Pseudoscalar-Meson Mass Differences in a Self-Consistent Model

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A bootstrap model of the P (pseudoscalar) and V (vector) meson octets is considered, in which the P and V are bound states or resonances in two-particle states of the type PV and PP, respectively. Unitary symmetry is assumed. Several approximations are made in the many-channel, partial-wave dispersion relations of the model, in order that simple self-consistency relations among the various P and V meson mass ratios may be obtained. If it is required that the P octet not be degenerate or nearly degenerate, near self-consistency can be obtained only if the pion mass is small compared to the K and η masses. It is argued that the SU_3 scheme of P, V, and baryon octets, and a $j^P=\frac{3}{2}^+$ baryon decuplet, with mass splittings similar to those observed experimentally, is unusually well suited to a nondegenerate solution of a bootstrap model, because of the strong mutual coupling that exists among the lightest members of the various multiplets.

I. INTRODUCTION AND GENERAL PROCEDURE

Several theoretical works of the last two years have shown that there is a strong possibility that the SU_3 symmetry of the strongly interacting particles may be the result of the self-consistency requirements of a bootstrap dispersion theory.[1,2] However, the experimentally observed mass splittings of the particle multiplets are large. Therefore, the goal of bootstrap theory must be to predict a badly broken SU_3, rather than an exact SU_3.

The symmetry-breaking mechanism that fits in most naturally with bootstrap models is "spontaneous breakdown." This mechanism has been studied by many authors, and was first applied to SU_3 symmetry by Glashow.[3] Basically, the idea is simple. One supposes that the masses and interaction constants of the particles are described by a system of nonlinear equations. A basic symmetry (SU_3 symmetry in our considerations) is present in the sense that these equations possess a solution involving degenerate multiplets and exact interaction symmetry. We call this solution the degeneracy solution. It is also supposed that the equations possess at least (and perferably, at most) one solution involving mass splitting and a breaking of the interaction symmetry. This solution must correspond to reality. In order to apply this technique to a bootstrap model, one simply takes as the basic equations the self-consistency equations associated with the positions and residues of the poles identified with the particles.

The application of this technique to bootstrap models has been discussed in several papers.[4–6] However, calculations of mass splitting for realistic systems of particles usually have been confined to terms that are of first order in the deviations of the masses from the degeneracy-solution values.[5–8] Unfortunately, one can obtain no idea of the magnitude or the sign of a mass splitting from such a first-order calculation. However, one of the most striking features of the experimental mass spectrum is the fact that the mass splitting of the lightest (pseudoscalar meson) multiplet is almost as large as is physically possible. One may describe the observed spectrum of strongly interacting particles with the phrase, "unitary symmetry with a sore thumb." The sore thumb is the pi meson, which is so much lighter than the other particles that it transmits the forces of longest range, and in this sense "sticks out." The main purpose of this paper is to investigate the possibility that the small pion mass may result from the self-consistency requirements of a bootstrap dispersion theory.

We consider a simple model of a coupled P (pseudoscalar) meson octet and V (vector) meson octet. SU_3 symmetry is assumed; we are not concerned here with the dynamical origin of this symmetry. The V mesons are associated with resonance poles in the coupled PP states, and the P with poles in the PV states. (In this paper the term "resonance" is used frequently to refer both to bound states and resonances.) A more complete model involving both baryons and mesons is discussed briefly in Sec. V.

It is reasonable to assume that the forces in the $j=1$, PP states are transmitted by the exchange of the V mesons. On the other hand, it is not obvious whether P exchange (associated with the VPP vertices) or V exchange (associated with VVP vertices) should be most important in the PV states containing the P resonance poles. Since the VPP interaction in SU_3 symmetry is antisymmetric with respect to the interchange of corresponding members of any two of the three octets, the P-meson poles must occur in the antisymmetric octet combination of the P and V octets. It is interesting that the most attractive force resulting from either the V or P exchange mechanism occurs in the antisymmetric octet state, so that either assumption

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[1] R. H. Capps, Phys. Rev. Letters **10**, 312 (1963).
[2] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); a list of other references is given in Ref. 6, below.
[3] G. Jona-Lasinio and Y. Nambu, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961); J. Goldstone, Nuovo Cimento **19**, 154 (1961); M. Baker and S. L. Glashow, Phys. Rev. **128**, 2462 (1962); S. L. Glashow, *ibid.* **130**, 2132 (1963).
[4] R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1354 (1963).
[5] R. H. Capps, Phys. Rev. **134**, B460 (1964).
[6] R. H. Capps, Phys. Rev. **134**, B1396 (1964).
[7] R. E. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963).
[8] R. H. Capps, Phys. Rev. **132**, 2749 (1963).

is consistent with our model.[9] Unfortunately, no matter which assumption one makes, there is no accurate dispersion theoretic approximation for treating such tightly bound systems as the P mesons. Therefore, we will not specify the origin of the forces, but will make a simple approximation for all the forces in the model. Deviations from exact SU_3 symmetry of the forces will be neglected.

It has been shown previously that this type of approximation agrees closely with more detailed calculations of the dependence of the first-order V octet mass splitting on the first-order P octet mass splitting.[6] However, it does not follow that the approximation is accurate for the present calculation in which both V and P mass splittings are calculated and higher order terms in the deviations from degeneracy are included. Present-day dispersion techniques are too crude for an accurate calculation of such higher-order effects. However, the problem of calculating self-generating mass differences is of key importance in bootstrap theory. In order to make a start on this problem, we make some rather drastic assumptions that lead to simple equations for the mass splitting. We cannot argue that all neglected effects are small. The basic idea of the present calculation was suggested previously by the author.[10]

The general procedure is to first assume masses for the P mesons, calculate the V masses from the dispersion relations associated with the V resonance poles in the PP states, then calculate final P masses from the equations for the P poles in the PV states, and finally compare with the P masses assumed originally. We believe that the effects of the many unknown short-range forces that are neglected in any dispersion-theoretic calculation are not so important for the mass differences of particles within SU_3 multiplets as they are for the ratios of masses of particles in different multiplets. Accordingly, we do not attempt to calculate V/P mass ratios, but take the average V/P mass ratio from experiment.

The basic dynamical equation of the model is Eq. (23) of Sec. IIB. This equation is very simple. In Sec. IIA, it is shown that this equation follows from six plausible dynamical assumptions. Readers uninterested in the detailed assumptions may skip to Eq. (23), and then read through to the end of the paper (skipping Sec. IV) without loss of continuity. Section III contains the results, and Secs. IV and V contain discussions of possible modifications and extensions of the model.

II. DETAILS OF THE MODEL

A. Assumptions

The first assumption is,

(1) Isotopic spin and hypercharge are conserved.

