Theory of Finite Nuclei

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A shell-model potential is defined for calculations on the structure of finite nuclei by the reaction-matrix theory. The rearrangement potential is part of this shell-model potential. The contribution to the rearrangement energy from the short- and long-range parts of the nucleon-nucleon potential is calculated in infinite nuclear matter. The third-order rearrangement term is shown to be predominantly due to the strong short-range part, while the second-order term comes mainly from the long-range part. The dependence of the reaction matrix on the structure of the nuclear medium in which the nucleons interact is investigated. The results from the calculations of rearrangement energies is then of help. Part of the short-range repulsion is proportional to the reaction-matrix potential energies of the interacting nucleons. Another part is a function of the "local" density. The long-range part of the reaction matrix varies somewhat less with the density of an infinite system than does the short-range part. It seems, however, that it would be important to include its dependence on the medium, especially for studying small nuclei or the structure of the surface.

I. INTRODUCTION

THE Brueckner theory of nuclear matter has been reasonably successful when applied to infinite nuclear matter, as is reported in several publications.[1–3] There have been made both exact calculations (within the Brueckner theory) and approximate ones. The value of the approximation methods lies, apart from the simplicity of the calculations, in the improved understanding of the physics that they give.

Still, the problem of the infinite medium cannot be regarded as finished. It is to be expected, for one thing—and it has been shown explicitly[4]—that different nucleon forces, all of which do describe free scattering, do not necessarily give the same binding or saturation for infinite nuclear matter. Therefore the final calculation has to await an accurate experimental determination of nucleon forces and a satisfactory theory of these forces. It has further been found that the off-energy-shell effect is more important than perhaps first expected.[3] It has been demonstrated that certain higher order terms in the reaction matrix expansion that earlier were found to be negligible,[5] partly due to the off-energy-shell effect, are not so.[6,7] Fortunately there seem to be large cancellations among these higher order terms,[7] but this is a problem which still has to be clarified.

Of special importance in nuclear matter calculations is the short-range repulsion which is usually represented by a hard core. This repulsion also enters in a seemingly important way in the off-energy-shell effect. It still remains to investigate the importance for nuclear-matter calculations of replacing the mathematical hard core by a more physically plausible soft core. Some preliminary investigations of this have been made.[8]

The above are some of the problems still remaining in the infinite-nuclear-matter problem. We do, however, also want a theory that describes other properties of the many-body system, such as, e.g., appear in the semi-empirical Bethe-Weiszäcker mass formula. Especially, we would like to be able to calculate the surface energy, from assumed known nucleon forces, and connected with this the surface thickness, suitably defined. For such a calculation we may treat a semi-infinite nucleon system, by which we would understand the surface of a very large nucleus (neglecting Coulomb forces). However, from the structure of the reaction matrix from which the surface energy is to be calculated, one must be led to believe that the surface energy is not a constant quantity, but would rather depend on the specific nucleus, e.g., on its size.

We would also like the theory to reproduce binding energies, density distributions, and separation energies of single particles for specified finite nuclei.

It is certainly true that before trying to deal with these problems we should first fully understand the simpler problem of infinite nuclear matter. Some of the remaining dark points of this were mentioned above. However, even the simplest reaction matrix theory, ignoring higher order terms and off-energy-shell effects, contains many features that undoubtedly are physically important, and we would like to investigate the relation of these features to the physics of a finite nucleus.

In the infinite system, the model wave functions are (by assumption) plane waves, and the state of the infinite system is defined by only one parameter, e.g., the Fermi momentum or the density. In a finite system the theory also has to provide the spatial wave functions, and the number of parameters required to define the state of the medium is in principle infinite. This is the essential difference between the finite and the infinite problem. Once the model wave functions are

[1] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).
[2] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960).
[3] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).
[4] K. A. Brueckner and K. S. Masterson, Jr., Phys. Rev. **128**, 2267 (1962).
[5] H. S. Köhler, Ann. Phys. (N. Y.) **12**, 444 (1961).
[6] R. Rajaraman, Phys. Rev. **129**, 265 (1963).
[7] H. A. Bethe (to be published).
[8] H. S. Köhler and Y. Waghmare (to be published).

given, the procedure of calculating the reaction matrix, etc., is formally identical in the two cases.

However, the calculation of the reaction matrix is quite time consuming even for an infinite medium, and would be far more so in a finite medium. Thus, we also wish to have an approximate way of calculating the reaction matrix in a finite medium and (we might hope) to use results from the calculations in the infinite medium. Brueckner and co-workers used such a procedure and derived a density-dependent reaction matrix[9,10] and applied this method to some finite nuclei.[11,12] However, their approximation of the reaction matrix has no deeper justification—as they also pointed out. One may hope that by a better approximation one would be able to improve the agreement with experimental binding energies, etc., of the finite nuclei, which was only moderately good.

In this paper we shall study the definition of a shell-model potential and an approximation to the reaction matrix to use in finite nuclear calculations. To this end, we also calculate rearrangement energies, which also contribute to the shell-model energies.

II. A SHELL-MODEL POTENTIAL

We assume that the wave functions are solutions of a Schrödinger equation with some potential that we shall refer to as the shell-model potential. It is the object of the theory to provide this shell-model potential.

In the well-known Hartree-Fock theory a shell-model potential V_{HF} is usually derived by a variational method. We briefly review some steps of this derivation. With an interaction v given, the single-particle energy is

$$E_i=(\varphi_i|p^2/2M|\varphi_i)+\sum_j(\varphi_i\varphi_j|v|\varphi_i\varphi_j)+\text{exchange}, \quad (1)$$

and the total energy by

$$E=\sum_i(\varphi_i|p^2/2M|\varphi_i)+\tfrac{1}{2}\sum_{ij}(\varphi_i\varphi_j|v|\varphi_i\varphi_j)+\text{exchange}, \quad (2)$$

where p is the momentum and M the mass of the particle. The minimization of E by varying the wave functions φ_i while preserving the normalization,

$$(\varphi_i|\varphi_i)=1, \quad (3)$$

gives

$$(p^2/2M+V_{\mathrm{HF}})\varphi_i=\lambda_i\varphi_i, \quad (4)$$

where the Hartree-Fock potential V_{HF} is

$$V_{\mathrm{HF}}=\sum_j(\varphi_j|v\varphi_j)+\text{exchange}, \quad (5)$$

and where the eigenvalues λ_i are the Lagrangian multipliers to take care of the side condition (3). For future reference, we point out that there is no *a priori* reason to give any physical meaning to the eigenvalues λ. If, however, we operate with φ_i from the left in (4) we obtain with (5)

$$\lambda_i=(\varphi_i|p^2/2M|\varphi_i)+\sum_j(\varphi_i\varphi_j|v|\varphi_i\varphi_j)+\text{exchange}. \quad (4a)$$

Thus, we now find from (1) that $\lambda_i\equiv E_i$ and λ_i is thus now found indeed to have a physical meaning. That this is, however, an *a posteriori* finding is of value for the understanding of our more general shell-model potential now to be derived. (This point will be better clarified by a practical application in a paper soon to be published.)

The derivation of a shell-model potential in the theory of a finite nucleus is a very important point. It is necessary to realize that the many-body theory we are dealing with is approximate. The shell-model potential we shall derive thus enters into an approximate theory and will itself be approximate. The aim of any approximation is naturally to reduce as much as possible the error in the quantity one wants to calculate. However, it is equally natural that one can only estimate the goodness of an approximation. We further want to emphasize that the development of a theory should be guided by internal consistency and formal elegance or simplicity. We would also like to have quantities which are physically meaningful whenever possible.

We mention this because a shell-model potential is not prescribed by the reaction matrix theory, and there is in fact no unique way of defining it; it will depend rather on the approach we choose.

