult to understand how self-consistent solutions develop unless hidden identities are satisfied.

I. INTRODUCTION

RECENTLY, considerable interest has surrounded two theoretical approaches to an understanding of elementary particle masses and their coupling constants, viz., "bootstrap" calculations[1] and "supermultiplet" structures.[2] Both have experienced some success in the sense that little else has been particularly successful at all. The bootstrap calculations have attempted to exploit in terms of self-consistency the very restrictive constraint of crossing symmetry in an approximate way. This is usually done within the context of N/D with two-particle unitarity. The group-theoretical approach attempts to gather particles into supermultiplets where, for example, the particles are the generators of some Lie group and the coupling constants are proportional to the structure constants of the group. Certainly, the most favored group at the present time is SU_3.

Within the past year, several interesting questions have developed as combinations of these approaches have been considered; e.g., can a bootstrap calculation yield a particular symmetry as a self-consistent solution; if so, is it a unique solution, etc.? Generally speaking, all such analyses[3] assume equal masses everywhere and proceed by the standard techniques, but some effort has been devoted to the unequal mass case.[4] The case of nondegenerate, supermultiplets, i.e., unequal masses, within a self-consistent calculation (with the usual approximations) is difficult to handle primarily because of its complexity. However, it raises several interesting questions. For example, consider a supermultiplet of particles interacting among themselves to produce another multiplet, say, a set of scalar or pseudoscalar mesons bootstrapping a set of vector mesons. Some of the questions of interest are the following: (1) Do nondegenerate solutions exist? (2) If solutions exist, can one choose the scalar masses arbitrarily? (3) If one considers the scalar masses to satisfy some specific relation among themselves, i.e., a mass formula, must a similar mass formula hold for the vector masses? (4) Are the solutions unique? If one takes the viewpoint that all masses and coupling constants are determined by the requirements of self-consistency, then one should be able to answer these questions, among others. The crucial point of course is how in fact does the imposition of self-consistency yield such answers. In this paper, we will attempt to explicate the mechanism by which the self-consistency manifests itself.

We will study the specific example of two sets of interacting particles—vector particles and scalar (or pseudoscalar) particles. The paper is divided into two investigations. The first is a field-theoretic analysis (Sec. II) and the second is a bootstrap analysis (Sec. III). Our basic procedure will be the following: We assume in each case that an equal mass solution exists in which the coupling constants are proportional to the structure constants of some Lie group. In fact, we will consider eight particles in each set of particles and take the group to be SU_3, since this arrangement has the most impressive experimental credentials. Of course, our analysis need not be confined to SU_3 and the self-consistency problem for other groups can be studied with the same techniques.

The solutions are perturbed, i.e., the symmetry is broken, and we examine the consequences of the self-consistency requirement to first order in the mass and coupling constant shifts. Within each of the two contexts of field-theoretic and bootstrap self-consistencies and within our very limited approximations, we attempt to answer as many of the interesting questions as we can. In Sec. IV we give a summary of our conclusions.

* Work supported in part by the U. S. Air Force Office of Scientific Research and Development Command.

[1] See for example, the review article by F. Zachariasen in *Strong Interactions and High Energy Physics, Scottish Universities Summer School, 1963* (Oliver and Boyd, Edinburgh and London, 1964).

[2] M. Gell-Mann, California Institute of Technology Synchrotron Lab. Report CTSL-20, 1961 (unpublished); Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[3] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); Hong-Mo Chan, P. C. DeCelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963).

[4] R. H. Capps, Phys. Rev. **132**, 2749 (1963); R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1354 (1963).

II. FIELD-THEORETIC SELF-CONSISTENCY

Here, as in Sec. III, we will take as the basis of our model a system consisting of two sets of interacting particles:

(a) A set of eight vector particles r, s, $\cdots$ of masses m_r, m_s, $\cdots$ represented by the real fields A_r, A_s, $\cdots$.

(b) A set of eight scalar or pseudoscalar particles a, b, $\cdots$ of masses μ_a, μ_b, $\cdots$ represented by the fields ϕ_a, ϕ_b, $\cdots$.

Further, we will consider these two sets to interact via a derivative-type coupling of the form

$$L_I=\tfrac{1}{2}\sum_{a,b,r} g_{ab}{}^r(\phi_a\partial_\mu\phi_b-\phi_b\partial_\mu\phi_a)A_r{}^\mu.$$

Divergence difficulties will be dealt with simply by introducing momentum-space cutoffs (or by a modification to the vector propagator) wherever it is required (see Sec. IID). [It might be thought that it would be simpler to make the set (a) scalar particles and the set (b) pseudoscalar, so as to avoid any spin complications. But then there would be no possibility of trilinear SU_3-symmetric coupling in which both sets of particles are octets.]

In the context of Lagrangian field theory, it has been suggested[5] that the requirement for a particle to be composite is that its wave function and vertex renormalization constants both vanish. We shall consider these two conditions in the lowest nontrivial order of perturbation theory first for the set (a), then for the set (b), and finally for both sets simultaneously. This is what we mean when we speak of field-theoretic self-consistency. When we consider both sets simultaneously we will usually speak of this as full self-consistency.

At each stage we shall assume that these conditions are satisfied, and thus that a self-consistent solution exists, when all the masses of set (a) and set (b) are equal, say to m_0 and μ_0, respectively, and the coupling constants $g_{ab}{}^r$ are proportional to the structure constants of SU_3. We will call this the "symmetric solution." We do not assume that this is a unique solution, but only that this *is* a solution. The object of our analysis will be to examine, in a linear approximation, the possibility of the existence of solutions in which the masses within the sets (a) and (b) differ slightly from equality.

A. Lowest Order, Self-Consistency Equations

For a scalar particle A considered as a composite of the particles B and C with coupling constant G, the lowest order, self-consistency equations are

$$1=\frac{G^2}{\pi}\int_{(M_B+M_C)^2}^{\infty} dS'\frac{\rho_{BC}(S')}{(S'-M_A{}^2)^2} \tag{1}$$

$$G=\frac{G^3}{\pi}\int_{(M_B+M_C)^2}^{\infty} dS'\frac{\rho_{BC}(S')T_B(S')}{S'-M_A{}^2}, \tag{2}$$

where M_A, M_B, M_C are the masses of A, B, C, respectively, $\rho_{BC}(S)$ is the phase-space factor for particles B and C in an S state, and $T_B(S)$ is the Born approximation to S-wave (BC) scattering with A exchange. For the moment we shall ignore spin complications, and in succeeding sections we will make the appropriate modifications where necessary. The right-hand side of Eq. (1) may be looked on as a dispersion relation for the derivative of the self-energy of A evaluated on the mass shell, and the right-hand side of Eq. (2) may be looked on as a dispersion relation for the (ABC) vertex function evaluated on the mass shell.

Equation (1) may be "proved"[6] by using the identity

$$1=\sum_P |\langle P|\phi_A(x)|0\rangle|^2, \tag{3}$$

where ϕ_A is the (unrenormalized) A particle field, $|0\rangle$ is the physical vacuum, and the $|P\rangle$ are a complete set of physical states. Each term in the sum is the probability that the bare particle is contained in the state $|P\rangle$. The one-particle states contribute a term in the sum proportional to the wave-function renormalization constant for particle A which we take to vanish. If we include in addition only two-particle states, then we have

$$1=\sum_{\mathbf{P}_1,\mathbf{P}_2} |\langle P_1{}^B P_2{}^C|\phi_A(x)|0\rangle|^2. \tag{4}$$

Equation (4) yields Eq. (1) using $(\square^2-M_A{}^2)\phi_A=j_A$ and taking the matrix element of the current in lowest order to be G.

To generalize Eq. (1) we consider A replaced by a set of particles of the same spin and parity, and similarly B and C. Since we can always choose the particles within each set so that the bare particles are orthogonal, Eq. (3) is replaced by

$$\delta_{ij}=\sum_p \langle 0|\phi_{Ai}(x)|P\rangle\langle P|\phi_{Aj}(x)|0\rangle$$

and instead of Eq. (1) we obtain

$$\delta_{ij}=\sum_{\beta\gamma} G_{i\beta\gamma}G_{j\beta\gamma}\frac{1}{\pi}\int_{(M_\beta+M_\gamma)^2} dS'\frac{\rho_{\beta\gamma}(S')}{(S'-M_i^2)(S'-M_j^2)}, \tag{5}$$

where β and γ label particles in the B and C multiplets.

To "derive" Eq. (2) one notes that the vertex renormalization constant, to lowest order, is $Z_v=1-L$, where L is essentially the Feynman expression for the vertex diagram. If one requires $Z_v=0$, one obtains Eq. (2). This may also be obtained by assuming an unsubtracted dispersion relation for the vertex function with the scattering amplitude replaced by the Born approximation and the vertex function evaluated on the mass shell. The modification required when A, B, C

[5] A. Salam, Nuovo Cimento **25**, 224 (1962).

[6] For example, see W. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958).

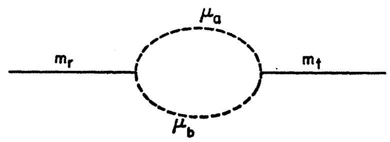


FIG. 1. Vector self-energy diagram.

are replaced by particle sets is

$$G_{i\beta\gamma}=\sum_{j\delta\epsilon} G_{j\beta\delta}G_{j\gamma\epsilon}G_{i\delta\epsilon}\frac{1}{\pi}\int_{(M_\delta+M_\epsilon)^2}^{\infty} dS'\frac{\rho_{\delta\epsilon}(S')T_B(S')}{S'-M_i^2}, \quad (6)$$

where T_B is now the S-wave Born approximation for $B_\beta+B_\gamma \to B_\delta+B_\epsilon$ with the exchange of A_j.

In the succeeding sections we will consider Eqs. (5) and (6) as implementing the requirement of compositeness with the modifications appropriate to the cases when the sets B and C are identical or when one of the sets has spin one rather than zero.

B. Self-Consistency of Vector Particles

Let us first consider the self-consistency of the set of vector particles. In terms of the notation introduced at the beginning of this section, the equations corresponding to Eqs. (5) and (6) are

$$\delta_{rt}=\sum_{ab} g_{ab}{}^r g_{ab}{}^t I(a,b;r,t) \quad (7)$$

$$g_{ab}{}^r=\sum_{cds} g_{ac}{}^s g_{bd}{}^s g_{cd}{}^r J(c,d,s;a,b,r), \quad (8)$$

where the functions $I(a,b;r,t)$ and $J(c,d,s;a,b,r)$ correspond to the Feynman diagrams given in Figs. 1 and 2, respectively, without coupling constants. These functions of course depend on the particle masses and this is indicated by the index of the particle in question.