The mass variables of the model are the π, K, and η P-meson masses and the ρ, K^* (called here the M), and φ V-meson masses. The ρ is coupled to $\pi\pi$ and $K\bar{K}$ states, the M to πK and ηK states, the φ to $K\bar{K}$, the π to $\pi\rho$, $K\bar{M}$, and $\bar{K}M$, the K to πM, ηM, $K\rho$, and $K\varphi$, and the η to $K\bar{M}$ and $\bar{K}M$. The basic method used is the matrix N/D dispersion method. At energies above the various thresholds, the partial-wave amplitudes connecting the P-wave PP states, and connecting the $j^P=0^-$ PV states, are related to the appropriate elements of the unitary S matrix by the equation, $T_{ij}=(ND^{-1})_{ij}=(S_{ij}-\delta_{ij})/(2iq_i^{3/2}\rho_i^{1/2}q_j^{3/2}\rho_j^{1/2})$, where $q_i^3\rho_i$ is a phase space factor and q_i is the magnitude of the i-channel particle momentum in the center-of-mass system. Since the resonating states are all P-wave states, the factor ρ_i contains no zeros or poles at the channel threshold. It is not necessary to specify the ρ_i exactly for the simple approximation of Secs. IIB and III.

The forces are assumed to satisfy the following three conditions:

(2) The forces (elements of the N matrices) are simple poles. The force poles for the various amplitudes coupled to a particular resonance are at the same energy.

(3) The ratios of the residues of the force poles are chosen so that an SU_3 symmetric solution involving a degenerate P octet and a degenerate V octet exists, and so that the forces vanish in all nonresonating PP and PV states.

(4) The average residues of the V and P force poles are chosen so that the calculated ratio of the magnitudes of the V and P octet mass-squared vectors agrees with experiment.[11]

Assumption (2) leads to the form listed below for the ND^{-1} equations for the channels coupled to the resonance pole associated with the V meson α, if only the PP states are considered in the unitarity condition. (Frequently, we write only the V pole equations. In all cases the P pole equations may be obtained simply by replacing the index V by P.) The equations are,

$$T_{ij}{}^\alpha=\sum_k N_{ik}{}^\alpha \operatorname{cof}_{jk}(D^\alpha)/|D^\alpha|\,, \tag{1}$$

$$N_{ij}{}^\alpha=F_{ij}{}^\alpha/(s-s_\alpha)\,, \tag{2}$$

$$D_{ij}{}^\alpha=\delta_{ij}-F_{ij}{}^\alpha H_i{}^\alpha(s)\,, \tag{3}$$

$$H_i{}^\alpha(s)=\frac{s-s_\alpha}{\pi}\int_{q_i=0}^{\infty}\frac{ds' q_i'^3\rho_i^V(s')}{(s'-s_\alpha)^2(s'-s-i\epsilon)}\,, \tag{4}$$

where $\operatorname{cof}_{jk}(D^\alpha)$ and $|D^\alpha|$ denote the jk cofactor and the determinant of D^α, s is the square of the total energy in the center-of-mass system, and s_α and $F_{ij}{}^\alpha$ are constants. Time-reversal invariance implies that $F_{ij}{}^\alpha=F_{ji}{}^\alpha$. The ij representation is the representation of the physical two-particle states.

We now show that the ratios of the force residues $F_{ij}{}^\alpha$ are determined by assumption (3). We consider the degeneracy solution, in which all the phase space factors $(q_i^3\rho_i^V)$ for the $j=1$, PP states are equal. It has been shown previously that the requirement that the

[9] This is demonstrated for the V exchange mechanism by R. H. Capps, Nuovo Cimento 30, 340 (1963).

[10] Reference 5, Sec. V.

[11] R. H. Capps, Phys. Rev. 134, B649 (1964).

resonance α correspond to a simple pole in some of the $T_{ij}{}^{\alpha}$ implies that the residues $r_{ij}{}^{\alpha}$ of this pole satisfy a factorizability condition,

$$r_{ij}{}^{\alpha}=-C^{V}\gamma_i{}^{\alpha}\gamma_j{}^{\alpha}, \tag{5}$$

where C^V is a positive constant independent of i, j, and α, and $\gamma_i{}^{\alpha}$ is the coupling constant of α to the two-particle state i.[6] If the pole corresponds to a resonance rather than a bound state, we make the small width approximation, i.e., we neglect the imaginary parts of the dispersion integrals when determining the position and residue of the pole. In general, the above factorizability condition does not imply that the force residues $F_{ij}{}^{\alpha}$ must satisfy a similar condition. However, our assumption that the forces vanish in the nonresonating states does imply not only that the $F_{ij}{}^{\alpha}$ are factorizable, but also that they are proportional to the $r_{ij}{}^{\alpha}$, i.e., $F_{ij}=C'\gamma_i{}^{\alpha}\gamma_j{}^{\alpha}$. In order to verify this assertion, one need only write the degeneracy-solution matrix amplitude T^{α} in the representation in which it is diagonal (in which representation all elements are zero except that associated with the resonance) and then transform to the representation of the physical particles. Since the $\gamma_i{}^{\alpha}$ are proportional to the Clebsch-Gordan coefficients $A_i{}^{\alpha}$ of SU_3, we write

$$F_{ij}{}^{\alpha}=g_V A_i{}^{\alpha}A_j{}^{\alpha}, \tag{6}$$

where g_V is a positive constant.

It has been shown that the V-meson exchange force in the PP states does satisfy the factorizability condition.[1] An equivalent statement is that this force is zero in the nonresonating PP states of the representations 10 and 10*. The P-meson exchange force in the PV states also vanishes in the representation 10 and 10*, but not in the nonresonating symmetric representations of dimensions 1, 8, and 27. It has also been shown that to first order in the mass differences, such a factorizability assumption on the forces is superfluous, for the nonresonating representations are not coupled to the resonating representation.[12] However, in a higher order calculation, such a coupling can exist. Since we do not know the nature of the exchange force in the PV states, we do not know the effect of this coupling. Hence, we neglect this coupling.