We now define the shell-model potential by a variational procedure, as in the Hartree-Fock theory, minimizing the energy of the system in question. One must realize that this is not an exact procedure, since the energy we minimize is approximate. This minimization should thus be distinguished from the variation of the wave functions of the exact Hamiltonian which by a well-known theorem does give the exact wave function when the energy is minimized.

In principle, of course, one has to be careful in picking the expression for the approximate energy to minimize. The choice often depends on what we wish to calculate. We usually require a good approximation not only for the energy, but also for the change of energy with the variation of the wave functions, and thus for the wave functions at the energy minimum.

For a very large system (infinite nuclear matter), the only function to vary is, as previously mentioned, the density or Fermi momentum, i.e., the wave function at the top of the Fermi sea. In fact the accepted procedure for calculating the saturation density is to calculate the energy as a function of density and search for the minimum. Thus our definition of the shell-model wave functions agrees with that for infinite nuclear matter in the limit of a vary large nucleus. The agreement is brought out more explicitly if, instead of searching for the energy minimum as is usually done, we calculate the energy of a real hole as a function of momentum at some

[9] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).
[10] K. A. Brueckner, and D. T. Goldman, Phys. Rev. **116**, 424 (1959).
[11] K. A. Brueckner, A. M. Lockett, and M. Rotenberg, Phys. Rev. **121**, 255 (1961).
[12] K. S. Masterson, Jr., and A. M. Lockett, Phys. Rev. **129**, 776 (1963).

trial density. Then we choose the next density by taking as the Fermi momentum the momentum at which the energy of a real hole agrees with the total energy. We then iterate until the density is stationary. This procedure is equivalent to the conventional one by the well-known theorem on the equality between total energy and the energy of a nucleon at the Fermi momentum, at saturation density.

In the theory of infinite nuclear matter, the energy is given in Brueckner theory by the K-matrix approximation

$$E=\sum_i(\varphi_i|\,p^2/2M\,|\,\varphi_i)+\tfrac{1}{2}\sum_{ij}(\varphi_i\varphi_j|K|\,\varphi_i\varphi_j) \\ +\text{exchange}, \quad (6)$$

with the reaction matrix K defined as usual by

$$K=v+vGK, \quad (7)$$

where v is the interaction and G the propagation in nuclear matter. As has been described, the K matrix is designed to sum certain terms or graphs up to third order in a K-matrix expansion of the energy by suitable definition of the propagator.[2,3,6] It has to some extent been shown that the corrections to (6) are small at the saturation density. (In a recent publication by Bethe,[7] it is, however, estimated that higher order K-matrix graphs are very important although important cancellations occur.) It is perhaps not so clear that the change of energy with density is well approximated from the expansion (6). So although the energy may be well approximated by (6) we can perhaps not expect the same good agreement for the saturation density.

But this is a well-known dilemma of quantum mechanics as applied, for example, in atomic physics. Wave functions for a system of electrons that approximate the energy well, often do not give the same good agreement when one calculates other quantities like transition probabilities. The only sensible procedure is to include more terms in the approximation to the Hamiltonian in order to improve on the wave functions.

By the same token, our first aim in the calculation on nuclei would be to calculate binding energies. Densities, radii, or other properties of the wave functions may however be changed by smaller changes in the approximate Hamiltonian. Mathematically, this is clearly because of the minimization. The energy changes only relatively slowly around the minimum as we change the wave functions.

Although the K-matrix approximation (6) for the total energy is argued to be a good approximation for calculations on infinite nuclear matter, it is not necessarily true that this approximation is good for the treatment of a finite system. It is a matter of numerical calculation to investigate whether any higher order K-matrix terms should be included, just as one has previously investigated the importance of, e.g., hole-particle, hole-hole, and third or higher order terms for infinite nuclear matter.

To emphasize the difference between the finite and the infinite medium we point out that in the infinite case we are concerned only with varying the wave functions at the top of the Fermi sea. In a finite nucleus, however, the lower lying states also come into play in the variational procedure.[13] Thus it can be, at least in principle, that terms in the K-matrix expansion of the energy which are unimportant for the calculation on an infinite system are not so for a finite system, since the surface of a finite system is also built up of lower lying states. That this may be true in practical calculations is made plausible by the important state dependence or non-locality of the K matrix.

When carrying through this scheme in practice, it may of course happen that we find terms in the K-matrix expansion of the energy which contribute insignificantly to the energy but significantly to the shell-model potential. But this can again only be investigated by detailed numerical work.

In our work to be presented now, we shall, as a natural first step, assume that for a finite nucleus also, the K-matrix approximation (6) is good.

In order to be able to perform the variational procedure in practice, we should now investigate the dependence of the reaction matrix on the wave functions. In fact, we have to do more. As pointed out in the Introduction, the numerical solution of (7) is already a complicated task for an infinite system, and the solution for a finite system does not (at present) seem, in general, possible. Therefore we are also compelled to solve (7) approximately for wave functions of a finite system, and in the following sections we shall deal with this problem.

III. PREVIOUS INVESTIGATIONS ON THE MEDIUM DEPENDENCE OF THE K MATRIX

The reaction matrix is defined by (7). Taking matrix elements between some set of states of the unperturbed medium, one gets

$$K_{kl,ij}=v_{kl,ij}+\sum_{mn}v_{kl,mn}G_{mn,ij}K_{mn,ij}, \quad (8)$$

and

$$G_{mn,ij}=(E_i^*+E_j^*-E_m^*-E_n^*)^{-1}, \quad (9a)$$

if m, n are unoccupied states, and

$$G_{mn,ij}=0, \quad (9b)$$

for m, n occupied. The virtual energies E_i^* are determined by

$$E_i^*=p_i^2/2M+\sum_j(K_{ij,ij}-K_{ij,ji}), \quad (10)$$

for hole states, while the energies for particle states m, n are determined similarly to (10) from a matrix $K^{(1)}$ calculated with a propagator off the energy shell, as has been explained elsewhere.[1,3]

The dependence of the K matrix on the density of an infinite system was investigated by Brueckner, Gammel, and Weitzner,[9] and we briefly review their procedure.

[13] H. S. Köhler, Bull. Ann. Phys. Soc. **9**, 504 (1964).

They concluded that the density variation is important only for angular momentum $l=0$ states. Further, they investigated separately the density dependence of the core part and of the attractive part. The core part is proportional to the slope of the wave function at the core edge. Thus they put

$$K=K_{\text{core}}+K_a. \tag{11}$$

It was concluded that K_a depends so little on density that this dependence could be ignored. Next they assumed a parametrical dependence on density for the core part and fitted the parameters to their calculations of the slope of the wave function at the core edge. This density-dependent $K_{\text{core}}(\rho)$ together with the attractive part K_a (calculated at the saturation density of infinite nuclear matter) has been used in extensive calculations on finite nuclei.[11,12] The density then put into $K_{\text{core}}(\rho)$ was the density at the center of mass of the two interacting nucleons, and therefore this approximation is referred to as the local density approximation. However, a closer inspection shows that core repulsion depends in a more complicated way on the structure of the medium in which the nucleons interact, and it may not be permissible to treat K_a as independent of the medium either. As the correlation between the nucleons, especially the correlation due to the core, is so important for the saturation of nuclear matter, we have to assume that an exact treatment of this correlation is also necessary for the structure of a finite nucleus.