Now the demand that the vector particles be in fact composite particles [i.e., Eqs. (7) and (8)] cannot be satisfied for arbitrary masses and coupling constants, at least not within the approximations used to obtain Eqs. (7) and (8). Thus, within the present context of an attempt to obtain a self-consistent solution for the vector set (a), we may regard Eqs. (7) and (8) as equations from which we can determine the masses of set (a) and the coupling constants $g_{ab}{}^r$ as functions of the masses of the scalar set (b) (and any cutoff parameters necessary). Our basic assumption regarding these equations is the following: When all the scalar masses are set equal, to μ_0 say, these equations possess a solution in which all the vector masses are equal, to m_0 say, and the coupling constants $g_{ab}{}^r$ are proportional to

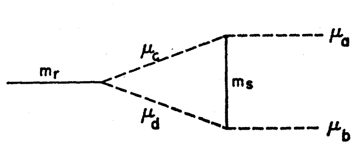


FIG. 2. Vertex function diagram.

the structure constants of SU_3,

$$g_{ab}{}^r=GC_{abr}, \quad (9)$$

where it is possible to choose the C_{abr} antisymmetric in all three indices and normalized so that

$$\sum_{ab} g_{ab}{}^r g_{ab}{}^s=2G^2\delta_{rs}. \quad (10)$$

From the Jacobi identity for the C_{abr} we can write

$$\sum_s (g_{ac}{}^s g_{bd}{}^s-g_{ad}{}^s g_{bc}{}^s)=\sum_s g_{ab}{}^s g_{cd}{}^s, \quad (11)$$

and using the antisymmetry, we have

$$\sum_{cds} g_{ac}{}^s g_{bd}{}^s g_{cd}{}^r=\tfrac{1}{2}\sum_{cds}(g_{ac}{}^s g_{bd}{}^s-g_{ad}{}^s g_{bc}{}^s)g_{cd}{}^r$$
$$=\tfrac{1}{2}\sum_s g_{ab}{}^s\sum_{cd} g_{cd}{}^s g_{cd}{}^r$$

so that, by use of Eq. (10), we have

$$\sum_{cds} g_{ac}{}^s g_{bd}{}^s g_{cd}{}^r=G^2 g_{ab}{}^r. \quad (12)$$

The antisymmetry property and Eqs. (10), (11), and (12) constitute all the group properties we shall use in the ensuing discussion.

When $\mu_a=\mu_b=\mu_c=\mu_d=\mu_0$ and $m_r=m_s=m_t=m_0$ let $I(a,b;r,t)=I_0$ and $J(c,d,s;a,b,r)=J_0$, then Eqs. (7) and (8) reduce to

$$I_0=1/2G^2, \quad (13)$$

$$J_0=1/G^2. \quad (14)$$

These equations may now be solved to obtain G^2 and m_0/μ_0 (perhaps as functions of cutoff parameters). The content of our assumption is that Eqs. (13) and (14) in fact have at least one solution, the symmetric solution.

Given the existence of the symmetric solution, we now consider the possibility that solutions exist when the scalar masses, the input, are not equal, and ask, if solutions exist, what are the values of the vector masses and the coupling constants, the output. We in fact consider this possibility when the scalar masses differ only slightly from equality

$$\mu_i^2=\mu_0^2+\delta_i$$

and we shall work only to first order in δ_i. If solutions exist in the unequal mass case, then the results of the self-consistency requirement will yield

$$m_i^2=m_0^2+\Delta_i,$$

and

$$g_{ab}{}^r=GC_{abr}+\gamma_{ab}{}^r=g_{ab}{}^r(0)+\gamma_{ab}{}^r.$$

Since $g_{ab}{}^r$ represents the coupling of two scalar (or pseudoscalar) particles to a vector particle, it must be antisymmetric in the indices a and b. If follows that $\gamma_{ab}{}^r$ has this property; however, unlike $g_{ab}{}^r(0)$, it may not be antisymmetric under the interchange $a\leftrightarrow r$ or $b\leftrightarrow r$.

We assume that Δ_r and $\gamma_{ab}{}^r$ are small, and that we need only consider first-order terms in Δ and γ as well as δ. These are certainly strong assumptions and certainly are not satisfied in the real world, but we believe that such an analysis may lead to some understanding of the restrictions imposed in the real world by self-consistency.

We now expand Eqs. (7) and (8) about the symmetric solution and obtain

$$\sum_{ab} g_{ab}{}^r g_{ab}{}^t[(\delta_a+\delta_a)I^\mu+(\Delta_r+\Delta_t)I_m] +I\sum_{ab}[g_{ab}{}^r\gamma_{ab}{}^t+\gamma_{ab}{}^r g_{ab}{}^t]=0, \quad (15)$$

$$\gamma_{ab}{}^r=J\sum_{cds}(\gamma_{ac}{}^s g_{bd}{}^s g_{cd}{}^r+g_{ac}{}^s\gamma_{bd}{}^s g_{cd}{}^r+g_{ac}{}^s g_{bd}{}^s\gamma_{cd}{}^r) +\sum_{cds} g_{ac}{}^s g_{bd}{}^s g_{cd}{}^r[(\delta_c+\delta_d)J^\mu+\Delta_s J^m] +\frac{1}{J}g_{ab}{}^r[(\delta_a+\delta_b)J_\mu+\Delta_r J_m], \quad (16)$$

where all functions are evaluated at the symmetric solution, and we have dropped this explicit indication. Here we have defined

$$(I^\mu,I_m)=\left(\frac{\partial}{\partial\mu_a{}^2},\frac{\partial}{\partial m_r{}^2}\right)I(a,b;r,t),$$
$$(J^\mu,J_\mu,J^m,J_m)=\left(\frac{\partial}{\partial\mu_c{}^2},\frac{\partial}{\partial\mu_a{}^2},\frac{\partial}{\partial m_s{}^2},\frac{\partial}{\partial m_r{}^2}\right)J(c,d,s;\,a,b,r), \quad (17)$$

all derivatives being evaluated at the symmetric solution.

To solve Eqs. (15) and (16) we proceed as follows: Let us suppose we have found solutions to the equations, i.e., the Δ's and γ's, for different sets of δ's, say two different sets. We may think of these as 8-dimensional mass vectors, $\boldsymbol{\delta}^1$ and $\boldsymbol{\delta}^2$. Then, since the equations are linear, the solution corresponding to the mass vector $(x\boldsymbol{\delta}^1+y\boldsymbol{\delta}^2)$, where x and y are arbitrary constants, is obtained simply by superposition. Thus, to find the solution corresponding to an arbitrary mass vector $\boldsymbol{\delta}$, it is sufficient to know the solutions corresponding to eight linearly independent vectors, $\boldsymbol{\delta}^1, \cdots, \boldsymbol{\delta}^8$. We take these vectors to be the solution of the equations

$$\sum_{ab} g_{ab}{}^r g_{ab}{}^t\delta_a=\frac{\epsilon}{2I}\delta_r\delta_{rt}, \quad (18)$$

and we shall refer to them as the "eigenvectors" and to ϵ as the corresponding eigenvalue. Using the known structure constants for SU_3, Eq. (18) can be solved explicitly, and the eigenvalues are

$$\left.\begin{matrix}\epsilon=2\\ \epsilon=1\\ \epsilon=\frac{1}{3}\end{matrix}\right\}\text{nondegenerate,}$$
$$\epsilon=-\tfrac{2}{3}\}\text{fivefold degenerate.}$$

If the basis of the 8-dimensional mass space is identified with the physical particles as

$$\left.\begin{matrix}1\\2\end{matrix}\right\}\text{charged } \pi\text{'s}$$
$$3 \ \text{neutral } \pi$$
$$\left.\begin{matrix}4\\5\end{matrix}\right\}\text{charged } K\text{'s}$$
$$\left.\begin{matrix}6\\7\end{matrix}\right\}\text{neutral } K\text{'s}$$
$$8 \ \eta,$$

then the masses of the bracketed pairs of particles must be equal by charge conjugation. The eigenvectors which satisfy this symmetry are

$$\begin{matrix}\epsilon=2 & 1 & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\\
\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\1\\1\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-1\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\\1\\-1\\-1\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{9}\\\frac{1}{9}\\\frac{1}{9}\\-\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\\0\end{pmatrix} & \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\\0\\0\\0\\0\\0\end{pmatrix}\end{matrix},$$

which are in fact orthogonal. It is seen that for $\epsilon=2$ the masses are not split and that for $\epsilon=\frac{1}{3}$ or one of the $\epsilon=-\frac{2}{3}$ eigenvectors, the masses are split within isospin multiplets. It can be shown that Eq. (18) is solved by any of the Clebsch-Gordan coefficients in the decomposition of $8\otimes8$ which happen to be diagonal in the coordinate systems used. For the solutions, ϵ is the corresponding recoupling coefficient, or crossing-matrix element. In fact, $\epsilon=2$, 1, $-\frac{2}{3}$ are, respectively, the crossing matrix elements for the exchange of an octet (antisymmetrically coupled) in the singlet, octet (symmetrically coupled), and twenty-seven dimensional representations.