We conclude that assumption (3) determines the residues of the force poles in terms of two constants g_V g_P. These constants are to be determined later from assumption (4). The fifth assumption is not really an assumption since it is necessary for self-consistency. It is,

(5) *The positions s_α of the various force poles are determined so that the values of any particular VPP interaction constant determined from the residues of the resonance poles corresponding to the three particles in the vertex are consistent with each other.*

The condition that a resonance or bound state occur at an energy-squared s_r is that $|D^{\alpha}(s_r)|$ vanish if the integrals $H_i{}^{\alpha}$ are replaced by their real parts. These integrals are dimensionless, and thus must be functions of ratios of the quantities s, s_α, $\mu_{ia}{}^2$, and $\mu_{ib}{}^2$, where μ_{ia} and μ_{ib} are the masses of the two particles of the state i. We make the following simple approximation for the integrals,

(6) *The integrals* $\mathrm{Re}H_i{}^{\alpha}$ *for the V resonances are approximated by*

$$\mathrm{Re}H_i{}^{\alpha}=H_0{}^V+H_1{}^V\frac{s}{\mu_i{}^2}+H_2{}^V\frac{s_\alpha}{\mu_i{}^2}, \tag{7}$$

where

$$\mu_i{}^2=\tfrac{1}{4}(\mu_{ia}+\mu_{ib})^2, \tag{8}$$

and $H_0{}^V$, $H_1{}^V$, *and* $H_2{}^V$ *are constants common to all the V poles.*

We discuss briefly the justification of this approximation. The dispersion integrals $H_i{}^{\alpha}$ depend on μ_{ia} and μ_{ib} only by means of the phase space factors, $\rho_i q_i{}^3$. The most important factors are the $q_i{}^3$; these are related to the particle masses by the equation,

$$4q_i{}^2=s-2(\mu_{ia}{}^2+\mu_{ib}{}^2)+(\mu_{ia}{}^2-\mu_{ib}{}^2)^2/s.$$

The approximation of Eq. (8) is equivalent to replacing the above equation by $4q_i{}^2=s-4\mu_i{}^2$, and is thus very accurate for values of the integration variable close to threshold.

The approximation of Eq. (7), that $\mathrm{Re}H_i{}^{\alpha}$ depends linearly on s and s_α, is reasonable since the dependence of the integrals of Eq. (4) on the s and s_α factors in the products $(s'-s_\alpha)^{-2}(s'-s)^{-1}$ is not as important for bound states and low-lying resonances as the dependence on the $(s-s_\alpha)$ factor. Equation (7) is a modification of the effective range approximation. This equation is not accurate if $s\gg\mu_i{}^2$. The effect of this inaccuracy is discussed after Eq. (23) and in Sec. IVA.

B. The Basic Equation

The assumptions of Sec. IIA lead to a simple equation relating the mass of a resonating particle to the masses of the particles in the two-particle states coupled to the resonance. The first step in deriving this equation is to rewrite Eqs. (1) and (2) in the form given below,

$$T_{ij}=t_{ij}/(s-s_\alpha)|D|, \tag{9}$$

$$t_{ij}=\sum_k F_{ik}\,\mathrm{cof}_{jk}(D), \tag{10}$$

where the index α on the quantities t, T, and D has been suppressed. If Eq. (10) is multiplied by H_i and combined with Eq. (3), the result is

$$H_i t_{ij}=\sum_k(\delta_{ik}-D_{ik})\,\mathrm{cof}_{jk}(D). \tag{11}$$

The quantity $\sum_k D_{ik}\,\mathrm{cof}_{jk}(D)$ is equal to the determinant of a matrix formed by replacing the j row of D by the i row, and is thus equal to $\delta_{ij}|D|$. Therefore, Eq. (11) may be reduced to the form

$$H_i t_{ij}=\mathrm{cof}_{ji}(D)-\delta_{ij}|D|. \tag{12}$$

[12] Reference 6, Appendix.

In order to obtain a further simplification we make use of Eq. (6), concerning the factorizability of the F_{ij}. This condition implies that the determinant of a submatrix of the matrix $\mathfrak{F}_{ij}=F_{ij}H_i$ can fail to vanish only if the submatrix has but one element. Because of this fact, many of the terms in $|D|$ and in H_it_{ij} vanish; these quantities become simply

$$|D|=1-\sum_k F_{kk}H_k, \tag{13}$$

$$H_it_{ij}=H_iF_{ij}, \quad \text{or} \quad t_{ij}=F_{ij}. \tag{14}$$

The final equation for T_{ij} is obtained by combining Eqs. (9), (13), and (14), and then using Eq. (6) to write the F_{ij} in terms of SU_3 Clebsch-Gordan coefficients. The equation is,

$$T_{ij}^\alpha=\frac{g_VA_i^\alpha A_j^\alpha}{(s-s_\alpha)[1-g_V\sum_k(A_k^\alpha)^2H_k^\alpha(s)]}. \tag{15}$$

This equation is almost as simple as in the degeneracy approximation, the only extra complication being the fact that the H_k^α are different in different channels.

The self-consistency conditions associated with the position and residues of the pole α are obtained easily from Eq. (15). The resonance energy is the energy at which the second factor in the denominator of Eq. (15) vanishes, if the imaginary parts of the integrals H_k^α are neglected, i.e.,

$$\sum_k(A_k^\alpha)^2\,\mathrm{Re}H_k^\alpha(m_\alpha^2)=g_V^{-1}. \tag{16}$$

The relation between the residues of the resonance poles and the coupling constants is given in Eq. (5). Applied to Eq. (15), this relation is,

$$\gamma_i^\alpha\gamma_j^\alpha=A_i^\alpha A_j^\alpha/(C^VZ^\alpha), \tag{17}$$

$$Z^\alpha=(m_\alpha^2-s_\alpha)\sum_k(A_k^\alpha)^2[\partial(\mathrm{Re}H_k^\alpha)/\partial s]_{s=m_\alpha^2}. \tag{18}$$

If the ratio of coupling constants for two channels coupled to the same V or P resonance pole is taken, the result is,

$$\gamma_i^\alpha/\gamma_j^\alpha=A_i^\alpha/A_j^\alpha. \tag{19}$$

This ratio is unaffected by deviations from degeneracy in our approximation, and corresponds to exact SU_3 symmetry. We may now make use of the self-consistency requirement concerning the residues, i.e., assumption (5).[13] This requirement, together with Eq. (19), is sufficient to establish that the ratios of all five of the VPP interaction constants must correspond to exact SU_3 symmetry. The φKK, ρKK, $M\pi K$, and $M\eta K$ constants are all associated with the K pole, and thus must correspond to exact symmetry because of Eq. (19). Furthermore, the interactions $\rho\pi\pi$ and ρKK are both coupled to the ρ pole, so the ratios of all five ($\rho\pi\pi$, ρKK, φKK, $M\pi K$, and $M\eta K$) interaction constants correspond to exact symmetry. It is seen from Eqs. (17) that this condition implies that the various Z^α for each of the two sets of poles must be equal, i.e.,

$$Z^\varphi=Z^\rho=Z^M(=Z^V), \tag{20a}$$

$$Z^\eta=Z^\pi=Z^K(=Z^P). \tag{20b}$$

This is regarded as a condition on the subtraction constants s_α.