In a previous publication,[14] we did in fact show that an important part (part of the dispersion term of the separation method) stems from a density dependence of the short-ranged part of the K matrix that is mainly proportional to the sum of potential energies of the two interacting nucleons. The potential energies of two nucleons in a finite nucleus are, however, not related to the "local" density. In fact, the density averaged over the whole of the nucleus would be more relevant. As has been pointed out,[14] the center-of-mass coordinate on the other hand feels mainly the center of the nucleus where the density is high. As the repulsion due to the core increases with density, we expect the local-density approximation to overestimate this repulsion and give too little binding. In an infinite medium, the part of the dispersion term we discuss contributes about +6 MeV to the energy at normal density. We may thus expect that an erroneous treatment of this term can underestimate the binding of a finite nucleus by some MeV. Further, the shell-model potential derived from the K matrix would be in error. Only by actual solution for the wave functions, however, can the effect of this be estimated numerically.

Brueckner and co-workers do consider a separation of the K matrix into the core and attractive parts, as in Eq. (11). The dispersion term from the separation method results from a short-range part that includes some of the attractive part, so there is no direct connection between the K_{core} of (11) and the dispersion term. However, we have previously[14] shown that our dispersion term is numerically comparable with the K_{core} of Brueckner *et al.* This is because the wave function for relative motion in nuclear matter is quite insensitive to changes in density, except right at the core edge where the slope of the wave function enters in an important way into the calculation of the K matrix. Actually, this circumstance is responsible for the success of the separation-method approximation as developed in Ref. 15, where in first approximation the wave function at small distances was approximated by the wave function for free scattering.

However, the other terms in the separation-method approximation are also density-dependent, and K_a of Eq. (11) is not completely independent of density either. Even a few percent error in the potential energy can mean an error of MeV's in the total energy. In the separation method, there appear terms connected with a long-range correlation which is appreciable in the $^3S_1$ state[9] due to the tensor force, the long-range part of which contributes appreciably in second-order Born approximation.[16] It may not be sufficient to treat only the short-range part of the K matrix as depending on the medium. We shall try to throw some light on this problem in the following and shall then use mostly the separation-method approximation.[2,15]

IV. THE SEPARATION METHOD

The separation-method approximation has been found to be semiquantitative,[4] and is physically illustrative. It is acceptable for the studies we are now going to make. This method gives[15]

$$K=v_l+K_s{}^F+(\Omega_s{}^F-1)(e_0-e)(\Omega_s{}^F-1)+2v_lQ(\Omega_s{}^F-1)+v_l(Q/e)v_l+(\Omega_s{}^F-1)e(Q-1)(\Omega_s{}^F-1), \tag{12}$$

where v_l and v_s are the long- and short-ranged parts of the interaction, respectively, and Q is the Pauli operator.

The separation should be chosen so that the diagonal element of $K_s{}^F$ defined by

$$\Omega_s{}^F=1+(1/e_0)K_s{}^F, \tag{12a}$$

$$K_s{}^F=v_s\Omega_s{}^F, \tag{12b}$$

is small or zero. The propagator $1/e_0$ contains only kinetic energies. We refer to the first and the third to last terms of (12) as the first order, the dispersion, the interference, the second-order Born term, and the Pauli term, respectively.

The separation method has been extensively discussed previously. We only wish to point out here how this fits into the K-matrix theory, as we feel this has been neglected hitherto.

[14] H. S. Köhler, Nucl. Phys. **38**, 661 (1962).

[15] H. S. Köhler, Ann. Phys. (N. Y.) **16**, 375 (1961).

[16] B. L. Scott and S. A. Moszkowski, Ann. Phys. (N. Y.) **14**, 107 (1961).

By introducing a K matrix, one sums to all orders certain types of important terms of a perturbation expansion. Now the separation method emphasizes that for the long-range part of the interaction it was really not necessary to do the summation over all orders, so one makes a re-expansion of the K matrix in powers of this part. We shall refer to this procedure as the separation-method *expansion*. In actual calculations one has kept only the first few terms in this expansion. One has also made approximations on these terms, and the goodness of these has been investigated.[15] We refer to the final expression (12) as the separation-method *approximation*. In Fig. 1 we show the graphs that this method includes.

Work with the K matrix and the separation method has shown that it is practical to split the nucleon force into three parts which have different influences on the two-particle correlation.

(i) The short-range part $v_s(r)$, $r \leq 1.1$ F. This contains the hard core which is the main reason for strong two-particle correlations. However, the strength of this correlation depends to an important degree on the medium through

(ii), the long-range central part $v_{lC}(r)$, $r > 1.1$ F. This part gives the main contribution v_l to the K matrix in (12), because of the cancellations between repulsion and attraction in v_s. It is also characterized by giving a small second-order Born term.

(iii) The long-range tensor force $v_{lT}(r)$, $r > 1.1$ F. This gives no first-order term, but the second-order Born term gives several MeV to the binding and is therefore important. Associated with this, it gives long-range $^3S_1$ correlations between two nucleons.

In the following investigation we shall often discuss contributions from these parts of the force separately.

V. REARRANGEMENT ENERGIES OF INFINITE MEDIUM

If the K matrix depended only very little on the medium or, in other words, if the two-particle correlation were mostly independent of the medium, we might neglect that dependence and replace the actual K matrix by a matrix calculated at some average state of the medium. However, it is known that the density dependence is important for the saturation of infinite nuclear matter, and such a procedure would not seem hopeful for a finite system either. Also, for a finite nucleus, it would result in neglect of the contributions to the shell-model potential from the medium dependence of the K matrix. An important part of these contributions is related to the often discussed rearrangement energy.[17] This is the energy shift, due to the implicit dependence on the medium of the K matrix, when a particle is physically removed.

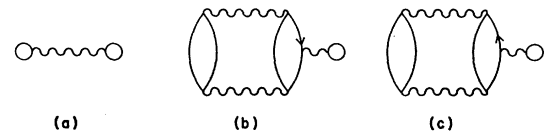

FIG. 1. Graphs summed over explicitly in the separation-method approximation. The interactions (wavy lines) are essentially $v_l + K_s^F$. If the interactions were K we would have only the Fig. 1(a), but in the separation method we also have to sum explicitly over 1(b) and 1(c). We distinguish between 1(b) and 1(c), which have a bubble interacting with a hole and a particle state, respectively.

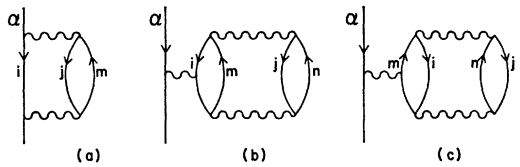

FIG. 2. Rearrangement contributions to energies of holes states. The interactions are K-matrix interactions. Splitting the K matrix into a short- and a long-range part, we get the contributions from them and from the interference between them.

We shall therefore be concerned in this section with rearrangement energies. Such have been calculated by Brueckner, Gammel, and Kubis for an infinite medium at normal density.[18] Their calculations are exact, based on an exact calculation of the K matrix. However, we wish to investigate how the different parts of the interaction discussed at the end of the last section contribute to the rearrangement energies. This is of help in understanding how the K matrix depends on the medium.

The K matrix depends on the medium through the Pauli operator and through the energy denominator in the propagator. The change in the Pauli operator due to the creation of a hole when a particle is removed produces a second order (in K) rearrangement term [Fig. 2(a)], and one gets

$$V_R^{(2)}(\alpha) = \sum_{ijm} \frac{n_i n_j (1-n_m) K_{m\alpha,ij}^2}{E_i^* + E_j^* - E_m^* - E_\alpha^*}, \tag{13}$$

where i, j, α are hole states, and m a particle state, and the n's are occupation numbers. The energies E^* are to be calculated from an on-the-energy-shell K matrix as described by Brueckner and Goldman.[17] This makes energy-conserving transitions possible, and leads to a finite lifetime for the hole. It also leads to a large numerical value for $V_R^{(2)}$. The main contribution comes from small momentum transfers. In actual calculations, the principal value is taken to obtain the real part of $V_R^{(2)}$. The energy denominator ΔE is replaced according to

$$\Delta E \rightarrow [(\Delta E)^2 + \Gamma^2]/(\Delta E). \tag{14}$$

After making this replacement in (13) $V_R^{(2)}$ is calculated in the limit $\Gamma \rightarrow 0$. When the hole energy graph is inserted into an energy diagram the particle energies E_m^* are to be calculated from an off-the-energy-shell

[17] K. A. Brueckner and D. T. Goldman, Phys. Rev. **117**, 207 (1960).