Equations (15) and (16) are now solved for a given eigenvector, $\boldsymbol{\delta}^i$, by assuming that the Δ's and γ's are proportional to the δ's and then determining the constants. In particular, the ansatz is

$$\boldsymbol{\Delta}^i=-\frac{I^\mu}{2I_m}(\epsilon_i+\lambda_i)\boldsymbol{\delta}^i=K_i\boldsymbol{\delta}^i, \quad (19)$$

$$(\gamma_{ab}{}^r)^i=g_{ab}{}^r[(\delta_a{}^i+\delta_b{}^i)k_i+\delta_r{}^i k_i'], \quad (20)$$

where ϵ_i is the eigenvalue corresponding to $\boldsymbol{\delta}^i$ and λ_i, k_i and k_i' are to be determined. The observant reader may have noticed that in fact the set of equations given in Eqs. (15) and (16) overdetermine the solutions. In particular, Eq. (16) gives just the correct number of equations to determine the $\gamma_{ab}{}^r$, but Eq. (15) overdetermines the remaining eight unknowns. The ansatz given in Eq. (20) eliminates the difficulty immediately, since it may be easily checked, by use of this equation with Eqs. (10) and (18), that Eq. (15) is identically satisfied when $r\neq t$. This leaves eight equations. Substitution of Eqs. (19) and (20), into Eqs. (15) and (16) with repeated application of Eqs. (7), (8), (10), (12), and (18) yields

$$\frac{J_\mu}{J}+\frac{\epsilon_i}{2}\left[2k_i'-\frac{I^\mu}{2I_m}\frac{J^m}{J}(\epsilon_i+\lambda_i)\right]=0, \quad (21)$$

$$-\frac{I_\mu}{2I_m}\frac{J_m}{J}(\epsilon_i+\lambda_i)+\epsilon_i\left[2k_i+\frac{J^\mu}{J}-k_i'+\frac{1}{2}\frac{I^\mu}{2I_m}\frac{J^m}{J}(\epsilon_i+\lambda_i)\right]=0, \quad (22)$$

$$k_i'=\frac{\lambda_i}{2}\frac{I^\mu}{I}-\epsilon_i k_i. \quad (23)$$

These equations give

$$\lambda_i=-\frac{2I_m}{I^\mu}\times\frac{1}{\epsilon_i}\frac{(1+\epsilon_i/2)J_\mu+\frac{1}{2}\epsilon_i^2\left[J^\mu-\frac{I^\mu}{2I_m}(J^m+J_m)\right]}{2I_m-\frac{1}{2}(J^m+J_m)}, \quad (24)$$

$$k_i=\frac{\lambda_i}{2\epsilon_i}\frac{I^\mu}{I}-\frac{1}{2\epsilon_i}\frac{I^\mu}{2I_m}\frac{J^m}{J}(\epsilon_i+\lambda_i)+\frac{J_\mu}{\epsilon_i^2 J}, \quad (25)$$

$$k_i'=\frac{1}{2}\frac{I^\mu}{2I_m}\frac{J^m}{J}(\epsilon_i+\lambda_i)-\frac{1}{\epsilon_i}\frac{J_\mu}{J}. \quad (26)$$

Thus, given an arbitrary scalar mass vector, $\boldsymbol{\delta}=\sum_i X_i\boldsymbol{\delta}^i$, we determine a vector mass vector $\boldsymbol{\Delta}$ as the same linear combination, $\boldsymbol{\Delta}=\sum_i X_i\boldsymbol{\Delta}^i$, and a set of coupling constant shifts $\gamma_{ab}{}^r=\sum_i X_i(\gamma_{ab}{}^r)^i$ which satisfy the self-consistency demand. It is important to emphasize at this point that self-consistency *does not* impose a restriction on the mass or coupling constant shifts—*solutions exist for arbitrary splitting.*

It is of interest however to examine some special cases. First, one would certainly think that the basic equations, Eqs. (7) and (8), should be scale invariant in the sense that, if a set of scalar masses, $\mu_1, \cdots, \mu_8$ yields a solution $m_1, \cdots, m_8$, with certain coupling constants, then the set, $\alpha\mu_1, \cdots, \alpha\mu_8$, will also yield a solution $\alpha m_1, \cdots, \alpha m_8$, with the same coupling constants, where α is any positive constant. The integrals represented by I and J require cutoffs, and whether or not the equations in fact are scale invariant depends on how these cutoffs are introduced. If the cutoffs are introduced such that scale invariance is present, then the following identities are satisfied

$$I^\mu/I_m=-m_0^2/\mu_0^2, \quad (27)$$

$$(J^\mu+J_\mu)/(J^m+J_m)=-\tfrac{1}{2}(m_0^2/\mu_0^2). \quad (28)$$

Using these equations, Eq. (24) reduces to

$$\lambda_i=-\frac{(2-\epsilon_i)(1+\epsilon_i)}{2\epsilon_i}\frac{2I_m}{I^\mu}\frac{J_\mu}{2I_m-\frac{1}{2}(J^m+J_m)}. \quad (29)$$

Now if all the scalar masses are equally shifted then $\boldsymbol{\delta}$ is in fact proportional to the eigenvector corresponding to $\epsilon=2$. In this case, $\lambda=0$ and also $k'=-2k$. It follows at once that

$$\Delta_a=-(I^\mu/I_m)C=(m_0^2/\mu_0^2)C,$$

where C is a constant, $\delta_a=C$, or

$$m^2/\mu^2=(m_0^2+\Delta)/(\mu^2+\delta)=m_0^2/\mu_0^2$$

and

$$\gamma_{ab}{}^r=0,$$

as one would expect.

As another case, consider that $\boldsymbol{\delta}$ is given by any combination of the $\epsilon=2$ and $\epsilon=1$ eigenvectors; then the scalar masses would obey the Gell-Mann–Okubo mass formula, and it follows from Eq. (19) that the vector masses will also obey such a relation. In general, if $\boldsymbol{\delta}$ is given by the sum of the $\epsilon=2$ eigenvector and any one of the other eigenvectors, a specific mass formula is satisfied by the scalar and vector masses. However, at this point these eigenvectors have been introduced as a mathematical device by which we solve the equations.

C. Self-Consistency of Scalar Particles

Here we wish to consider the self-consistency of the set of scalar particles in the sense defined in Sec. IIA. The conditions that the scalar particles be self-consistent are, in analogy to Eqs. (7) and (8),

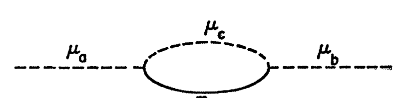
$$\delta_{ab}=\sum_{cr} g_{ac}{}^r g_{bc}{}^r \tilde{I}(c,r;a,b), \quad (30)$$

$$g_{ab}=\sum_{cds} g_{ac}{}^s g_{bd}{}^s g_{cd}{}^r \tilde{J}(c,d,s;a,b,r). \quad (31)$$

Equations (30) and (31) correspond to Figs. 3 and 2, respectively. In fact, $J=\tilde{J}$ and Eqs. (8) and (31) are identical since this is the condition for the vanishing of the scalar-scalar-vector renormalization constant.

In complete analogy to Sec. IIB, we now regard Eqs. (30) and (31) as equations to determine the coupling constants and one set of masses in terms of the other set of masses. Again we assume the existence of a symmetric solution with $\mu_i=\mu_0$, $m_i=m_0$ and, as in Eq. (9), $g_{ab}{}^r=GC_{abr}$. If we define $\tilde{I}_0$ and $\tilde{J}_0$ as the values of $\tilde{I}$ and $\tilde{J}$ for the symmetric solution, then the

FIG. 3. Scalar self-energy diagram.



existence of such a solution implies

$$\tilde{I}_0 = 1/2G^2, \tag{32}$$

$$\tilde{J}_0 = 1/G^2. \tag{33}$$

With the same approximations as those in Sec. IIB with respect to first-order terms, etc., the linear equations for the deviations from the symmetric solution are

$$\sum_{cr} g_{ac}{}^r g_{bc}{}^r [\delta_a \tilde{I}_\mu + \delta_c \tilde{I}^\mu + \Delta_r \tilde{I}^m] + I \sum_{cr} (g_{ac}{}^r \gamma_{bc}{}^r + \gamma_{ac}{}^r g_{bc}{}^r) = 0 \tag{34}$$

together with Eq. (16). In Eq. (34) we have defined,

$$(\tilde{I}^m, \tilde{I}^\mu, \tilde{I}_\mu) = \left(\frac{\partial}{\partial m_r{}^2}, \frac{\partial}{\partial \mu_c{}^2}, \frac{\partial}{\partial \mu_a{}^2}\right) \tilde{I}(c,r;a,b), \tag{35}$$

where the derivatives are evaluated at the symmetric solution.

To solve these equations we again look for solutions in which the sets of masses are eigenvectors in the sense of Eq. (18) and, corresponding to Eqs. (19) and (20), we make the ansatz

$$(\gamma_{ab}{}^r)^i = g_{ab}{}^r [\tilde{k}_i (\delta_a{}^i + \delta_b{}^i) + \tilde{k}_i' \delta_r{}^i], \tag{36}$$

$$\Delta_a{}^i = -\frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}^m} \delta_a{}^i, \tag{37}$$

where $\tilde{k}_i$, $\tilde{k}_i'$, and η_i are constants to be determined. Substituting these into Eqs. (16) and (34) we obtain,

$$\tilde{k}_i' = \frac{\epsilon_i - 2}{\epsilon_i} \frac{\tilde{I}_\mu}{2\tilde{I}_0} - \frac{\eta_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{2\tilde{I}_0} - \tilde{k}_i \left(\frac{\epsilon_i + 2}{\epsilon_i}\right) \tag{38}$$

$$\epsilon_i \left[2\tilde{k}_i - \tilde{k}_i' + \frac{\tilde{J}^\mu}{\tilde{J}_0} + \frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{2\tilde{I}_m} \frac{\tilde{J}^m}{\tilde{J}_0}\right] - \frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{J}_m}{\tilde{J}_0} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}^m} = 0, \tag{39}$$

$$\epsilon_i \left[\tilde{k}_i' - \frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{2\tilde{I}_m} \frac{\tilde{J}^m}{\tilde{J}_0}\right] + \frac{\tilde{J}_\mu}{\tilde{J}_0} = 0, \tag{40}$$

from which we find,

$$\frac{\epsilon_i}{\epsilon_i + \eta_i} = \frac{\dfrac{\epsilon_i + 2}{2\epsilon_i}\left[\tilde{J}^\mu + \dfrac{3\epsilon_i + 2}{\epsilon_i(\epsilon_i + 2)} \tilde{J}_\mu\right] - \left[\tilde{I}^\mu + \dfrac{2}{\epsilon_i} \tilde{I}_\mu\right]}{\left[\dfrac{1}{2}\left(\tilde{J}^m + \dfrac{\epsilon_i + 2}{\epsilon_i{}^2} \tilde{J}_m\right) - \tilde{I}^m\right]\left(\dfrac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}_m}\right)}, \tag{41}$$

$$\tilde{k}_i = \frac{\epsilon_i}{2(\epsilon_i + 2)} \frac{\eta_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}_0} - \frac{\epsilon_i - 2}{2(\epsilon_i + 2)} \frac{\tilde{I}_\mu}{\tilde{I}_0} + \frac{\epsilon_i}{\epsilon_i + 2} \frac{\tilde{J}_\mu}{\tilde{J}_0} - \frac{\epsilon_i{}^2}{2(\epsilon_i + 2)} \frac{1}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}_m} \frac{\tilde{J}^m}{\tilde{J}_0}, \tag{42}$$

$$\tilde{k}_i' = -\frac{\tilde{J}_\mu}{\tilde{J}_0} + \frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{2\tilde{I}_m} \frac{\tilde{J}^m}{\tilde{J}_0}. \tag{43}$$

As before, the solution for an arbitrary $\boldsymbol{\delta}$ can be obtained as a linear combination of eigensolutions from Eqs. (36) and (37). Further, there are no restrictions placed on $\boldsymbol{\delta}$ by the self-consistency requirements; any given $\boldsymbol{\delta}$ leads to a $\boldsymbol{\Delta}$ and a set of $\gamma_{ab}{}^r$ which satisfy the self-consistency equations.