We now make use of our simple approximation for $\mathrm{Re}H_i$, Eq. (7). It is seen that with this approximation Eqs. (18) and (20a) lead to a simple equation for the subtraction energy, i.e.,

$$(s_\alpha/\mu_\alpha^2)=(m_\alpha^2/\mu_\alpha^2)-(Z^V/H_1^V), \tag{21}$$

where

$$(1/\mu_\alpha^2)=\sum_i[(A_i^\alpha)^2/\mu_i^2].$$

Since $(A_i^\alpha)^2$ is the probability of the two-particle state i in the wave function for the particle α that corresponds to exact symmetry, μ_α^{-2} is essentially the average value of μ_i^{-2} for the particle α.

The pole-position equation, Eq. (16), for our simple choice of $\mathrm{Re}H_i^\alpha$, is

$$H_1^V\frac{m_\alpha^2}{\mu_\alpha^2}+H_2^V\frac{s_\alpha}{\mu_\alpha^2}=g_V^{-1}-H_1^V. \tag{22}$$

If the variable s_α/m_α^2 is eliminated from this equation by means of Eq. (21), the resulting equation for the mass-squared of the V meson may be written in the form,

$$m_\alpha^2=N^V\left[\sum_i\frac{(A_i^\alpha)^2}{\mu_i^2}\right]^{-1}. \tag{23}$$

The sum is over the two particle states coupled to the pole α; A_i^α is the appropriate SU_3 Clebsch-Gordan coefficient, and μ_i is the average mass of the two particles of the state i. The normalization constant N^V depends on the coupling constant g_V and on the other constants introduced in this subsection; it is positive for reasonable choices of these constants, and is the same for all V mesons. The method used for determining the normalization constant is explained at the end of this section.

Equation (23) and the corresponding equation for the P masses (obtained by substituting N^P for N^V) are the basic equations of the model. If the terms in these equations are expanded in powers of the deviations of the individual P and V square masses from the respective average square masses, the linear approximation is the probability matrix approximation of Eq. (15) of Ref. 5 [with the constant α_{PV} of Ref. 5 set equal to $|\beta^V|^{1/2}/(|\beta^V|^{1/2}+|\beta^P|^{1/2})$]. The principal effect of the higher order terms of Eq. (23) is that when the difference between the masses of two states coupled to the same resonance pole is increased, the lighter state plays a larger role in determining the properties of the resonance pole. This is certainly a real physical effect. Our assumption that it is important is consistent with the most popular philosophy used in applying dispersion relations.

[13] The possible utility of this requirement in complete bootstrap models is discussed in detail in Ref. 6.

It was pointed out in Sec. IIA that the approximate form assumed for the dispersion integral, Eq. (7), is not accurate if $s/\mu_i^2 \gg 1$. In fact, Eq. (7) blows up if s/μ_i^2 approaches infinity, while the actual integral, Eq. (4), remains finite. This catastrophe does not occur in our results, however, as the only range of s/μ_i^2 that plays a role in the equation for the pole α is that immediately around m_α^2/μ_i^2. However, it follows from Eq. (23) that m_α^2/μ_i^2 is bounded from above by $N^{V,P}/(A_i^\alpha)^2$.

In order to make clear the behavior of our approximation in the extreme limit $\mu_i^2 \to 0$, we consider the ρ mass equation following from Eq. (23), i.e.:

$$\frac{1}{m_\rho^2}=\frac{1}{4N^V}\left(\frac{2}{3\mu_\pi^2}+\frac{1}{3\mu_K^2}\right).$$

If $\mu_\pi \to 0$, this equation leads to $m_\rho \to 0$, whereas the linear approximation $[m_\rho^2 = 4N^V(\frac{2}{3}\mu_\pi^2 + \frac{1}{3}\mu_K^2)]$ leads to $m_\rho^2 \to \frac{4}{3}N^V(\mu_K^2)$. Thus, the K meson plays no role in determining the ρ mass in this limit of our approximation. (If the ρ meson were coupled only the π, $\mu_\pi \to 0$ would necessarily lead to $m_\rho \to 0$, since only the ratio m_ρ/μ_π would be determinable.) In an exact model, the decoupling of the K meson would not be complete in the limit $\mu_\pi \to 0$. The present approximation is not unreasonable, though, as is obvious from the fact that most bootstrap treatments of the ρ meson neglect the K coupling completely, simply because the μ_K/μ_π mass ratio is so large.

Finally, we discuss the exact method used to determine the normalization constants $N^{V,P}$. The normalization is in terms of the V and P octet mass-squared vectors $\boldsymbol{\beta}^V$ and $\boldsymbol{\beta}^P$.[11] These vectors are defined by

$$\boldsymbol{\beta}^{V,P}=\sum_{i=1}^{8}\beta_i^{V,P}\mathbf{E}_i\,, \tag{24}$$

where $\beta_i^{V,P}$ is the square of the mass of the i member of the V or P octet and the $\mathbf{E}_i$ are defined to be orthogonal unit vectors in an eight-dimensional space. For each choice of original P masses, N^V is chosen so that the calculated $|\boldsymbol{\beta}^V|/|\boldsymbol{\beta}^P|$ ratio corresponds to experiment, and N^P is chosen so that the calculated value of $|\boldsymbol{\beta}^P|$ agrees with that assumed originally.

In order to determine the experimental value of $|\boldsymbol{\beta}^V|$, one must specify the combination of the experimental ω and φ particles that is to be identified with the isoscalar member of the V octet. The only strong argument for $\omega-\varphi$ mixing results from the assumption that the Gell-Mann–Okubo (GO) sum rule is accurate for the V octet.[14] However, the justification of this rule depends on the assumption that the linear approximation to mass splitting may be made. On the other hand, a nondegenerate solution to the dynamic equations of a bootstrap model can exist only if higher order terms in mass splitting are comparable to the linear terms for at least one multiplet. Therefore, we assume pure ω and φ states, in which case the φ must be identified with the isoscalar octet member, since it is coupled strongly to the $K\bar{K}$ state. (An SU_3 singlet V meson is not coupled strongly to PP states.) The ratio $|\boldsymbol{\beta}^V|/|\boldsymbol{\beta}^P|$ plays a nontrivial role in our model only when the masses of the PV states coupled to the P poles are computed.