[18] K. A. Brueckner, J. L. Gammel, and J. T. Kubis, Phys. Rev. **118**, 1438 (1960).

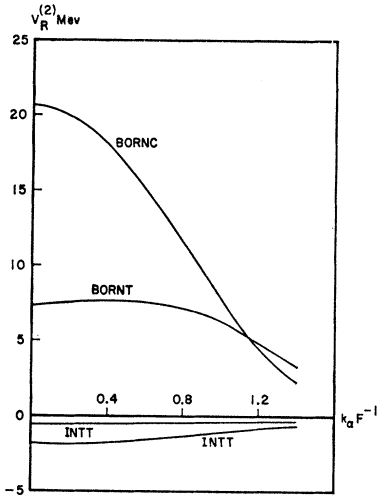

FIG. 3. The second-order rearrangement term $V_R^{(2)}$. In the K-matrix expansion (12) the Born and interference terms contribute to this. We have plotted the contribution from a central (C) and a tensor (T) force as described in the Appendix.

$K^{(1)}$ matrix and the corresponding contributions to the inserted hole energies are much smaller.[17]

In our variational-principle definition of the shell-model potential, the main contribution again comes from the energy-conserving transitions, i.e., with E_m^* calculated from an on-the-energy-shell K matrix. This may seem somewhat strange. However, the singularity we then obtain has the same origin as the well-known singularity in the K matrix of any two nucleons with opposite momenta close to the Fermi surface. This singularity, which may give rise to the Bardeen-Cooper-Schrieffer phenomenon,[19] is believed to be unimportant for the total energy calculations, and is usually avoided by introducing a gap in the virtual energy-conserving transitions. This would correspond to our Γ in (14). That the two singularities are related is best seen if we consider the infinite-nuclear-matter problem. Then we vary only the density or occupation numbers at the Fermi surface. The change in the Pauli principle is then most important for nucleons close to the Fermi surface which can make transitions with ΔE small or zero.

Now in the expansion (12) the terms that contain the Pauli principle are the Born, the interference, and the Pauli terms. We have calculated the contributions to $V_R^{(2)}$ from these different terms in infinite nuclear matter of normal density. Details of the calculation are in the Appendix We use the effective-mass approximation with $M^*/M=0.67$ at the Fermi surface.[1] The result of this calculation is presented in Fig. 3. We do expect a large contribution to come from small-momentum transfers and therefore the second-order Born term with central force (BORNC) contributes significantly compared to the second-order Born term with tensor force (BORNT), especially for small momenta, although the tensor force contributes significantly more to the total energy than the central in second Born approximation. The interference terms contribute significantly less, owing to the one short-range interaction involved. The Pauli terms contribute insignificantly and are not shown in the figure. From the figure, we read the values 25.5 and 4.8 MeV at the bottom and top of the Fermi sea, respectively, for the total contribution to $V_R^{(2)}$. These are to be compared with the values 26.8 and 4.4 MeV obtained by Brueckner, Gammel, and Kubis. Thus, the calculations agree remarkably although the interactions are not exactly the same in the two cases.

Calculations of the density dependence of $V_R^{(2)}$ gave as a result that $V_R^{(2)}$ is nearly proportional to density.

The result of this calculation is thus that the long-range part of the central force gives a strongly momentum-dependent contribution to the second-order rearrangement term. The long-range tensor force gives a considerably smaller contribution for small momenta but gives the main contribution at the Fermi surface.

We defer a discussion of the implications of this result on the calculations on finite nuclei until the next section, after we have treated third-order rearrangement terms.

A calculation of third-order rearrangement terms [Figs. 2(b),(c)] was done in Ref. 20 in connection with the binding energy of a Λ particle in nuclear matter, and also in Ref. 14 in connection with a calculation on oxygen 16. It was there found to be of special advantage to split the third-order rearrangement term into two, the hole ($V_{Rh}^{(3)}$) and the particle ($V_{Rp}^{(3)}$) rearrangement terms. [In Ref. 14 we referred to these as V_{st} and V_ρ. Actually the V_{st} of Eq. (35) in Ref. 14 contains two terms. Only the last of these is the rearrangement term. The first probably has no physical meaning but appears only as a result of the variational procedure. Thus we see, as mentioned after (4a), that the shell-model eigenvalues do not always of necessity have a physical meaning.] The hole and particle rearrangement terms are shown, respectively, in Figs. 2(b) and 2(c).

The third-order rearrangement contribution to the energy of a hole can be written[18]

$$V_R^{(3)}(\alpha)=\tfrac{1}{2}\sum_{ijmn} n_i n_j(1-n_m)(1-n_n)K_{mn,ij}{}^2 \times\frac{\partial}{\partial n_\alpha}\frac{1}{E_i^*+E_j^*-E_m^*-E_n^*}, \quad (15)$$

where the virtual energies were defined by Eq. (10) and thereafter. Thus we get

$$V_R^{(3)}(\alpha)=-\tfrac{1}{2}\sum_{ijmn} n_i n_j(1-n_m)(1-n_n)K_{mn,ij}{}^2 \times\frac{K_{i\alpha,i\alpha}+K_{j\alpha,j\alpha}-K_{m\alpha,m\alpha}{}^{(1)}-K_{m\alpha,n\alpha}{}^{(1)}+\text{exchange}}{(E_i^*+E_j^*-E_m^*-E_n^*)^2}, \quad (15a)$$

[19] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 248 (1960).

[20] Janusz Dabrowski and H. S. Köhler, Phys. Rev. **136**, B162 (1964).

or with a Hermitian K matrix

$$V_R^{(3)}(\alpha)=-\sum_{ijmn} n_i n_j(1-n_m)(1-n_n)K_{mn,ij}^2 \times\frac{K_{i\alpha,i\alpha}-K_{i\alpha,\alpha i}-K_{m\alpha,m\alpha}^{(1)}+K_{m\alpha,\alpha m}^{(1)}}{(E_i^*+E_j^*-E_m^*-E_n^*)^2}, \quad (15b)$$

or

$$V_R^{(3)}(\alpha)=-\sum_i C_i n_i(K_{i\alpha,i\alpha}-K_{i\alpha,\alpha i}) +\sum_m(1-n_m)C_m'(K_{m\alpha,m\alpha}^{(1)}-K_{m\alpha,\alpha m}^{(1)}), \quad (15c)$$

where

$$C_i=\sum_{jmn} n_j(1-n_m)(1-n_n)K_{mn,ij}^2 \times\frac{1}{(E_i^*+E_j^*-E_m^*-E_n^*)^2}, \quad (16a)$$

and

$$C_m'=\sum_{ijn} n_i n_j(1-n_n)K_{mn,ij}^2 \times\frac{1}{(E_i^*+E_j^*-E_m^*-E_n^*)^2}. \quad (16b)$$

This result is understood physically as follows: The first term of (15c) is the correction to the first order (in K) single-particle energy due to the deletion of state i [Fig. 2(b)], and C_i is the amplitude for this deletion. Similarly C_m' is the amplitude for finding a nucleon excited into state m [Fig. 2(c)] due to the correlated wave function. We then get

$$V_{Rh}^{(3)}(\alpha)=-\sum_i C_i n_i(K_{i\alpha,i\alpha}-K_{i\alpha,\alpha i}), \quad (17a)$$

$$V_{Rp}^{(3)}(\alpha)=\sum_m(1-n_m) \times C_m'(K_{m\alpha,m\alpha}^{(1)}-K_{m\alpha,\alpha m}^{(1)}). \quad (17b)$$