Again, if scale invariance is present, then the identity

$$(\tilde{I}^\mu + \tilde{I}_\mu)/\tilde{I}_m = -(m_0{}^2/\mu_0{}^2) \tag{44}$$

is satisfied as well as Eq. (28), and it can be verified that the $\epsilon = 2$ eigenvector results only in a change of scale in the symmetric solution.

D. Full Self-Consistency

We now wish to address ourselves to the following question. Is it possible that masses and coupling constants exist which satisfy both self-consistency requirements, i.e., can self-consistency for both the scalar and vector set be obtained simultaneously? Within our present framework, i.e., approximation, this means that we must attempt to satisfy Eqs. (7), (8), and (30) simultaneously. In order to do this, we must make the additional assumption that a simultaneous symmetric solution exists. Even if the separate symmetric solutions exist it is not obvious that a simultaneous symmetric solution also exists. We must certainly construct our approximation so that scale invariance is present, and thus we have two unknowns, $m_0{}^2/\mu_0{}^2$ and G^2, and three equations. On the other hand, we have no criteria for determining the cutoff parameters. At most, we could have three cutoff parameters and at a minimum only one. We will take the point of view that the cutoff parameter (or parameters) is chosen so that a simultaneous symmetric solution exists. It is certainly true that this solution is probably not unique, but since we wish to examine the existence of solutions with unequal masses given some equal-mass solution, we will not concern ourselves with this important question here.

Once we have assumed the existence of a simultaneous symmetric solution, the question of full self-consistency within the linear approximation reduces to an examination of simultaneous solutions of Eqs. (15), (16), and (34). We have seen that the most general scalar mass-splitting vector may be written

$$\boldsymbol{\delta} = \sum_i X_i \boldsymbol{\delta}^i. \tag{45}$$

Then the corresponding mass-splitting vector for the vector particles is

$$\boldsymbol{\Delta} = \sum X_i \boldsymbol{\Delta}^i, \tag{46}$$

where to satisfy Eqs. (15) and (16) $\boldsymbol{\Delta}^i$ is given by Eq. (19) and to satisfy Eqs. (16) and (34) $\boldsymbol{\Delta}^i$ is given by Eq. (37). Thus for full self-consistency with $\boldsymbol{\delta}$ given by Eq. (45), it is certainly necessary that

$$\frac{\epsilon_i}{\epsilon_i + \eta_i} \frac{\tilde{I}^\mu + \tilde{I}_\mu}{\tilde{I}^m} = (\epsilon_i + \lambda_i) \frac{I^\mu}{2I_m}, \tag{47}$$

for all i with $X_i \neq 0$. In the case of scale invariance this reduces to

$$\frac{\epsilon_i}{\epsilon_i+\eta_i}=\tfrac{1}{2}(\epsilon_i+\lambda_i). \tag{48}$$

Since λ_i and η_i depend only on ϵ_i and the constants I_μ, $\tilde{I}_\mu$, J_μ, etc., Eq. (48) represents a separate condition for each of the eigenvalues ϵ_i. However, since we regard I_μ, $\tilde{I}_\mu$, J_μ, etc., as known constants fixed by the symmetric solution, it follows that in general we should not expect Eq. (48) to be satisfied for any of the allowed values of $\epsilon_i(2, 1, \frac{1}{3}, -\frac{2}{3})$. Actually, when scale invariance is present we must have Eq. (48) satisfied for $\epsilon=2$, and this is in fact true [put $\eta_i=\lambda_i=0$ in Eq. (48)].

We have not yet exhausted all the conditions which must be satisfied for full self-consistency. We still have to investigate the condition that the coupling constant shifts arrived at in Secs. IIB and IIC are the same. The coupling constants corresponding to ϵ_i are the same if

$$k_i=\tilde{k}_i, \tag{49}$$

$$k_i'=\tilde{k}_i'. \tag{50}$$

However, it may be verified that Eqs. (49) and (50) are in fact a *consequence of* Eq. (47). The reason for this rests on the fact that in our approximation the condition for the vanishing of the vertex renormalization constant is the same for scalar and vector self-consistency, i.e., the self-consistency of the scalar particles introduces only *one* new set of conditions, Eq. (30). Thus, full self-consistency is obtained, in addition to the trivial solution $\epsilon=2$, if Eq. (48) above is satisfied for one of the allowed eigenvalues, ϵ_i.

At this point, then, within our approximations, we can conclude two things: First, in general, the requirements of self-consistency are more restrictive on nondegenerate solutions infinitesimally close to the symmetric solution than they are on the symmetric solution itself. Secondly, strictly speaking, there will in general be no solution infinitesimally close to the symmetric solution.

However, conclusions of this sort must be strongly tempered by a consideration of the approximations involved. There are at least two points of view here. One could argue that we would certainly not expect to obtain a solution which gave the correct magnitudes for the splitting of the vector set given some scalar mass splitting. On the other hand, given a set of ratios for the scalar mass splitting, we might hope to obtain a fully self-consistent set of ratios for the vector set, although the magnitudes given by Eqs. (19) and (37) might differ considerably. This is in fact easy to accomplish by choosing $\boldsymbol{\delta}$ as a sum of the $\epsilon=2$ eigenvector and any one of the other eigenvectors. Since it is rather natural to believe that an arbitrary $\boldsymbol{\delta}$ will not lead to self-consistent solutions, one can argue on the basis of this analysis (so that at least mass ratios are self-consistent) that only those $\boldsymbol{\delta}$'s which are an arbitrary sum of just two of the $\boldsymbol{\delta}^i(\epsilon=2$ and one other of $\epsilon=1, \frac{1}{3}$, or $-\frac{2}{3})$ will yield self-consistent solutions. It then follows immediately that one or another of certain mass relations must be satisfied. For instance, if $\boldsymbol{\delta}$ is a sum of contributions from the $\epsilon=2$ and $\epsilon=1$ eigenvectors, both sets of masses will satisfy the Gell-Mann–Okubo mass formula. The eigenvectors introduced in Sec. IIB as simply mathematical devices now take on a physical significance—they represent the only mass ratios which will yield self-consistent solutions. Nothing can be said about which of the four possible mass formulas is the self-consistent one, nor about the magnitude of the shifts.

A second viewpoint is to reinterpret Eq. (48) by considering ϵ as a continuous variable and determining a value for ϵ; Eq. (48) is in fact a quartic equation in ϵ. We know that $\epsilon=2$ is a solution and we look for others. The best situation, of course, occurs when there is only one real root in addition to $\epsilon=2$, and it is close to $\epsilon=1$, $\frac{1}{3}$ or $-\frac{2}{3}$. One then "concludes" that a better calculation might have yielded a root even closer to the eigenvalue in question and that the exact analysis might yield the eigenvalue. In this case, the approximate solution yields definite vector mass shifts and coupling-constant shifts for a given set of scalar mass shifts. Further, the shifted scalar masses (the vectors as well) must satisfy a specific mass formula which one argues is the only self-consistent solution.

A somewhat more pessimistic situation would be if there were no roots in ϵ even close to the allowed eigenvalues other than $\epsilon=2$, for instance if the only other real root were $\epsilon=50$. In this case, one can only say either that the approximation is so bad that one can conclude nothing about specific solutions or that the approximation is reasonably good and broken SU_3 multiplets do not lead to self-consistent solutions. This, of course, hinges on the validity of the approximations, and at this point it is well to emphasize that there are really several crucial approximations here, viz., two-particle states, Born approximation, and the linear approximation for the perturbed solutions. Actually, as is well known, the first two approximations have met with at least some success. But of course the essential point is that virtually nothing is known about the effects of second- and higher order terms in the mass shifts.

III. BOOTSTRAP SELF-CONSISTENCY

We now wish to carry out an analysis in parallel to the discussion of Sec. II with exactly the same physical system. However, the self-consistency equations that we now use are those of the "bootstrap." The analysis of the bootstrap equations can be carried through using the same ansatz as in Sec. II, however, new features do arise to alter our conclusions.

A. Scalar-Scalar Scattering

Here we wish to examine the self-consistency requirements arising from a bootstrap calculation of a set of vector mesons in the two-particle scattering of a set of scalar particles. Let $T_{(ab)(cd)}(S)$ be the P-wave scattering amplitude for the process

$$a+b\leftrightarrow c+d\,, \tag{1}$$

where a, b, c, d label members of the scalar set of particles and S is the square of the center-of-mass energy. The brackets are placed around (ab), (cd) to emphasize that a state is labeled by a pair of letters, irrespective of the order. The set of amplitudes constitutes a matrix in a space labeled by the states (ab). Since two identical bosons of spin 0 cannot be in a P state, a and b must be different. Since the sets consist of eight particles, the space has $8\times7/2=28$ dimensions.