[14] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

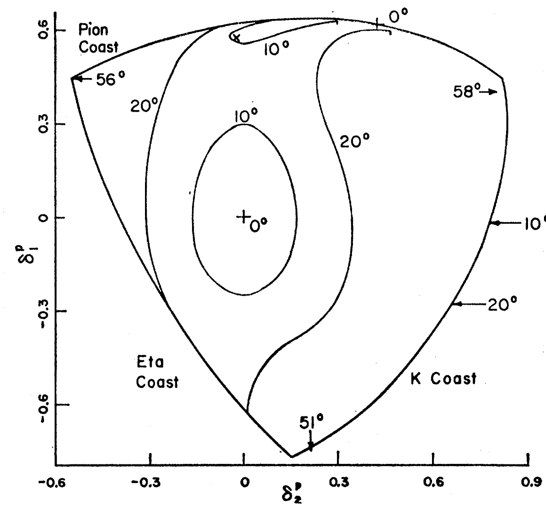

FIG. 1. Self-consistency angle λ for different P mass assumptions. The diagonal $\times$ is the physical point. The curves corresponding to 10° and 20° near the top of the figure follow close to the π and K coasts to the points marked on the K coast.

III. RESULTS

The self-consistency of a particular assumption concerning the P masses may be expressed conveniently in terms of the P octet mass-squared vector, introduced in Ref. 11 and defined in Eq. (24) above. Since the absolute magnitude of $\boldsymbol{\beta}^P$ is not computed, the angle λ between the $\boldsymbol{\beta}^P$ assumed originally and that calculated in the last stage of the calculation is a measure of the self-consistency. Exact consistency corresponds to $\lambda=0°$. Since isotopic spin conservation is assumed, there are only the three (π,K,η) independent P masses, i.e., $\boldsymbol{\beta}^P$ is confined to a three-dimensional subspace of the eight-dimensional octet mass-squared space. A convenient set of three orthogonal components for $\boldsymbol{\beta}^P$ is the following:

$$\begin{aligned}\beta_0^P &= (1/8)^{1/2}(3\beta_\pi+4\beta_K+\beta_\eta)\,,\\ \beta_1^P &= (1/5)^{1/2}(2\beta_K+\beta_\eta-3\beta_\pi)\,,\\ \beta_2^P &= (3/40)^{1/2}(\beta_\pi+3\beta_\eta-4\beta_K)\,,\end{aligned} \tag{25}$$

where $\beta_i = m_i^2$. Similar components may be defined for the V mesons. The components β_1^P and β_2^P measure the mass splittings that satisfy and violate the GO sum rule, respectively.[14] (The sum rule is the statement that $\beta_2^P=0$.) We define normalized mass-squared variables by the equation $\delta_i = \beta_i/|\boldsymbol{\beta}|$, so that $\delta_0^2+\delta_1^2+\delta_2^2=1$. The angle λ is then given by $\cos\lambda = \sum_{i=0,1,2}\delta_i^P\delta_i^{P'}$; we con-

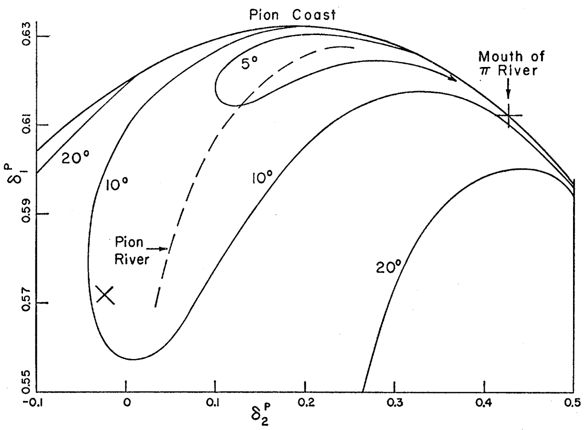

FIG. 2. Large-scale version of the region of near self-consistency near the boundary.

sistently use a prime to distinguish the P mass-squared vector calculated at the final stage of the procedure.

The self-consistency angle λ calculated for different assumed values of $\delta_1{}^P$ and $\delta_2{}^P$ is shown in Fig. 1. The boundary of the rounded triangular region in $\delta_1{}^P-\delta_2{}^P$ space corresponds to zero mass for one of the three P mesons; the points where the slope changes correspond to two zero P masses. The plot in Fig. 1 resembles a relief map of a triangular island, so we have used geographical terms. That part of the boundary corresponding to $\beta_\pi=0$ is called the pion coast, etc. Two $+$ marks show the positions of the two exactly self-consistent solutions. Curves corresponding to $\lambda=10°$ and 20°, and several peaks corresponding to large values of λ, are also shown.

The solution corresponding to $\delta_1{}^P=\delta_2{}^P=0$ is the degeneracy solution. The other exactly consistent solution occurs on the pion coast and corresponds to $\delta_1{}^P=0.612$, $\delta_2{}^P=0.43$. The fact that this solution occurs exactly on the boundary is a consequence of the crudeness of the calculation. In a more exact treatment this point would be expected to move either out or in. If the point moved out, there would be no nondegenerate solution. Hence, the calculation is not sufficiently accurate to indicate whether or not a nondegenerate solution really exists. However, the calculation does indicate that if such a solution exists, it is likely to occur in the region of low pion mass.

We assume that a self-consistent solution to a more accurate treatment of the $V=(PP)$, $P=(VP)$ model does exist, and that the present approximation is sufficiently accurate that the angle λ defined above should be small for the P mass values of this solution. The only region corresponding to $\lambda<10°$, other than that centered around the degeneracy point, is the thumblike region extending down from the $\beta_\pi=0$ boundary. We cannot estimate from our model how close to zero $\beta_\pi/|\boldsymbol{\beta}^P|$ should be, but we can estimate the range of values of β_η/β_K that corresponds to approximate self-consistency. An enlargement of this region is shown in Fig. 2. In this figure the "pion river" lies along the valley defined by the condition that λ possess a minimum as a function of β_η/β_K, for a fixed low value of $\beta_\pi/|\boldsymbol{\beta}^P|$.

The components of the experimental P mass-squared vector are $8^{1/2}\beta_\pi/|\boldsymbol{\beta}^P|=0.09$, $8^{1/2}\beta_K/|\boldsymbol{\beta}^P|=1.20$, $8^{1/2}\beta_\eta/|\boldsymbol{\beta}^P|=1.49$, or, equivalently, $\delta_1{}^P=0.572$, $\delta_2{}^P=-0.023$. This point is indicated with a diagonal $\times$ in Figs. 1 and 2; it lies close to the pion river. Explicitly, the minimum value of λ for $8^{1/2}\beta_\eta/|\boldsymbol{\beta}^P|=0.09$ occurs at $\beta_\eta/\beta_K=1.45$ (or $\delta_2{}^P=0.046$). The physical ratio of β_η/β_K is 1.24. Figure 2 illustrates the possibility that the accuracy of the GO sum rule for the P octet may be to some extent an accident. Nonlinear terms in the mass splitting are essential in our model, yet most of the pion river lies fairly close to the $\delta_2{}^P=0$ axis.