Replacing the K matrices in (16a) and (16b) by wave functions, one finds that C_i is proportional to the volume of the distortion of the model wave function created by the correlation, while C_m' depends on the Fourier transform of this hole. In Ref. 20, C_i was computed by such a procedure with the Bethe-Goldstone solution for the wave function. We then found C_i to vary only by some 30% as a function of momentum state i at normal density. Thus we could take an average of C over i. We obtained C (averaged over i) $=\frac{3}{4}\rho\times1.16=0.15$, where ρ is the normal density ($\rho=0.172$ nucleons/F^3). In Ref. 14 we obtained $C=\frac{3}{4}\rho\times0.5=0.065$. The discrepancy between these two calculations may be due to an underestimate in that calculation using the wave function for free scattering corresponding to the wave operator Ω_s^F in Eq. (12). But it may also be due to a too slow healing[21] of the Bethe-Goldstone solution. It is anyway clear that the calculation of C_i is sensitive to the wave function. In Ref. 14, we introduced the effective core radius c_e and we then obtained $c_e=0.50$ F. The calculation in Ref. 20 gives $c_e=0.6$ F. It would be of value to perform this calculation with a more accurate wave function.

[21] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (N. Y.) 3, 241 (1958).

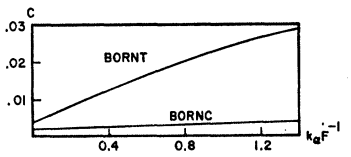

FIG. 4. The coefficient $C(k_\alpha)$ as defined by (16a) and entering into the third order rearrangement term $V_R^{(3)}$. The contributions from the Born terms are shown with a central (C) and tensor (T) force as described in the Appendix.

It is clear from these calculations that the main contribution to C_i comes from the short-range interaction K_s^F, since this can make the necessary excitations. In the expansion (12), this corresponds to calculating C_i from the dispersion term, as we also did in Ref. 14. However, the Born and Pauli terms also depend on the energy denominator e. Estimates show that the Pauli term would contribute completely negligibly. The smallness of the second-order Born term with a central interaction shows that this part can also give only a small contribution to C_i. The tensor force gives a much larger second-order Born term, however. Therefore we also calculated the contributions to C_i from this term in (12). This corresponds to replacing the K's in (16a) by the long-range part of a tensor force. We also did the same calculation with the long-range part of a central interaction. The results of these calculations are shown in Fig. 4, while some details of the calculations are found in Appendix A. The result is about as we can expect from previous discussion.

We approximate the contribution as $C(k_\alpha)$ from the tensor Born term in Fig. 4 by

$$C(k_\alpha)=0.0041+0.0259k_\alpha-0.00609k_\alpha^2. \quad (18)$$

We further assume an effective-mass approximation for the K matrix, and thus obtain a quadratic momentum dependence for this. We deduce the two parameters of this dependence from the result of K-matrix potential energy given in Ref. 18, by fitting at the values -112 and -70.3 MeV at the bottom and top of the Fermi sea, respectively. We put this into (17a), do the summation and get a contribution to the third-order rearrangement term (in MeV) from the long-range part of the tensor force.

$$V_{Rh,\mathrm{BORNT}}=2.66-0.51k_\alpha^2. \quad (19)$$

The calculations of C in Ref. 14 and 20 were only with central interactions. As the long-range central term contributes so little (Fig. 4), we can state that those calculations of C were both due to the short-range part, i.e., due to the dispersion term. We therefore call that contribution $V_{Rh,\mathrm{DISP}}^{(3)}$ and

$$V_{Rh,\mathrm{DISP}}^{(3)}=-C_s U(k_\alpha), \quad (20)$$

where C_s has been previously calculated ($0.065\leq C_s$

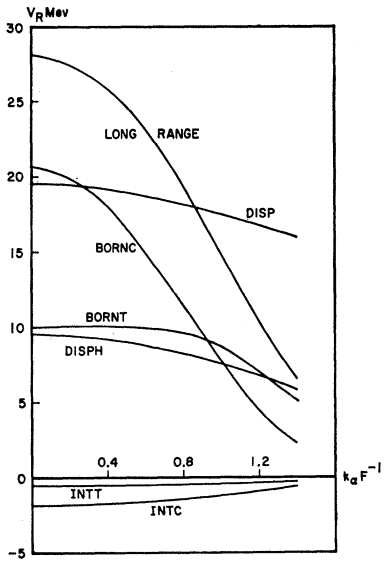

FIG. 5. The contribution to the rearrangement term V_R from the different terms of the K matrix expansion (12), with a central (C) and tensor (T) force as described in the Appendix. We have also plotted separately the sum of all contributions from the terms containing the long-range interaction, and the contribution from the dispersion term. Shown separately is also the dispersion-term contribution to $V_{Rh}^{(3)}$, denoted by DISPH.

≤ 0.15) and $U(k_\alpha)$ is the first-order K-matrix potential energy of momentum k_α. The contribution (19) is smaller than (20) but not completely negligible.

The size of $V_{Rp}^{(3)}$ in (17b) is indeed uncertain. It depends critically both on the high-frequency Fourier transforms of the correlated wave function and on the off-the-energy-shell $K^{(1)}$ matrix for large momenta, as is brought out explicitly in Refs. 14 and 20. However, it can be regarded as independent of hole state α because the states m [Fig. 2(c)] are all of such large momentum that the relative momentum between m and α is practically independent of α.

In the work of Bethe *et al.*[3] it was found that, under the assumption that the hard core works in all angular-momentum states, the single-particle virtual energy for off-the-energy-shell interaction has an effective mass less than the real mass even for very large excitations. The single-particle potential energy is then positive for these excitations, and this leads to a positive $V_{Rp}^{(3)}$. We can calculate this term as in Ref. 20. If we neglect the difference in reduced masses due to the difference in masses of lambdas and nucleons, and further realize that the attraction in angular-momentum states $l>0$ is smaller than in the Λ-nucleon case, we end up with [Ref. 20, Eqs. (101), (102)]

$$V_{Rp}^{(3)} \sim 10 \text{ MeV}. \tag{21}$$

However, this figure is rough and therefore only indicative. Actually we expect V_{Rp} to be equal to the contribution to the energy from the third-order bubble-particle graph [Fig. 1(c)], which may perhaps be as large as 10 MeV due to off-energy-shell propagation.

We found the tensor force to contribute significantly to C_i of (16a) because of its effectiveness in scattering out of the Fermi sea. Therefore, it would also contribute to C_m' (16b) and thus to $V_{Rp}^{(3)}$ according to (17b). However, most of these scatterings will be to momenta just outside the Fermi sea. The single-particle potential energies there are relatively small compared to the single-particle potential energies to which the short-range force scatters. (Due to the off-energy-shell effect the particle potential energies increase rapidly with momentum.[3]) Therefore the tensor force contributes relatively little to $V_{Rp}^{(3)}$. Because of this, and because of the uncertainty of the calculated number 10 MeV of Eq. (21), C_m' was not calculated for the tensor (or for the central) force.

Thus we get

$$\begin{aligned} V_R^{(3)}(\text{bottom}) &\approx 18\text{–}28 \text{ MeV}, \\ V_R^{(3)}(\text{top}) &\approx 15\text{–}20 \text{ MeV}, \end{aligned} \tag{22}$$

for the total third-order rearrangement term at the bottom and top of the Fermi sea, respectively. These values are considerably larger than the (assumed) exact values of Brueckner, Gammel, and Kubis,[18] who obtain 9.4 and 6.1 MeV, respectively. The disagreement is probably because of an underestimate of the off-energy-shell effect, so that we should compare our values for $V_{Rh}^{(3)}$ with their total third-order term. Then our result agrees with theirs if $C_s \approx 0.06$, which is close to our estimate in Ref. 14. We shall therefore use this value in subsequent calculations. In Fig. 5, we have plotted the contributions to the rearrangement energy from the different terms in the expansion (12).