As usual we write the matrix equation

$$\mathbf{T}=\mathbf{N}\mathbf{D}^{-1} \tag{2}$$

and restrict ourselves to the determinantal approximation, where $\mathbf{N}$ is the matrix of the P-wave projections of the scattering amplitudes in Born approximation, $T^{B}{}_{(ab)(cd)}(S)$, corresponding to Fig. 4. Of course, in $\mathbf{N}(S)$ there is a summation over all the exchanged vector particles, labeled by t in Fig. 4. We may write the Born amplitude for a given exchange, t, as

$$T^{B}{}_{(ab)(cd)}(S,t)=g_{ac}{}^{t}g_{bd}{}^{t}f(c,d,t\,;a,b,S)\,, \tag{3}$$

where f is a known function and depends on the particle masses which we again simply labeled by their indices. Therefore we have

$$N_{(ab)(cd)}(S)=\sum_t g_{ac}{}^{t}g_{bd}{}^{t}f(c,d,t\,;a,b,S)\,. \tag{4}$$

The matrix $\mathbf{D}$ is then given by

$$D_{(ab)(cd)}(S)=\delta_{(ab)(cd)}-\sum_t\left[g_{ac}{}^{t}g_{bd}{}^{t}\alpha(a,b,t\,;c,d,S)-g_{ad}{}^{t}g_{bc}{}^{t}\alpha(a,b,t\,;d,c,S)\right], \tag{5}$$

with

$$\alpha(a,b,t\,;c,d,S)=\frac{S-S_0}{\pi}\int dS'\frac{\rho_{(cd)}(S')f(c,d,t\,;a,b,S')}{(S'-S)(S'-S_0)}, \tag{6}$$

where $\rho_{(cd)}(S')$ is the two-particle phase-space appropriate to the masses $\mu_a{}^2$, $\mu_b{}^2$, and S_0 is the value of S at which $\mathbf{D}$ is normalized to the unit matrix.

As before, we begin by a consideration of the equal-mass case and look for the symmetric solution. The coupling constants are taken as proportional to the structure constants of SU_3 and Eqs. (10), (11), and (12) of Sec. II are still valid. In this case, it can be shown[7] that the matrices $\mathbf{N}$ and $\mathbf{D}$ are both diagonalized by an orthogonal matrix $O_{(ab)}{}^{r}$ whose columns are the

[7] Hong-Mo Chan, P. C. DeCelles, and J. E. Paton, Nuovo Cimento 33, 70 (1964).

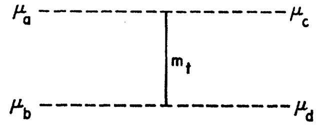


FIG. 4. Born approximation diagram for scalar-scalar scattering.

(normalized) eigenvectors $\psi_{(ab)}{}^{r}$ of the matrix

$$V_{(ab)(cd)}=\sum_t(c_{ac}{}^{t}c_{bd}{}^{t}-c_{ad}{}^{t}c_{bc}{}^{t})\,, \tag{7}$$

where the c's are the structure constants of SU_3. Thus eight of the $\psi_{(ab)}{}^{r}$ are the $c_{(ab)}{}^{r}$, and it can be shown that the remaining 20(28–8) eigenvectors correspond to the eigenvalue zero.

We denote by $f_0(S)$ and $\alpha_0(S)$ the functions f and α when scalar and vector sets have the same masses, μ_0 and m_0, respectively, and when the g's are proportional to the SU_3 structure constants. The requirement that the vector-meson poles occur at $S=m_0{}^2$ is, in the diagonalized form, easily seen to yield

$$1=G^2\alpha_0(m_0{}^2)\,. \tag{8}$$

Further, the requirement that the residue of the pole is correctly given by crossing symmetry leads to

$$f_0=-G^2\alpha_0{}'(m_0{}^2)\,, \tag{9}$$

where $\alpha_0{}'(m_0{}^2)$ is the derivative of $\alpha_0(S)$ evaluated at $S=m_0{}^2$. We have here chosen a particular normalization of the $T_{(ab)(cd)}$, which are the P-wave scattering amplitudes with kinematical factors removed, so that they are finite and nonzero at threshold. As before, we now assume that Eqs. (8) and (9) yield at least one solution for G^2 and $m_0{}^2$. At this point in the discussion, the subtraction point S_0 is considered to be a parameter determined by other means, e.g., so that the discontinuity across the left-hand cut is correctly given at one point. We will adopt a different viewpoint later.

Let us turn now to the question of other solutions infinitesimally close to the symmetric solution. The situation is not quite as transparent as it was in Sec. II. In principle, it should be possible to write down equations relating masses and coupling constants in the general mass case and then expand these equations to first order. These would yield a set of linear first-order equations for the vector mass shifts, $\mathbf{\Delta}$, and the coupling constant shifts, $\gamma_{ab}{}^{r}$ in terms of the scalar mass shifts, $\boldsymbol{\delta}$. Again, since the equations would be linear, we could start by looking for solutions in which $\boldsymbol{\delta}$ was an eigenvector, and the most general solution could be expressed in terms of these.

In practice, however, this program is difficult to carry through, and we instead expand $\mathbf{N}(S)$ and $\mathbf{D}(S)$ to first order in the δ's, making the same ansatz which was successful in Sec. II [Eqs. (19) and (20)]. If one uses this ansatz and transforms $\mathbf{N}$ and $\mathbf{D}$ with the matrix $\mathbf{O}$ introduced in the discussion preceding Eq. (7), then it is possible to extract, correct to first order, the pole and residue conditions corresponding to Eqs. (8) and (9) in the equal-mass case.

Let us define,

$$(\alpha^{\mu}(S),\alpha_{\mu}(S),\alpha^{m}(S))=\left(\frac{\partial}{\partial\mu_c{}^2},\frac{\partial}{\partial\mu_a{}^2},\frac{\partial}{\partial m_r{}^2}\right)\alpha(c,d,r;a,b,S)_{|\mu_i{}^2=\mu_0{}^2,\ m_r{}^2=m_0{}^2}, \tag{10}$$

$$\alpha_m=\alpha_0'(m_0^2)=\frac{\partial}{\partial S}\alpha(a,b,r;c,d,S)_{|\mu_i{}^2=\mu_0{}^2,\ m_r{}^2=m_0{}^2}, \tag{11}$$

$$(\alpha^{\mu},\alpha_{\mu},\alpha^{m})=(\alpha^{\mu}(m_0^2),\alpha_{\mu}(m_0^2),\alpha^{m}(m_0^2)), \tag{12}$$

$$(f^{\mu}(S),f_{\mu}(S),f^{m}(S))=\left(\frac{\partial}{\partial\mu_c{}^2},\frac{\partial}{\partial\mu_a{}^2},\frac{\partial}{\partial m_r{}^2}\right)f(c,d,r;a,b,S)_{|\mu_i{}^2=\mu_0{}^2,m_r{}^2=m_0{}^2}, \tag{13}$$

$$f_m=\frac{\partial}{\partial S}f(c,d,r;a,b,S)_{|\mu_i{}^2=\mu_0{}^2,\ m_r{}^2=m_0{}^2}, \tag{14}$$

$$(f^{\mu},f_{\mu},f^{m})=(f^{\mu}(m_0^2),f_{\mu}(m_0^2),f^{m}(m_0^2)), \tag{15}$$

$$(\alpha^{\mu}{}_{,m};\alpha_{\mu,m};\alpha^{m}{}_{,m};\alpha_{m,m})=\frac{\partial}{\partial S}(\alpha^{\mu}(S);\alpha_{\mu}(S);\alpha^{m}(S);\alpha'(S))_{|\mu_i{}^2=\mu_0{}^2,\ m_r{}^2=m_0{}^2}. \tag{16}$$

To first order, one obtains an expansion of $\mathbf{D}$,

$$\begin{aligned}D_{(ab)(cd)}(S)=\delta_{(ab)(cd)}&-\sum_r(g_{ac}{}^rg_{bd}{}^r-g_{ad}{}^rg_{bc}{}^r)\alpha_0(S)\\&-\sum_r[\gamma_{ac}{}^rg_{bd}{}^r+g_{ac}{}^r\gamma_{bd}{}^r-\gamma_{ad}{}^rg_{bc}{}^r-g_{ad}{}^r\gamma_{bc}{}^r]\alpha_0(S)\\&-\sum_r(g_{ac}{}^rg_{bd}{}^r-g_{ad}{}^rg_{bc}{}^r)[\alpha^{\mu}(S)(\delta_a+\delta_b)+\alpha_{\mu}(S)(\delta_c+\delta_d)+\alpha^{m}(S)\Delta_r]\end{aligned} \tag{17}$$

and $N_{(ab)(cd)}$ can be obtained by changing the sign of the right-hand side of Eq. (17), omitting the unit matrix and replacing f by α wherever it occurs. We now transform $\mathbf{N}$ and $\mathbf{D}$ by the orthogonal matrix $\mathbf{O}$ to give the matrices

$$\mathbf{N}'=\mathbf{O}^T\mathbf{N}\mathbf{O},\quad \mathbf{D}'=\mathbf{O}^T\mathbf{D}\mathbf{O}, \tag{18}$$

where $\mathbf{N}'$ and $\mathbf{D}'$ are matrices labeled by a single index [in contrast to the labeling, e.g., (ab), of $\mathbf{N}$ and $\mathbf{D}$], running from 1 to 28, denoting the state (antisymmetric) which is being coupled to the decomposition $8\otimes8=1+8+8'+10+\overline{10}+27$. Only the 8, 10, and $\overline{10}$ states are allowed. Let us denote this index by t or u when it lies in the range of the octet states and x otherwise. Using as our ansatz Eqs. (19) and (20) of Sec. II one obtains, after extensive use of the Jacobi identity and the other relations,

$$D'_{tu}(S)=\delta_{tu}-G^2\delta_{tu}\{\alpha_0(S)+\epsilon_i\delta_t{}^i[2k_i\alpha_0(S)+\alpha^{\mu}(S)+\alpha_{\mu}(S)+\tfrac{1}{2}\epsilon_i(\epsilon_i-1)\delta_t{}^i[2k_i'\alpha_0(S)+K_i\alpha^m(S), \tag{19}$$

$$N'_{tu}(S)=G^2\delta_{tu}\{f_0(S)+\epsilon_i\delta_t{}^i[2k_if_0(S)+f^{\mu}(S)+f_{\mu}(S)]+\tfrac{1}{2}\epsilon_i(\epsilon_i-1)\delta_t{}^i[2k_i'f_0(S)+K_if^m(S)]\}, \tag{20}$$

$$N'_{xu}(S)=G\sum_{(ab)}O_{ab}{}^xg_{ab}{}^u(\delta_a{}^i+\delta_b{}^i)\{f^{\mu}(S)+k_if_0(S)+\tfrac{1}{2}\epsilon_i[2k_i'f_0(S)+K_if^m(S)]\}, \tag{21}$$

$$N'_{ux}(S)=G\sum_{(ab)}g_{ab}{}^uO_{ab}{}^x(\delta_a{}^i+\delta_b{}^i)\{f_{\mu}(S)+k_if_0(S)+\tfrac{1}{2}\epsilon_i[2k_i'f_0(S)+K_if_m(S)]\}. \tag{22}$$

We do not write down all the matrix elements of $\mathbf{D}'$ since we shall require only those between octet states. We see that, to first order, the submatrices of $\mathbf{N}'$ and $\mathbf{D}'$ between octet states are diagonal.