The choice of the experimental P masses for $\boldsymbol{\beta}^P$ leads to the following computed parameters: $\delta_1{}^V=0.53$, $\delta_2{}^V=0.13$, $\delta_1{}^{P'}=0.46$, $\delta_2{}^{P'}=0.08$, and $\lambda=9.6°$. The experimental V mass deviations, (based on ρ, M, and φ masses of 750, 888, and 1019 MeV), are $\delta_1{}^V=0.195$, $\delta_2{}^V/\delta_1{}^V=0.35$. It is seen that the calculated V mass splitting is much too large, but the calculated ratio $\delta_2{}^V/\delta_1{}^V$, which measures the deviation from the GO sum rule, is of the right sign and approximately the right magnitude. A possible explanation for the small experimental V mass splitting is given in Sec. IVA.

The manner in which the V and P mass splitting depend on the $\delta_1{}^P$ assumed initially (for $\delta_2{}^P=0$) is shown in Fig. 3. The salient features of these curves may be understood if the basic equation of the model is expanded in powers of deviations from degeneracy. If such an expansion of Eq. (23) and the corresponding P-pole equation is made, the results may be expressed in terms of the $\delta_1-\delta_2$ components, i.e.,

$$\delta_1{}^V=\tfrac{1}{2}\delta_1{}^P+O_h\,, \tag{26a}$$

$$\delta_2{}^V=-\tfrac{1}{3}\delta_2{}^P+O_h'\,, \tag{26b}$$

$$\delta_1{}^{P'}=\tfrac{1}{2}(R_P\delta_1{}^P+R_V\delta_1{}^V)+O_h''\,, \tag{26c}$$

$$\delta_2{}^{P'}=-\tfrac{1}{3}(R_P\delta_2{}^P+R_V\delta_2{}^V)+O_h'''\,, \tag{26d}$$

$$R_P=|\boldsymbol{\beta}^P|^{1/2}/(|\boldsymbol{\beta}^P|^{1/2}+|\boldsymbol{\beta}^V|^{1/2}),\quad R_V=1-R_P\,.$$

The symbols O_h, O_h', etc., represent all terms of order greater than the first in the δ. These higher order terms are such as to lead to positive $\delta_2{}^V$ and $\delta_2{}^{P'}$. The calculated ratio $\delta_2{}^V/\delta_2{}^{P'}$ is greater than one because the linear term $-\frac{1}{3}R_V\delta_2{}^V$ in Eq. (26d) partially cancels the higher order term in this equation. This conclusion does not depend on the fact that $\delta_2{}^P$ has been set equal to zero in Fig. 3. A calculation shows that if $\delta_1{}^P=0.57$, the choice $\delta_2{}^P=0.06$ leads to $\delta_2{}^{P'}=\delta_2{}^P$ and to a $\delta_2{}^V/\delta_2{}^{P'}$ ratio of 1.6.

IV. POSSIBLE MODIFICATIONS OF THE MODEL

The approximation of Secs. II and III does not take into account the fact that the P mesons are much more tightly bound systems than the V mesons, except when the masses of the PV states are computed. In this section we discuss some simple modifications that take the $V-P$ mass difference into fuller account. We do not discuss the origin of the $V-P$ mass difference, but assume it results primarily from a mechanism similar to that discussed previously by the author, i.e., the fact that crossing matrix elements tend to be greater for forces in states of low statistical weight, and hence low spin.[15] We wish to make only one point related to V resonances appreciably above threshold (Sec. IVA) and one point that applies to the very tightly bound P mesons (Sec. IVB).

A. Resonances Appreciably Above Threshold

If the mass of a resonating particle is appreciably greater than the threshold of a particular channel, Eq. (7) is not a good approximation for the dispersion integral associated with that channel. In fact, if m_α^2 is sufficiently greater than $4\mu_i^2$ that the region $(s'<s)$ is important in the integral, $\mathrm{Re}H_i^\alpha(m_\alpha^2)$ may be an increasing function of μ_i^2. {Since Eq. (7) must apply to bound states as well as resonances, the constants in this equation must be chosen so that $\partial[\mathrm{Re}H_i^\alpha(s)]/\partial\mu_i^2$ is negative.}

In order to study this "high-resonance" effect, we modify assumption (6) of Sec. IIA. We substitute the phase space factor $\rho_i^V=s^{-1/2}$ into Eq. (4) and evaluate the integral explicitly, assuming that $s_\alpha<0$. The result is

$$8\pi\,\mathrm{Re}H_i^\alpha(s)=1-\frac{J_y-1}{y}+\frac{2(x-1)(J_y-J_x)}{x+y},$$

where

$$\begin{aligned}&x=\tfrac{1}{4}s/\mu_i^2,\quad y=-\tfrac{1}{4}s_\alpha/\mu_i^2,\\&J_y=[(y+1)/y]^{1/2}\ln[(1+y)^{1/2}+y^{1/2}],\\&J_x=[(1-x)/x]^{1/2}\arctan[x/(1-x)]^{1/2},\quad 0<x<1,\\&J_x=[(x-1)/x]^{1/2}\ln[(x-1)^{1/2}+x^{1/2}],\quad x\geqq 1.\end{aligned}\tag{27}$$

For simplicity, we assume that s_α is large and negative, so that $\ln y\gg 1$ and $\ln y\gg\ln x$. In this case the leading terms of $\mathrm{Re}H_i^\alpha$ and $\partial(\mathrm{Re}H_i^\alpha)/\partial_x$ are

$$8\pi\,\mathrm{Re}H_i^\alpha=1+\ln(4y)(x-\tfrac{3}{2})y^{-1},\tag{28}$$

$$8\pi(x+y)\partial(\mathrm{Re}H_i^\alpha)/\partial_x=\ln(4y)+x^{-1}-1-J_x(2+x^{-1}).\tag{29}$$

If we assume that the V pole equations are a part of a complete bootstrap model of the V and P mesons, we may apply the coupling-constant self-consistency condition of Eq. (20a). (This condition does not depend on the assumed form of H_i^α.) It is convenient to express this condition on s_α in terms of the two parameters m_0^2 and s_0, rather than g_V and Z^V. The m_0^2 and s_0 are the values of m_α^2 and s_α that occur in the degeneracy solution corresponding to g_V and Z^V. Combination of Eqs. (20a), (18), (27), and (29) leads to the equation for s_α,

$$\begin{aligned}\ln(s_\alpha/s_0)&=[\textstyle\sum_i(A_i^\alpha)^2h(\tfrac{1}{4}s_\alpha/\mu_i^2)]-h(\tfrac{1}{4}m_0^2),\\h(x)&=J_x(2+x^{-1})-x^{-1},\end{aligned}\tag{30}$$

where we have normalized the P masses so that $\mu_i^2=1$ in the degeneracy solution.