VI. DISCUSSION OF THE REARRANGEMENT-ENERGY CALCULATIONS

The rearrangement energies in the previous section were calculated in a system of infinite nuclear matter. However, we expect qualitatively the same results for a finite system as far as the momentum dependence is concerned. In configuration space, we usually refer to this as a nonlocality. Thus we can deduce from our calculations on the infinite system that in a finite system we should get rearrangement contributions to the shell-model potential accordingly:

(i) From the short-range part of the nucleon interaction, including the core, we get a third-order (hole) contribution, which is proportional to the first-order K-matrix (Hartree-Fock) contribution [Eq. (20)]. We estimated the proportionality factor to be $0.06 \times (\rho/\rho_0)$ where ρ_0 is the normal density (end of last section). We get another third-order particle contribution involving the short-range interaction which is local (independent of momentum), and at normal density about 10 MeV [Eq. (21)]. The third-order term varies about as density-squared.

(ii) The long-range part of the central potential contributes an amount in second order which is at normal density just a few MeV at the top of the Fermi sea (and therefore probably negligible for central densities of a very large nucleus), but is strongly momentum-dependent (nonlocal). Thus, it might be

quite important to include this part of the shell-model potential in a shell nucleus or in studies of a surface.[13]

(iii) The long-range part of the tensor force contributes roughly equal amounts in second and third order. And the nonlocality seems to be roughly like the Hartree-Fock potential as seen in Fig. 5.

However, these qualitative findings do not tell us anything about the spatial dependence of the rearrangement contributions to the shell-model potential in a finite nucleus. To investigate this we must first understand how the K matrix depends on the spatial distribution of nucleons, or more generally on the state of the medium. We can understand this somewhat by studying our results of rearrangement energies.

It is first of all clear that if rearrangement energies were all small compared to first-order K-matrix energies then the K matrix could be represented by some "static" (or effective) K matrix.

In an infinite medium we are concerned only with the density, as the only parameter to describe the state of the system. Thus if (at a specified density) a specific part of the K matrix leads to a negligible single-particle rearrangement energy at the Fermi momentum, that part of the K matrix can be treated as static when calculating the saturation of infinite nuclear matter. Thus our findings indicate that we can probably treat the second-order Born term with central interaction, together with the corresponding interference term as well as the Pauli term, as a static part.

However, we find a large contribution to come from the third-order graph and specifically from excitations due to the short-range part, i.e., the dispersion term. The contribution is 4–14 MeV depending on the off-energy-shell treatment. We conclude from this (and previous calculations of saturation densities) that one must include the density dependence of the dispersion part. This necessity is further brought out by the circumstance that the dispersion part is nearly proportional to the density. If important in an infinite medium, it should also be important in a finite medium.

Next in importance at normal density is the $^3S_1$ Born term (tensor part), contributing ≈ 3.3 MeV in second order and ≈ 1.7 MeV in third order.

The $^1S_0$ Born term (central part) gives a small second-order rearrangement term at the Fermi momentum which is, however, strikingly momentum-dependent. As mentioned above, it should be sufficient to treat this term as static in an infinite medium, but in a finite nucleus we are also concerned with specifying the lower lying states. Thus we conclude that it would, at least in principle, be more important to consider the medium dependence of the $^1S_0$ Born term in a finite than in an infinite medium. Associated with this we get, as previously mentioned, a large contribution to the nonlocality of the shell-model potential.

The importance of the lower lying states is, however, suppressed by the phase-space factor. There is always room for more nucleons in the higher lying states, and it may be that it is after all not so important to include, e.g., the medium dependence of the $^1S_0$ Born term.

How do our conclusions compare with the investigation of Brueckner and co-workers[9] leading to the approximation of Sec. III? First, they only calculated the K matrix as a function of density in an infinite medium. This would correspond to our deducing the medium dependence of the K matrix only from looking at the rearrangement energies at the Fermi momentum. However, we have already argued that also the lower lying states have to be changed to form a finite nucleus. Further, we then miss the nonlocality of the rearrangement potential. The phase-space factor mentioned above may reduce the first error. The second was approximately investigated by Masterson and Lockett and was found to be not very important.[12]

However, it was also concluded in the previously mentioned work that their long-range part K_a [Eq. (11)] is static as defined above. However, this conclusion was reached by varying the density of an infinite system, and we have already commented about that procedure. Further, however, the K_a does vary with density and it is not actually clear from the calculations that this can be neglected, as mentioned in Sec. III. Thus, we can conclude that there is no contradiction between our results and those of Brueckner and coworkers. We only look into more detail on the medium dependence of the K matrix. (See the next section.)

In order to investigate quantitatively the possible importance of these points just discussed, numerical calculations on finite nuclei are clearly necessary. The outcome of this depends in a fairly complicated way on self-consistency conditions on both the shell-model potential and the K-matrix virtual energies. In a later publication we shall present some such calculations.

VII. MEDIUM DEPENDENCE OF THE K MATRIX

Our previous investigation showed that the important medium-dependent terms of the expression (12) are the dispersion and Born terms, while the rest of the K matrix can be considered as static. The procedure in actual calculation would then be to calculate the K matrix at some average density, and then separately the dispersion and Born terms, which are then subtracted from the calculated K matrix to obtain the static part. We believe this is a possible procedure in practice. In a calculation on a finite nucleus, one then has to use these parts of the K matrix to calculate the total energy. The most complicated phase of this is the calculation of the Born terms, because this involves by necessity a sum over intermediate excited states. The calculation of the contribution to the shell-model potential from the implicit dependence of the Born terms on the medium would be still more complicated. It is, however, not clear how important this part of the shell-model potential is, and it may be enough to make a crude approximation

to it. Some preliminary calculations indicate that this is so.

The dispersion term, however, can be approximated in a sensible way, as well as the contributions to the shell-model potential. We actually showed this in a previous publication.[14] For completeness, we shall now discuss this approximation again, especially with regard to the off-energy-shell effect, the importance of which was stressed in Ref. 3.

There is actually a further reason to treat the medium dependence of the dispersion term properly. It is associated with the third-order graphs, and is a correction for the inclusion of the bubbles in these graphs, and thus contains the potential-energy factor e_0-e [Eq. (12)]. Therefore the dispersion K-matrix term is nearly proportional to the density, and positive. Thus it is very important for saturation.

We now compute the matrix element of the dispersion term [third term of (12)] between two single-particle states i, j, described by spatial wave functions, $\varphi_i(\mathbf{r}_1)$ and $\varphi_j(\mathbf{r})$. We drop the indices on the wave operator and obtain for the dispersion term K^D,

$$K_{ij,ij}{}^D=(\varphi_i\varphi_j|\,(\Omega-1)\,|\,(e_0-e)\,|\,(\Omega-1)\,|\,\varphi_i\varphi_j). \quad (23)$$

The operator $\Omega-1$ operates on the state of relative motion between i and j, and depends on the relative angular momentum. It is important only for relative S states and can be well approximated by zero for higher angular-momentum states. This is because the core perturbs only the relative S-state motion appreciably. Associated with this is the finding that only the S-state part of the K matrix is density dependent.[9] We now separate into relative and center-of-mass coordinates and get

$$(\Omega-1)\,|\,\varphi_i(\mathbf{r}_1)\varphi_j(\mathbf{r}_2)\rangle = (\Omega-1)\,|\,\varphi_i(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\varphi_j(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\rangle. \quad (24)$$

Now the operator $(\Omega-1)$ is very short-ranged ($\sim$0.5 F) in coordinate space. The range is short compared to typical wavelengths of nucleons. Thus, it is a good approximation to neglect the $\mathbf{r}$ dependence in φ_i and φ_j. We then obtain from (24)

$$(\mathbf{Rr}|\,(\Omega-1)\,|\,\varphi_i\varphi_j\rangle=\chi(r)\varphi_i(\mathbf{R})\varphi_j(\mathbf{R}), \quad (25)$$

where $\chi(r)$ is the correlation function. In a more exact treatment χ would also depend on the state of relative motion. This dependence is weak, however, as seen, e.g., from Ref. 20, Eq. (42), or from Ref. 2, Table IV.