Consider now the condition for a pole in the transformed amplitude

$$\mathbf{T}'=\mathbf{N}'\mathbf{D}'^{-1}. \tag{23}$$

The elements of $\mathbf{T}'$ will have a pole at a particular value of S if $\det\mathbf{D}$ has a zero. The matrix $\mathbf{D}'$ is of the form

$$\mathbf{D}'=\begin{pmatrix}\mathbf{\Lambda}&\mathbf{A}\\\mathbf{B}&\mathbf{C}\end{pmatrix}, \tag{24}$$

where $\mathbf{\Lambda}$ is diagonal (8×8), $\mathbf{A}$ and $\mathbf{B}$ are the zero matrix, and $\mathbf{C}$ is the unit matrix, plus first order (and therefore presumed small) corrections. In the symmetric case, $\mathbf{\Lambda}$ is truly diagonal with equal elements which vanish when $S=m_0^2$. To see where the poles move, we need only examine the zeros of the diagonal elements of $\mathbf{\Lambda}$. This follows since the leading elements of $\mathbf{D}'$ are its diagonal elements, and correct to first order $\det\mathbf{D}$ is just the product of these diagonal elements. These zeros occur, by the self-consistency requirement, when the energy is given by $S=m_0^2+\Delta_r$, which yields the condition

$$K_i=[2k_i\alpha_0-(1-\epsilon_i)k_i'\alpha_0+\alpha^\mu+\alpha_\mu]/[\tfrac{1}{2}(1-\epsilon_i)\alpha^m-(1/\epsilon_i)\alpha_m]. \quad (25)$$

We now wish to determine the implications of self-consistency for the residues of the poles at $S=m_0^2+\Delta_r$. The residues of the elements of $\mathbf{T}'$ may be obtained by expanding the matrix product $\mathbf{N}'\mathbf{D}_0'^{-1}$ to first order where

$$(\mathbf{D}_0'^{-1})_{ts}=\lim_{S\to m_0^2+\Delta_r}[(S-m_0^2-\Delta_r)(\mathbf{D}^{-1})_{ts}]. \quad (26)$$

$\mathbf{D}_0'^{-1}$ is of the form

$$\begin{pmatrix}\mathbf{M} & 0\\ 0 & 0\end{pmatrix},$$

where $\mathbf{M}$ is diagonal and 8×8, with elements given as

$$M_{rr}=\left(\frac{S-m_0^2-\Delta_r}{D_{rr}'}\right)_{S=m_0^2+\Delta_r}.$$

The residues of the elements of $\mathbf{T}$ can now be obtained from those of $\mathbf{T}'$ by transforming back with the orthogonal matrix $\mathbf{O}$. Self-consistency is then obtained by equating the residues of $T_{(ab)(cd)}$ to the sum $\sum_{[r]}(g_{ab}{}^r+\gamma_{ab}{}^r)(g_{cd}{}^r+\gamma_{cd}{}^r)$ where the summation over $[r]$ is over all those values of r which have the same shifted, vector-particle mass.

Thus for the residue of $T_{(ab)(cd)}$ one obtains

$$\mathrm{Res}T_{(ab)(cd)}=-\frac{1}{G^2}\sum_{[t]}g_{ab}{}^tg_{cd}{}^t\left\{\frac{f_0}{\alpha_m}+\frac{\delta_t{}^i}{\alpha_m}[\epsilon_i(2k_if_0+f^\mu+f_\mu)+\tfrac{1}{2}\epsilon_i(\epsilon_i-1)(2k_i'f_0+K_if^m)+K_if_m]\right.$$
$$\left.-\delta_t{}^i\frac{f_0\epsilon_i}{\alpha_m{}^2}\left[\alpha^\mu{}_{,m}+\alpha_{\mu,m}-\tfrac{1}{2}K_i(1-\epsilon_i)\alpha^m{}_{,m}+\frac{K_i}{\epsilon_i}\alpha_{m,m}\right]\right\}$$
$$-\frac{1}{G^2}\sum_{[t]}g_{ab}{}^tg_{cd}{}^t\frac{(\delta_a{}^i+\delta_b{}^i-\epsilon_i\delta_t{}^i)}{\alpha_m}[f_\mu+k_if_0+\tfrac{1}{2}\epsilon_i(2k_i'f_0+K_if^m)], \quad (27)$$

which for self-consistency must be equated to

$$\sum_{[t]}g_{ab}{}^tg_{cd}{}^t+\sum_{[t]}g_{ab}{}^tg_{cd}{}^t\times[k_i(\delta_a{}^i+\delta_b{}^i+\delta_c{}^i+\delta_d{}^i)+2k_i'\delta_t{}^i]. \quad (28)$$

A casual inspection of these two expressions shows that they are inconsistent. To see this, one notes that expression (28) is symmetric under the interchange $(ab)\leftrightarrow(cd)$, as it should be since the matrix $\mathbf{T}$ should be symmetric by time-reversal invariance. However, Eq. (27) does not exhibit this property. This is a well-known difficulty within the determinantal method when one has unequal masses in a multichannel problem. This disease can be cured in several ways all of which have their advantages and disadvantages. However, in this case they all yield the same result, viz., Eq. (27) is modified by the substitutions

$$\delta_a{}^i+\delta_b{}^i\to\tfrac{1}{2}(\delta_a{}^i+\delta_b{}^i+\delta_c{}^i+\delta_d{}^i), \quad (29)$$

$$f^\mu\to\tfrac{1}{2}(f^\mu+f_\mu). \quad (30)$$

The second substitution is in fact trivial since $f_\mu=f^\mu$, but formally we will maintain the distinction.

We now equate the two expressions for the residues, and if we are allowed to equate the coefficients of $(\delta_a{}^i+\delta_b{}^i+\delta_c{}^i+\delta_d{}^i)$ and δ_t, we obtain two further conditions satisfied by the quantities k_i, k_i', K_i, in particular,

$$k_i-\epsilon_ik_i'=\frac{f^\mu+f_\mu}{2f_0}+\tfrac{1}{2}\epsilon_iK_i\frac{f^m}{f_0} \quad (31)$$

$$-\epsilon_ik_i+(2+\epsilon_i)k_i'=\epsilon_i\frac{f^\mu+f_\mu}{2f_0}-\tfrac{1}{2}\epsilon_iK_i\frac{f^m}{f_0}+\frac{K_if_m}{f_0}-\frac{\epsilon_i}{\alpha_m}\times\left[\alpha^\mu{}_{,m}+\alpha_{\mu,m}-\tfrac{1}{2}K_i(1-\epsilon_i)\alpha^m{}_{,m}+\frac{K_i}{\epsilon_i}\alpha_{m,m}\right]. \quad (32)$$

In fact, one is justified in equating the coefficients for all of the eigenvectors, $\boldsymbol{\delta}^i$, except for that corresponding to $\epsilon=2$, i.e., the case where all the masses remain degenerate. In this case the equations allow a determination of the parameter K_i and the combination $(2k_i+k_i')$ only. This is reasonable since it is only the combination $(2k_i+k_i')$ which enters into the definition of the $\gamma_{ab}{}^r$.

Thus, just as in Sec. IIB, the problem is, in principle, solved. By means of linear combinations of eigenvectors, one can in principle determine Δ_r and $\gamma_{ab}{}^r$ in terms of the given δ_a.

One can again inquire into the consequences of scale invariance. This gives the identities:

$$2(f^\mu+f_\mu)\mu_0^2+(f^m+f_m)m_0^2=-f_0, \quad (33)$$

$$(\alpha^\mu+\alpha_\mu)/(\alpha^m+\alpha_m)=-(m_0^2/2\mu_0^2), \quad (34)$$

$$2(\alpha^\mu{}_{,m}+\alpha_{\mu,m})\mu_0^2+(\alpha^m{}_{,m}+\alpha_{m,m})m_0^2=-\alpha_m, \quad (35)$$

from which it is easy to show that there is a solution with $\epsilon=2$ and the coupling constants and mass ratio unchanged.

B. Scalar-Vector Scattering

In close analogy to the bootstrap of the vector mesons, we bootstrap the scalar mesons by considering scalar-vector scattering. As our model of scalar-vector scattering we take a set of amplitudes of the form

$$\sum_{b,t} N_{ar,bt}(D^{-1})_{bt,cu}\,, \tag{36}$$

where, as before, $\mathbf{N}$ is a matrix of Born approximation amplitudes. The first of each pair of indices refers to a scalar particle, the second to a vector particle.

These Born amplitudes correspond to the diagrams which involve the scalar-scalar-vector couplings only, i.e., the diagram given in Fig. 5. With this assumption we can write

$$N_{ar,bt}(S)=\sum_c g_{ac}{}^t g_{bc}{}^r h(a,r,c;\,b,t,S)\,, \tag{37}$$

$$D_{ar,bt}(S)=\delta_{ar,bt}-\sum_c g_{ac}{}^t g_{bc}{}^r \beta(a,r,c;\,b,t,S)\,. \tag{38}$$

In the equal-mass case $\mathbf{N}$ and $\mathbf{D}$ can be diagonalized by the 64×64 matrix $\mathbf{Q}$ whose columns are the set of Clebsch-Gordan coefficients for the coupling of $8\otimes 8$ to any of the 64 states of the various irreducible representation of SU_3. In contrast to the situation in Sec. IIIA, both symmetrical and antisymmetrical couplings are now allowed. As before, eight columns of $\mathbf{Q}$ will consist of the $g_{ab}{}^r/G$. The others we denote by $\bar{g}_{ab}{}^x/G$.