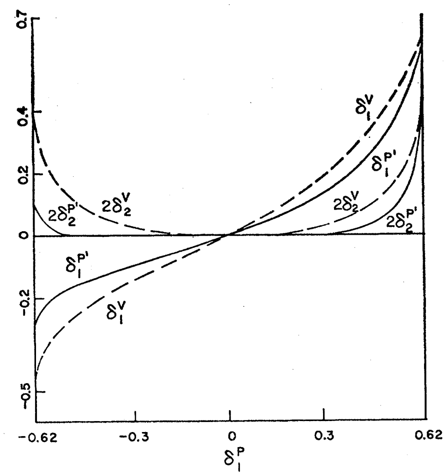

FIG. 3. Dependence of calculated mass splittings on δ_1^P, for $\delta_2^P=0$.

The pole-position equation, Eq. (16), may also be expressed in terms of s_0 and m_0^2. If use is made of Eq. (28), the pole-position equation leads to the relation,

$$m_\alpha^2-6\textstyle\sum_i(A_i^\alpha)^2\mu_i^2=(s_\alpha/s_0)(m_0^2-6).\tag{31}$$

If (s_α/s_0) is eliminated from Eqs. (30) and (31), the basic equation of the modified model for the V masses results.

We first consider the simple case $m_0^2=6$, in which case

$$m_\alpha^2=6\textstyle\sum_i(A_i^\alpha)^2\mu_i^2.\tag{32}$$

If the P masses are chosen to be the experimental values, this choice leads to the ratio of lengths of the V and P mass-squared vectors of 4.95, rather than the experimental value of 3.69. On the other hand, agreement with experiment of the calculated V mass ratios is improved; the calculation leads to the results, $m_\rho^2/m_M^2=0.50$, $m_\varphi^2/m_M^2=1.30$, or, alternately, $\delta_1^V=0.315$, $\delta_2^V=0.045$.

An alternate procedure is to choose the parameter m_0^2 in Eq. (31) equal to 5.37; this value leads to agreement with experiment of the calculated $|\beta^V|/|\beta^P|$, if the experimental P mass values are used. The results of this calculation are very close to those of Sec. III, i.e., $\delta_1^V=0.53$, $\delta_2^V=0.11$.

The reason for the decrease of the V mass splitting that results from Eq. (32) is that the ρ and M masses are sufficiently higher than the $\pi\pi$ and πK thresholds that the values of the dispersion integrals associated with these channels are much smaller comparatively

[15] R. H. Capps, Nuovo Cimento (to be published). See also G. F. Chew, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), pp. 525–529.

than in the approximation of Sec. II. This is the effect discussed at the beginning of the present section. In the present approximation this effect is important only for an assumed value of $|\beta^V|/|\beta^P|$ larger than the experimental value. However, if we had chosen $|s_0|$ to be comparable to m_0^2, rather than very large, the low s' regions of the dispersion integrals would be more accentuated, and this effect of a decreased V splitting would be present for a smaller value of $|\beta^V|/|\beta^P|$.

If Eq. (32) for the V poles is combined with the approximation of Sec. II for the P poles, there is no region except the central region of the $\delta_1{}^P-\delta_2{}^P$ plot that corresponds to near self-consistency. In this combined approximation the initial assumption of experimental P masses leads to a calculated λ of 17.9°.

B. Tightly Bound Systems

We may not carry out an approximation for the P poles similar to that used above for the V poles, since the appropriate choice of $\rho_i{}^P$ in Eq. (4) would lead to a divergent integral. In this subsection, we wish only to illustrate the fact that if the approximation of Sec. II is modified, the results for tightly bound systems are very sensitive to the procedure used in writing the forces (elements of the N matrix).

We consider, for simplicity, a meson A that is a bound state of two identical mesons B. The spins of A and B need not be specified. The condition for a bound state at $m_A{}^2$ is of the form $H(m_A{}^2,m_B{}^2,t)=\text{constant}$, where H is a dispersion integral and t is a variable of dimension $(\text{mass})^2$ associated with the force. (In our model t could be the position of the force pole.) We assume that the A and B are members of larger multiplets, and consider first-order deviations Δ_i from the degeneracy solution values. Our notation is $A=A_0+\Delta_A$, etc., where A is shorthand for $m_A{}^2$, and the subscript 0 refers to the degeneracy solution. The first-order equation for Δ_A is,

$$\Delta_A=-(H_B/H_A)\Delta_B-(H_t/H_A)t, \qquad (33)$$

where $H_i=\partial H/\partial i$. Dilatational invariance (the fact that H is dimensionless) implies the relation,

$$H_AA_0+H_BB_0+H_tt_0=0. \qquad (34)$$

In the probability matrix approximation of Refs. 5 and 6, the H_t terms are neglected in both Eqs. (33) and (34). The result is simple, i.e.,

$$(\Delta_A/A_0)=(\Delta_B/B_0). \qquad (35)$$

An alternative procedure that might be used is to neglect the H_t term in Eq. (33) but not in Eq. (34), and to calculate H_B/H_A from an assumed definite form of H. If $A_0>|t_0|$, as is often the case for a loosely bound or unbound resonance, the H_tt_0 term of Eq. (34) is likely to be small, so that the results of this procedure are similar to those of the probability matrix approximation. On the other hand, if the system is tightly bound so that $A_0<|t_0|$, the value of Δ_A calculated in this way is likely to be very sensitive to the assumed form of H. In the present paper, the coupling-constant consistency condition was used to eliminate that t_0 dependence of the integrals from the problem; the linear approximation to the model is thus similar to Eq. (35). We feel that this consistency condition must be present in any reasonable model. The method used here is admittedly crude, however. We hope that in the near future more accurate dispersion methods for tightly bound systems will be developed.

V. CONCLUDING REMARKS

The results of Sec. III show that if one assumes that the mass splitting of a resonating multiplet depends primarily on the mass splitting of the multiplets of the coupled two-particle states, the most favored type of splitting in the simple P and V octet bootstrap model is primarily of the Gell-Mann–Okubo type, with the π and ρ being the lightest P and V mesons. The basic reason for this result is the strong $\rho\pi\pi$ coupling constant that results from SU_3 symmetry. Essentially, that type of mass splitting is most favored in our model which leads to the strongest mutual coupling of the lightest members of the various multiplets.