Now in (23) there appears the potential-energy term e_0-e, and we use the notation

$$V_i=\textstyle\sum_k(K_{ik,ik}-K_{ik,ki}), \quad (26)$$

for hole states i, and

$$V_l=\textstyle\sum_k(K_{lk,lk}{}^{(1)}-K_{lk,kl}{}^{(1)}), \quad (26a)$$

for particle states l. Then (23) gives

$$K_{ij,ij}{}^D=-(V_i+V_j)\int\chi^2(r)d\mathbf{r}\int\rho_i(\mathbf{R})\rho_j(\mathbf{R})d\mathbf{R} + \int\sum_{lm}\varphi_i(\mathbf{R})\varphi_j(\mathbf{R})\chi(r)\,|\,\varphi_l\varphi_m)(V_l+V_m)(\varphi_l\varphi_m\,| \times\varphi_i(\mathbf{R})\varphi_j(\mathbf{R})\chi(r)d\mathbf{R}d\mathbf{r}. \quad (27)$$

The first term corresponds to Fig. 1(b) and the second to 1(c). The first contains the correlation volumes $\int\chi^2(r)d\mathbf{r}$ which entered into the calculation of C_s, (20), and for which we have obtained the value 0.5–1.16 F^3. It further contains a factor $\int\rho_i(\mathbf{R})\rho_j(\mathbf{R})d\mathbf{R}$, which is the amplitude for two nucleons to interact via a short-ranged force. It is also proportional to the K-matrix potential energies of the interacting nucleons. This first term is readily evaluated without further approximations for any medium, given the single-particle wave functions.

To evaluate the second term, we first note that the sum over intermediate states l and m produced by the short-ranged function χ involves excitations with momenta of about 4 F^{-1}, which is quite high compared to the average momentum in a nucleus. Thus we can to good approximation put $V_l=V_m$. Further, due to the short range of $\chi(r)$, the sum over the intermediate states explores the nucleon medium only around the center-of-mass coordinate $\mathbf{R}$. Thus the second term depends, for fixed $\mathbf{R}$, on the nuclear structure only at and around the point $\mathbf{R}$. This circumstance suggests the use of a Thomas-Fermi type of approximation. The finite system is imagined to be built up at point $\mathbf{R}$ of plane waves with a maximum momentum k_F related to the density $\rho(\mathbf{R})$ by $\rho=\frac{2}{3}\pi^2k_F{}^3$. We then Fourier-analyze the function $\chi(r)$ into components k and assume in accordance with the discussion above that the particles i, j are scattered into states of opposite momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, with $k_1=k_2$. If we assume the center-of-mass moment to be $k_F/2$, we then obtain

$$k_1=(\tfrac{1}{4}k_F{}^2+k^2)^{1/2}. \quad (28)$$

Thus we are now concerned with the potential energy of a nucleon with momentum k_1 in infinite nuclear matter of density ρ and we denote this by $V(k_1,\rho)$. We insert this into the last term of (27) and get, with the index p referring to particle interaction,

$$K_{ij,ij}{}^{Dp}=2\int\int\rho_i(\mathbf{R})\rho_j(\mathbf{R})\bar{\chi}^2(k)V(k_1,\rho(\mathbf{R}))d\mathbf{R}d\mathbf{k}, \quad (29)$$

where k_1 is given by (28). The factor 2 comes in because we have two potentials V_l and V_m.

Thus we found that, to a fair approximation, the summation over the intermediate states can be reduced to essentially a problem of infinite nuclear matter, with the density evaluated at the point of interaction, i.e., a local-density approximation is appropriate. In con-

trast, the first term of (27), which we analogously refer to as K^{Dh}, does not depend on the medium structure at the center of mass but rather on the extension of the respective states of the interacting nucleons over the whole nucleus. This is so because of the potentials V_i and V_j.

The procedure and the conclusions we draw from the result are mainly the same as in Ref. 14. (We did not now include a dependence of χ on initial relative motion, because this is probably an unnecessary refinement.) [See text after Eq. (25).]

Actually, the approximation used in the summation over the excited states can be used in any calculation of the Green's function Q/e. Thus we may also use it to compute the Born-term part of the K matrix in (12). However, this involves a long-range correlation of the wave functions of relative motion, especially in the low-density region and with tensor force. Thus it is doubtful that a Thomas-Fermi-type approximation would be good, since the correlation ranges in question are of the order of the thickness of the nuclear surface.

In Ref. 14 we concluded that in an infinite system K^{Dp} was negative, and only about -10% of the first term of (27). However, Bethe and co-workers showed that the off-energy-shell effect makes the $K^{(1)}$ matrix potentials V_l and V_m quite large.[3] In fact they obtained (with Rajaraman's correction[6]) an effective mass $m^*=1-(4\pi/3)c^3\rho$, where c is the core radius, for large excitations. This makes K^{Dp} contribute about equally or more to the total energy. As mentioned after Eq. (21) K^{Dp} contributes in fact an amount equal to V_{Rp} which was estimated at 10 MeV.

The contributions to the shell-model potential from the dispersion terms are readily evaluated by variation of the wave functions. This was done in Ref. 14 and we refer to this paper for details.

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APPENDIX A: EVALUATION OF SECOND-ORDER REARRANGEMENT ENERGIES

When evaluating $V_R^{(2)}(\alpha)$ of (13) we use the effective-mass approximation for the K-matrix energies E^*. The excitations close to the Fermi surface are most important, so we use an effective mass of the Fermi surface, $M^*=0.67M$.

We use the following notations: $\mathbf{k}_m$, momentum of nucleon m; $\mathbf{k}_\alpha$, momentum of nucleon α; $\mathbf{k}$, relative momentum between i and j; $\mathbf{P}$, center-of-mass momentum; θ, angle between $\mathbf{k}_m$ and $\mathbf{k}_\alpha$; β, angle between $\mathbf{P}$ and $\mathbf{k}$.

When we consider S-state interactions only, we can put $K_{m\alpha,ij}\rightarrow K(k,k')$ and get

$$V_R^{(2)}(k_\alpha)=-\frac{3}{4}\frac{4\times 2M^*(2\pi)^2}{\hbar^2(2\pi)^6}\int K(k,k')\frac{1}{\Delta E}k_m^2dk_m \times k^2dkd(\cos\beta)d(\cos\theta), \quad \text{(A1)}$$

where

$$k'=\tfrac{1}{2}(k_m^2+k_\alpha^2-2k_mk_\alpha\cos\theta)^{1/2},$$

and

$$\Delta E=2k^2-2k'^2.$$

The integration over $\cos\beta$ is performed analytically. The limits of integration are

if $\quad 0<k\leqslant k_F-\tfrac{1}{2}P, \quad -1<\cos\beta<1,$

if $\quad k_F-P/2<k<(k_F^2-\tfrac{1}{4}P^2)^{1/2},$

$$0>\cos\beta>(k^2+\tfrac{1}{4}P^2-k_F^2)/kP,$$

where $P=(k_m^2+k_\alpha^2+2k_mk_\alpha\cos\theta)^{1/2}$. We must have $P<2k_F$, where k_F is the Fermi momentum, so that

$$k_m\leqslant k_m^{\max}=(k_\alpha^2\cos^2\theta-k_\alpha^2+4k_F^2)^{1/2}-k_\alpha\cos\theta,$$

and we get

$$V_R^{(2)}(k_\alpha)=\frac{6M^*}{\hbar^2(2\pi)^4}\int_{\cos\theta=-1}^{1}\int_{k_m=k_F}^{k_m^{\max}}\left[\int_{k=0}^{k_F-\frac{1}{2}P}\frac{2}{\Delta E}K^2(k,k')k^2dk + \int_{k=k_F-\frac{1}{2}P}^{(k_F^2-\frac{1}{4}P^2)^{\frac{1}{2}}}\frac{2}{\Delta E}\frac{k_F^2-k^2-\frac{1}{4}P^2}{kP}K^2(k,k')k^2dk\right]k_m^2dk_md(-\cos\theta). \quad \text{(A2)}$$