The existence of the symmetric solution is guaranteed by assuming that

$$1=G^2\beta_0(\mu_0{}^2)\,, \tag{39}$$

$$h_0=-G^2\beta_0{}'(\mu_0{}^2) \tag{40}$$

have a solution, where $\beta_0(S)$ and $h_0(S)$ are the functions $\beta(S)$ and $h(S)$, evaluated when all the scalar and vector masses have the values m_0 and μ_0, respectively; $\beta_0{}'$ is the derivative of β_0. Equations (39) and (40) may be established by diagonalizing $\mathbf{N}$ and $\mathbf{D}$ in the equal-mass case by use of $\mathbf{Q}$.

If we return to the case of nonequal mass, expanding to first order in the mass shifts and using the ansatz of Sec. IIC, Eqs. (36) and (37), one can still transform $\mathbf{N}$ and $\mathbf{D}$ by $\mathbf{Q}$. The resulting matrices, $\mathbf{N}'$ and $\mathbf{D}'$ will, as before, not be diagonal, but they will contain 8×8 diagonal submatrices. These 8×8 diagonal submatrices are

$$\begin{aligned} D'_{fj}=\delta_{fj}-G^2\beta_0(S)\delta_{fj}-G^2\delta_{fj}\delta_j{}^i(\tfrac{1}{2}\epsilon_i)\{\beta^\mu(S)+\beta_\mu(S) \\ +2\tilde{k}_i\beta_i(S)\epsilon_i+\tilde{K}_i[\beta^m(S)+\beta_m(S)] \\ +2\tilde{k}_i\beta_0(S)+(\epsilon_i-1)\beta'(S)\}\,, \end{aligned} \tag{41}$$

$$\begin{aligned} N'_{fj}=G^2h_0(S)\delta_{fj}+G^2\delta_{fj}\delta_j{}^i(\tfrac{1}{2}\epsilon_i)\{h^\mu(S)+h_\mu(S) \\ +2\tilde{k}_ih_0(S)\epsilon_i+\tilde{K}_i[h^m(S)+h_m(S)] \\ +2\tilde{k}_i{}'h_0(S)+(\epsilon_i-1)h'(S)\}\,, \end{aligned} \tag{42}$$

where we have defined

$$\left(\frac{\partial}{\partial\mu_a{}^2},\frac{\partial}{\partial\mu_b{}^2},\frac{\partial}{\partial\mu_c{}^2},\frac{\partial}{\partial m_r{}^2},\frac{\partial}{\partial m_t{}^2}\right)\begin{Bmatrix}\beta(a,r,c;\,b,t,S)\\ h(a,r,c;\,b,t,S)\end{Bmatrix} =\begin{Bmatrix}\beta^\mu(S),\,\beta_\mu(S),\,\beta'(S),\,\beta^m(S),\,\beta_m(S)\\ h^\mu(S),\,h_\mu(S),\,h'(S),\,h^m(S),\,h_m(S)\end{Bmatrix}. \tag{43}$$

As before, we will need some additional elements of $\mathbf{N}'$ to obtain the residue condition; the appropriate elements are

$$\begin{aligned} N'_{xj}(S)=\frac{1}{2G^2}\sum_{a,r,b,t,c}\bar{g}_{ar}{}^x g_{ac}{}^t g_{bc}{}^r g_{bt}{}^j \\ \times\{[h^\mu(S)+\tilde{k}_ih_0(S)]\delta_a{}^i+[h_\mu(S)+\tilde{k}_ih_0(S)]\delta_b{}^i \\ +[\tilde{K}_ih^m(S)+h_0\tilde{k}_i{}'(S)]\delta_r{}^i+[\tilde{K}_ih_m(S)+h_0(S)\tilde{k}_i{}']\delta_t{}^i \\ +[h'(S)+2h_0(S)\tilde{k}_i]\delta_c{}^i\}\,. \end{aligned} \tag{44}$$

The discussion of the zeros of $\det\mathbf{D}(S)$ is identical to that of the last section, and instead of Eq. (24), one obtains

$$\tilde{K}_i=-\frac{\{\beta_s+\frac{1}{2}\epsilon_i[\beta^\mu+\beta_\mu+(\epsilon_i-1)\beta'+2\tilde{k}_i\beta_0\epsilon_i+2\tilde{k}_i{}'\beta_0]\}}{\frac{1}{2}\epsilon_i(\beta^m+\beta_m)}\,, \tag{45}$$

where β_s is the derivative of β with respect to S and all derivatives are evaluated at the mass values of the symmetric solution.

In the same way as before, we can evaluate the residue of $\mathbf{T}$. It is

$$\begin{aligned} \mathrm{Res}(\mathbf{N}\,\mathbf{D}^{-1})_{\alpha\rho,\beta\tau}=\sum_{[j],btc}\frac{g_{\alpha c}{}^t g_{bc}{}^\rho g_{bt}{}^j g_{\beta\tau}{}^j}{r^{jj}} \\ \times\{(h^\mu+\tilde{k}_ih_0)\delta_\alpha{}^i+(h_\mu+\tilde{k}_ih_0)\delta_b{}^i \\ +(\tilde{K}_ih^m+h_0\tilde{k}')\delta_\rho{}^i+(\tilde{K}_ih_m+h_0\tilde{k}_i{}')\delta_t{}^i \\ +(h'+2h_0\tilde{k}_i)\delta_c{}^i+h_0+h_s\delta_j{}^i\} \end{aligned} \tag{46}$$

where

$$\begin{aligned} r_{jj}=-G^2\{\beta_s+\delta_j{}^i\beta_{s,s}+\delta_j{}^i(\tfrac{1}{2}\epsilon_i)[\beta^\mu{}_{,s}+\beta_{\mu,s}+2\epsilon_i\tilde{k}_i\beta_s \\ +\tilde{K}_i(\beta^m{}_{,s}+\beta_{m,s})+2\tilde{k}_i{}'\beta_s+(\epsilon_i-1)\beta_s{}']\}\,. \end{aligned} \tag{47}$$

The symbols with two subscripts signify second derivatives, evaluated at the symmetry masses. The "free" sums in Eq. (46) can be done. Again, an examina-

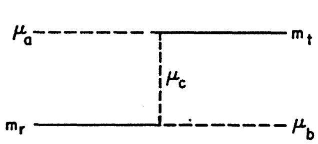


FIG. 5. Born approximation diagram for scalar-vector scattering.

tion of the residues of $T_{ar,bt}$ and $T_{bt,ar}$ at the various poles shows that they are not equal, so that $\mathbf{T}$ is not symmetric. As before, one can symmetrize, and then one obtains

$$\begin{aligned}\mathrm{Res}(\mathbf{N}\,\mathbf{D}^{-1})_{\alpha\rho,\beta\tau}=\sum_{[j]}\frac{g_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}}{r_{jj}}\{h_0+\tfrac{1}{2}(\delta_\alpha{}^i+\delta_\beta{}^i)\\ \times[h^\mu+h_\mu+\tilde{k}_ih_0+\tfrac{1}{2}\epsilon_i(\tilde{K}_i(h^m+h_m)-h^\mu-h_\mu\\ +2h_0\tilde{k}_i+h'+2\tilde{k}_i'h_0)]+\tfrac{1}{2}(\delta_\rho{}^i+\delta_\tau{}^i)\\ \times[\tilde{K}_i(h^m+h_m)+2\tilde{k}_i'h_0+\tfrac{1}{2}\epsilon_i(h^\mu+h_\mu-\tilde{K}_i(h^m+h_m)\\ +2h'-2h_0\tilde{k}_i'+6h_0\tilde{k}_i)]+\delta_j{}^i[h_s+\tfrac{1}{2}\epsilon_i(h^\mu+h_\mu\\ +\tilde{K}_i(h^m+h_m)-2h'+2h_0\tilde{k}_i'-2h_0\tilde{k}_i)]\}\,. \quad (48)\end{aligned}$$

Thus, to first order in the $\boldsymbol{\delta}^i$, one obtains from Eq. (48), after expanding r_{jj},

$$\begin{aligned}\mathrm{Res}(\mathbf{N}\,\mathbf{D}^{-1})_{\alpha\rho,\beta\tau}=\sum_{[j]}g_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}\left\{-\frac{h_0}{G^2\beta'}+A_i\frac{(\delta_\alpha{}^i+\delta_\beta{}^i)}{2}\right.\\ \left.+B_i\frac{(\delta_\rho{}^i+\delta_\tau{}^i)}{2}+C_i\delta_j{}^i\right\}\,, \quad (49)\end{aligned}$$

where the expressions for A_i, B_i, and C_i are rather cumbersome to write down, and we do not do so here.

Now, by self-consistency, Eq. (49) must be equal to

$$\sum_{[j]}g_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}+\sum_{[j]}g_{\alpha\rho}{}^{j}\gamma_{\beta\tau}{}^{j}+\sum_{[j]}\gamma_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}$$

or

$$\begin{aligned}\sum_{[j]}g_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}+\sum_{[j]}g_{\alpha\rho}{}^{j}g_{\beta\tau}{}^{j}\\ \times[(\delta_\alpha{}^i+\delta_\beta{}^i+2\delta_j{}^i)\tilde{k}_i+(\delta_\rho{}^i+\delta_\tau{}^i)\tilde{k}_i']. \quad (50)\end{aligned}$$

We see from Eq. (50) that the coefficients of δ_j and $\frac{1}{2}(\delta_\alpha+\delta_\beta)$ must be equal which implies, from Eq. (49),

$$A_i=C_i. \quad (51)$$

Further, self-consistency requires

$$A_i=2\tilde{k}_i, \quad (52)$$

$$B_i=2\tilde{k}_i'. \quad (53)$$

Thus, in contrast to the situation in Sec. IIIA, we now have four equations, Eqs. (45), (51), (52), and (53) to determine the three unknowns, $\tilde{k}_i$, $\tilde{k}_i'$, and $\tilde{K}_i$ for a given ϵ_i. In general, it will be impossible to satisfy all of these equations for the allowed values of ϵ. Strictly speaking, this does not prove that our self-consistency problem has no solutions. It is certainly possible that one of these relations is an unobserved identity, but the origin of such an identity is not clear to us. Another possibility is that, to obtain solutions to the self-consistency problem, one requires a more general ansatz for $\gamma_{ab}{}^r$ than that given by Eq. (36) of Sec. II. The most general linear form may be written as

$$\gamma_{ab}{}^r=g_{ab}{}^r[\tilde{k}_i(\delta_a{}^i+\delta_b{}^i)+\tilde{k}_i'\delta_r{}^i]+\tilde{\Gamma}_{ab}{}^r(\delta_x)\,, \quad (54)$$

where $\tilde{\Gamma}_{ab}{}^r(\delta_x)$ is an unknown linear function of the δ_x's, $x\neq a, b$, or r. We have not explored such possibilities in any depth, but it seems likely that each new constant introduced into the γ's by Eq. (54) will lead to a new constraint when the residue of $(\mathbf{N}\mathbf{D}^{-1})_{\alpha\rho,\beta\tau}$ is equated to $\sum_{[j]}(g_{\alpha\rho}{}^j+\gamma_{\alpha\rho}{}^j)(g_{\beta\tau}{}^j+\gamma_{\beta\tau}{}^j)$. If this is the case, then we shall still have one more equation than unknowns, and the possibility of a solution is still improbable.