We now extend this SU_3 model to include four multiplets, the P and V octets, a $j=\frac{1}{2}$ baryon octet (B), and a $j^P=(\frac{3}{2})^+$ baryon decuplet (B^*). It is assumed that the V, B, and B^* are coupled to PP, PB, and PB states, respectively, and that the P are coupled to a linear combination of PV and $B\bar{B}$. It is assumed that the PBB interaction angle θ is equal to 33°, as given by consideration of the $B-B^*$ reciprocal bootstrap model.[7] The terms of first order in the mass splitting for this model are discussed in Ref. 6.

If the π is taken to be the lightest P meson in this four-multiplet model, and the V, B, and B^* mass splittings are assumed to result from the P mass splitting, calculations to first order show that the lightest members of the other multiplets are the isotriplet V, an isodoublet B (which may be identified with the nucleon), and the isoquadruplet B^*.[6,7] The fact that this corresponds with experiment is not a compelling reason to believe that the P splitting causes the other splittings, as one could make a similar argument starting with a light ρ meson or a light nucleon. The significant point is that the mutual couplings among the lightest members of the four multiplets are large. The probabilities (calculated from exact symmetry) of the $\pi\pi$, πN, and πN states in the wave functions of the ρ, N, and N^*, respectively, are $\frac{2}{3}$, 0.69, and $\frac{1}{2}$. The probabilities of the $\pi\rho$ and $N\bar{N}$ in the PV and $B\bar{B}$ parts of the π wave function are $\frac{2}{3}$ and 0.46. Since the lightest particles are particularly important in dispersion theoretic models, it may be said that the pion became the sore thumb of SU_3 because it does so much of the work in pulling up on the bootstrap. In fact, it would be fairer to say that the pion is not really sore, only exceptionally sturdy.

For some Lie group schemes of particles, there are no types of mass splittings that allow such a strong mutual coupling among the lightest members of the multiplets. In order to illustrate this point, we consider a scheme based on SU_2, in which the P and V mesons are π and ρ triplets. The ρ is a $\pi\pi$ resonance, and the π is a $\pi\rho$ bound state. The particle-antiparticle pairs $\pi^\pm$ and $\rho^\pm$ are each degenerate, but the neutral and charged particles are assumed nondegenerate. Appropriate components of the π mass-squared vector are,

$$\beta_0{}^\pi=(\tfrac{1}{3})^{1/2}(2\beta_{\pi^+}+\beta_{\pi^0})\,,$$
$$\beta_1{}^\pi=(\tfrac{2}{3})^{1/2}(\beta_{\pi^+}-\beta_{\pi^0})\,.$$

The fractional mass-squared parameters are defined by the equation $\delta_i{}^\pi=\beta_i{}^\pi/[(\beta_0{}^\pi)^2+(\beta_1{}^\pi)^2]^{1/2}$. Similar equations may be written for the ρ mesons.

One may follow the procedure of Secs. II and III for this scheme, i.e., assume a value for $\delta_1{}^\pi$, use Eq. (23) to calculate $\delta_1{}^\rho$, and then use the corresponding P-pole equation to calculate $\delta_1{}^{\pi\prime}$. The self-consistency angle is given by $\cos\lambda=\sum_{i=0,1}\delta_i{}^\pi\delta_i{}^{\pi\prime}$. The value of this angle as a function of $\delta_1{}^\pi$ is shown in Fig. 4. In this case there is no extended region of small λ other than that around the degeneracy point $\delta_1{}^\pi=0$. A nondegenerate solution exists at the boundary point $\delta_1{}^\pi=-(\tfrac{2}{3})^{1/2}(\beta_{\pi^+}=0)$, but the region $\lambda<10°$ extends only a small distance inside the boundary.

The basic reason that self-consistency for nondegenerate multiplets is difficult to achieve in this scheme is that the mutual coupling among the lightest π and ρ is not particularly strong. The model favors light $\pi^\pm$ and ρ^0 particles; the charged π is the "thumb" of the model. The probability of the $\pi^\pm$ in the degeneracy-solution wave function $\psi(\pi^+)$ is only $\frac{1}{2}$, while the probability of the $\pi^\pm$ in $\psi(\pi^0)$ is one. The situation is not improved if one extends the model to include a nucleon doublet. Any nucleon splitting, being odd under charge reflection, is not coupled to the π and ρ splittings. Therefore, the $N\bar{N}$ part of the π wave function cannot help in maintaining the π splitting.

Similar considerations have not been applied to a large class of possible Lie-group schemes. However, the linear terms in mass splittings in $V-P$ models for a large number of schemes are examined in Ref. 5; it is shown there that for many schemes no type of mass splitting is as highly favored as the GO type of the SU_3 double-octet scheme. It is also shown in Ref. 5 that in the linear approximation to the double-octet scheme, isospin-violating deviations are less favored than the GO deviation. It is worth pointing out here that isospin-violating and isospin-preserving deviations are not coupled in models similar to the present one, even

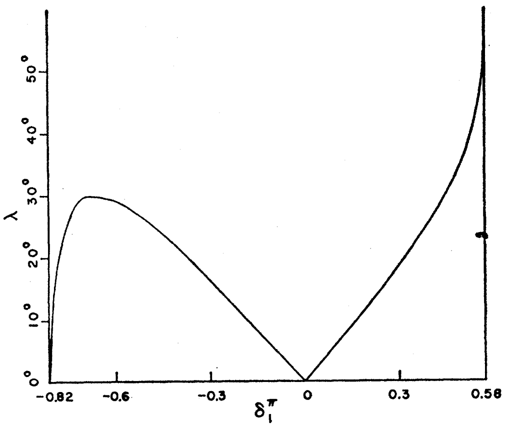

FIG. 4. Self-consistency angle as a function of π mass splitting for the ρ–π model.

through terms of higher order than the first order in the deviations are included.

The above comparison of different schemes illustrates the fact that in the long run the large experimental P mass splitting may be a blessing to the bootstrap hypothesis. One expects that in an accurate calculation within a particular group-representation scheme, the over-all magnitude of the mass splitting of a nondegenerate solution should be smaller if the splitting is highly favored, i.e., if the mutual coupling of the light members is large. As we have shown, the observed type of splitting is highly favored. Yet experimentally, the observed P mass splitting is almost as large as is physically possible; the pion mass is barely positive. It seems likely that if a dispersion method is developed that explains accurately the observed mass splitting of the P octet, the same method when applied to certain other Lie-group schemes, may not yield any nondegenerate solutions at all. In other words, the pressure of the bootstrap in some Lie-group particle schemes may break the thumb completely. We close with the speculation that the sturdy thumb of the SU_3 bootstrap model may be one of the crucial reasons that SU_3 is realized in nature. Of course, the significance of this type of argument would be increased if a simple reason were found for discarding solutions to bootstrap models that involve completely degenerate multiplets.

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