It is appropriate to summarize here why it is that our conclusions in this section differ from those in Sec. IIIA. In Sec. IIIA we found that a solution of the scalar-scalar scattering bootstrap exists for any given set of scalar masses infinitesimally close to the symmetric solution. We now find that within the present method the existence of *any* nondegenerate solution to the scalar-vector scattering bootstrap is unlikely. The reason is the following: We have assumed, as is necessary from general considerations, the antisymmetry of the coupling constants $g_{ab}{}^r$ under interchange of the particle labels a and b. This implies the equality of the coefficients of δ_a and δ_b in the ansatz of Eq. (54) for $\gamma_{ab}{}^r$. Now, the P-wave scalar-scalar scattering amplitude $T_{ab,cd}$ has an antisymmetry property with respect to the interchange of labels a and b, and also c and d, and therefore so does its residue. Because of this, the equality in Sec. IIIA, of the coefficients of δ_a and δ_b in $\gamma_{ab}{}^r$ did not lead to an extra constraint. Here it has led to an extra constraint because the scalar particles occur both internally and externally in scalar-vector scattering. No general property of the scalar-vector scattering amplitude guarantees the required symmetry property of its residue under the interchange of internal and external particle labels.

C. Full Self-Consistency

As in Sec. II, we now wish to investigate the possibility that the scalar and vector bootstraps are self-consistent simultaneously. For this purpose we shall simply ignore the difficulty encountered when we considered the scalar-vector scattering bootstrap. In other words, we shall assume, unlikely as it is, that Eqs. (45), (51), (52), and (53) can indeed be solved for the $\tilde{k}_i$, $\tilde{k}_i'$, and $\tilde{K}_i$. We shall encounter difficulties other than those of the scalar-vector bootstrap above.

The investigation proceeds in close analogy to that of Sec. IID. We first assume that a fully self-consistent symmetric solution exists, i.e., Eqs. (8), (9), (38), and (39) are simultaneously satisfied. If we again impose scale invariance, we have two unknowns $m_0{}^2/\mu_0{}^2$ and G^2. It is certainly possible that no pair of values will satisfy all four equations. There are however, two parameters, the two D function subtraction points. We will assume that they can be chosen so that a fully self-consistent symmetric solution exists. No doubt, such a solution is not unique, but hopefully one might be able to choose the subtraction points so that some vestige of crossing symmetry remains.

In any case, we are primarily interested here in the situation with unequal masses. Full self-consistency demands that the masses obtained in each bootstrap be equal. This is the case if

$$K_i=\tilde{K}_i. \tag{55}$$

Since all quantities are fixed in K_i and $\tilde{K}_i$, we would conclude that in general there are no fully self-consistent solutions other than the symmetric solution with $\epsilon=2$ (because of scale invariance). The discussion in Sec. IID is also appropriate here. Various points of view can be taken. It could be the case that Eq. (55) is approximately satisfied for one of the eigenvalues. This could indicate that a given set of scalar and vector mass shifts would yield a self-consistent solution. On the other hand, one could again argue that the most that such a result implies is that the scalar and vector masses both obey a particular mass formula, with no possibility of a discussion of magnitudes.

The above discussion has ignored the question of the equality of the coupling constants as calculated in the two self-consistency problems. This gives two further equations

$$k_i=\tilde{k}_i \tag{56}$$

$$k_i'=\tilde{k}_i' \tag{57}$$

to be satisfied for full self-consistency. Equations (56) and (57) are not implied by Eq. (55). This is different from the situation met with in the discussion of Sec. II. We recall that if the "mass condition," Sec. II, Eq. (48) were satisfied, then so were the "coupling-constant conditions," Sec. II, Eqs. (49) and (50). Here the problem would appear to be hopelessly overdetermined, since we must have all three of Eqs. (55), (56), and (57) satisfied for the same value of ϵ_i.

We conclude that it is quite improbable that a fully self-consistent solution with nondegenerate masses exists within the present model. No doubt the simplest and most convincing way to eliminate the present difficulty would be to discover identities between the equations, analogous to those of Sec. II. However, it is likely that none exist within our approximations, and we have not uncovered any.

The conclusion here differs from that of Sec. IID. There it was also concluded that, in general, a fully self-consistent solution with nondegenerate masses would not exist. However, for such a solution to exist only one additional constraint would have to be satisfied, and it seemed possible that an improvement in the self-consistency approximation or the inclusion of higher order terms might yield a solution with nondegenerate masses. We now find that in the bootstrap analysis three separate additional constraints must be satisfied by a solution with nondegenerate masses, so that the existence of such a solution is correspondingly more unlikely. This difference between the results of the field-theoretic and bootstrap analyses might come as a surprise to some, because Rockmore[8] has shown that the two analyses are equivalent within appropriate approximations. However, since his analysis was made for single-channel problems it does not apply here. It is precisely the presence of several channels with different particle masses which is the source of the difficulty.

IV. SUMMARY

Broadly speaking, we have obtained two results. The first concerns the possibility of limited self-consistency, i.e., the self-consistency of one set of particles. In Sec. II we showed, in the field-theoretic analysis, that it is possible to make either the vector or scalar set of particles self-consistent for arbitrary values of the masses of the scalar or vector sets, respectively. This is, of course, always under the assumption that the corresponding solution with exact SU_3 symmetry exists, and involved working to first order only in the mass shifts.

In Sec. III we investigated the question of whether a similar result is true in the bootstrap analysis. By considering the bootstrap of vector particles in scalar-scalar scattering, we did find that the same result is true for self-consistent vector particles. We also considered the bootstrap of scalar particles in scalar-vector scattering. We found that self-consistency now required more equations to be satisfied than there were unknowns to be determined. The reason for this is that the residue of the pole in $(\mathbf{ND}^{-1})_{ar,bs}$ (where a and b are scalar, and r and s are vector particle indices) does not automatically contain the correct symmetry property required of it if it is to be equated to a product of scalar-scalar-vector coupling constants. Because of this, it seems difficult to see how, in our approximations, a nondegenerate solution can exist to the scalar bootstrap problem. One would suspect that the same difficulty will arise in any approximate bootstrap calculation in which the extra constraint on the residue of $(\mathbf{ND}^{-1})_{ar,bt}$ is not identically satisfied.

Throughout the analysis referred to above, it was found that the solutions to the self-consistency problem with specific sets of input masses were particularly simple. For these input mass shifts (called eigenvectors), the output mass shifts were proportional to the input mass shifts, and the shifts in the coupling constants had the simple form of Sec. II, Eq. (20). One such set of mass shifts is such that the associated total masses obey the Gell-Mann–Okubo formula. So we have the result that if the input masses obey the Gell-Mann–Okubo formula, so do the output masses.

Our second result concerns the possibility of full self-consistency. We considered this problem within both the field-theoretic and bootstrap contexts again

[8] R. M. Rockmore, Phys. Rev. **132**, 878 (1963).

assuming the existence of a solution possessing exact SU_3 symmetry. Our first point was that the requirements of full self-consistency are most easily satisfied by sets of mass shifts proportional to an eigenvector. Masses obeying the Gell-Mann–Okubo formula fall into this class. Which particular eigenvector (if any) is favored is, however, a matter of detailed calculation and very probably is model-dependent.

Neither in the field-theoretic nor in the bootstrap framework does a solution exist, strictly speaking, to the full self-consistency problem with nondegenerate masses—at least within our scheme. In the field-theoretic analysis, however, because of certain identities, there was only one more equation than unknowns, and it was plausible that a more accurate calculation (including, for example, second-order terms) might yield a solution for one particular eigenvector. In the bootstrap analysis, even laying aside the difficulty encountered in the bootstrap of the scalar particles, it was difficult to see how a fully self-consistent solution could come about. There were three more equations to be satisfied than there were unknowns (plus a further equation to enable the scalar bootstrap itself to have a solution). None of these equations appeared, in our approximation at least, to reduce to an identity.

It should be emphasized that even if a fully self-consistent solution were to exist for some particular eigenvector, there is no requirement which selects this solution rather than the equal-mass, symmetry solution. It is very difficult to understand how criteria could be established which would in fact distinguish between the two solutions. Since this situation will always occur when one assumes the existence of a symmetric solution which is then perturbed (keeping any order terms), it may be that an understanding of how the self-consistency develops cannot be obtained with this assumption. This very well may be a fundamental deficiency of the approach presented in this paper.

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High-Energy Behavior in Field Theory and Dilatational Symmetry*

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The high-energy forward-scattering amplitude in $\lambda\phi^4$ theory is investigated by means of the Bethe-Salpeter equation. The class of irreducible diagrams which have dilatational symmetry at high energies is considered as kernels for the Bethe-Salpeter equation. For these kernels, the equation is solved using a Mellin transform. The solution is found to contain a term which, at high energies, behaves like $E^{n_0}/(\ln E)^c$. This term is caused by a branch cut in a complex four-dimensional Euclidean angular-momentum plane. A lower bound for n_0 is obtained from a simple diagram. Using the upper bound on n_0 which results from unitarity, an upper bound on the coupling constant is obtained: $\lambda < (3\sqrt{8})\pi^2$.

I. INTRODUCTION

RECENTLY, several investigations have been made into the high-energy dependence of scattering amplitudes in field theory.[1–3] In the case of pseudoscalar meson-fermion interactions[4–11] and $\lambda\phi^4$ theory, the high-energy scattering in a certain approximation was found to be dominated by an energy-independent branch point in the Regge plane. This gives rise to a power-law dependence (neglecting logarithmic factors) on energy for the scattering amplitude. As pointed out by Fubini,[12] these results are consequences of the fact that under

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