When $K(k,k')$ is the long-range part of the central interaction, $v_l(r)$, we put[2]

$$v_l(r)=Ve^{-\mu r}, \quad r>d,$$
$$v_l(r)=0, \quad r<d, \quad \text{(A3)}$$
$$\mu=2.083\ \text{F}^{-1}, \quad V=603.06\ \text{MeV}, \quad d=1.1\ \text{F}.$$

Then we get

$$K(k,k')=\frac{4\pi Ve^{-\mu d}}{2kk'}\left[\frac{\mu\cos k_-d-k_-\sin k_-d}{\mu^2+k_-^2} - \frac{\mu\cos k_+d-k_+\sin k_+d}{\mu^2+k_+^2}\right], \quad \text{(A4)}$$

where $k_+=k+k'$ and $k_-=k-k'$. The expression (A2) is evaluated with the replacement of ΔE as in Eq. (14).

When we deal with the tensor force we put in (13) $K_{m\alpha,ij}=K(q)$, where q is the momentum transfer in the interaction. We then get

$$V_R^{(2)}(k_\alpha)=C\int K^2(q)\frac{1}{\Delta E}d\mathbf{q}d\mathbf{k}_j, \tag{A5}$$

where

$$\Delta E=2k_\alpha qt_\alpha+2k_jqt_j,$$

t_j and t_α being the cosine functions for the angles between $\mathbf{k}_j$ and $\mathbf{q}$, and between $\mathbf{k}_\alpha$ and $-\mathbf{q}$, respectively. We then get with the replacement (14)

$$V_R^{(2)}(k_\alpha)=C\int K^2(q)\frac{qk_jt_j+qk_\alpha t\alpha}{(qk_jt_j+qk_\alpha t_\alpha)^2+\Gamma_0^2/4}q^2dqk_j^2dk_jdt_\alpha dt_j, \tag{A6}$$

where

$$\Gamma_0=\Gamma(2M^*/\hbar^2).$$

We can now integrate over

$$-1<t_\alpha<(k_F^2-k_\alpha^2-q^2)/2qk_\alpha=T_\alpha\leqslant 1,$$

and get, with $T_j=T_{\alpha\to j}$,

$$V_R^{(2)}(k_\alpha)=\frac{C}{k_\alpha}\int_{q=0}^{k_\alpha+k_F}\int_{t_j=-1}^{T_j}\int_{k_j=0}^{k_F}K^2(q)\times\ln\frac{(qk_jt_j+qk_\alpha T_\alpha)^2+\Gamma_0^2}{(qk_jt_j-qk_\alpha)^2+\Gamma_0^2}qdqk_j^2dk_jdt_j. \tag{A7}$$

Statistical factors, etc., give with Serber force

$$C=\frac{1}{4}\frac{(2\pi)^2}{(2\pi)^6}\frac{3}{16}\times4\times8\times(4\pi)^2\frac{2M^*}{\hbar^2}=\frac{3}{\pi^2}\frac{M^*}{\hbar^2}.$$

We have not included the exchange term.

We used for $K(q)$ the long-range part of the tensor force of the Gammel-Thaler potential and

$$K(q)=\int_d^\infty r^2dr\ V_T(r)j_2(kr), \tag{A8}$$

where j_2 denotes second-order spherical Bessel function,

$$V_T=V(e^{-\mu r}/\mu r),\quad V=-159.4\text{ MeV},\quad \mu=1.045\text{ F}^{-1}, \tag{A9}$$

and $d=1.0$ F.

We evaluated (A2) and (A7) with $\Gamma=3.0$ MeV and with $\Gamma=6.0$ MeV and extrapolated to $\Gamma=0$.

When calculating the contributions from the interference terms, the replacement $(1/\Delta E)K_{m\alpha,ij}=\Omega_s^F-1$ is made and, owing to the short range of this operator in coordinate space compared to v_l, we put

$$(\Omega_s^F-1)(k,k')=\text{constant}=-\tfrac{4}{3}\pi c_e^3, \tag{A10}$$

and we use[14] $c_e=0.5$ F.

APPENDIX B: EVALUATION OF THIRD-ORDER REARRANGEMENT ENERGIES

We are here concerned with calculating $C(k_\alpha)$ of (16a). We again use the effective-mass approximation. However, the particle energies are now calculated off the energy shell, and we then put an average for the gap Δ at the Fermi surface in the two-particle energy spectrum. We put[22] $\Delta=50$ MeV.

We now use the notations: $\mathbf{k}_j$, momentum of nucleon j; $\mathbf{k}_n$, momentum of nucleon n; $\mathbf{k}$, relative momentum between m and n; $\mathbf{k}'$, relative momentum between α and j; θ, angle between $\mathbf{k}_\alpha$ and $\mathbf{k}_j$; other notations are as before.

When we consider S-state interactions we have again $K_{mn,\alpha j}\to K(k,k')$, and get

$$C(k_\alpha)=\frac{3}{4}\frac{4}{(2\pi)^4}\int\frac{K^2(k,k')}{(\Delta E)^2}d\mathbf{k}_jd\mathbf{k}, \tag{B1}$$

where

$$k'=\tfrac{1}{2}(k_j^2+k_\alpha^2-2k_jk_\alpha\cos\theta)^{1/2},$$

and

$$\Delta E=(\hbar^2/2M^*)(2k^2-2k'^2+\Delta).$$

We can perform the angular integration over β. If $k>k_F+\frac{1}{2}P$, then $-1<\cos\beta<1$. If $(k_F^2-\frac{1}{4}P^2)^{1/2}<k<k_F+\frac{1}{2}P$, then $0<\cos\beta<(\frac{1}{4}P^2+k^2-k_F^2)/kP$, and

$$C(k_\alpha)=\frac{3}{(2\pi)^4}\int_{k_j=0}^{k_F}\int_{\cos\theta=-1}^{1}\left[\int_{k=(k_F^2-\frac{1}{4}P^2)^{1/2}}^{k_F+\frac{1}{2}P}K^2(k,k')\frac{2}{(\Delta E)^2}\frac{\frac{1}{4}P^2+k^2-k_F^2}{kP}k^2dk+\int_{k=k_F+\frac{1}{2}P}^{\infty}K^2(k,k')\frac{2}{(\Delta E)^2}k^2dk\right]k_j^2dk_jd(\cos\theta). \tag{B2}$$

With $K(k,k')$ from (A4), the contribution to C from the long-range part of the central force was calculated.

When we deal with the tensor force we get again with previous notations.

$$C(k_\alpha)=C\int\frac{K^2(q)}{(\Delta E)^2}d\mathbf{k}_jd\mathbf{q}, \tag{B3}$$

[22] G. E. Brown, G. T. Schappert, and C. W. Wong, Nucl. Phys. **56**, 191 (1964).

where

$$\Delta E = 2q^2 - 2qk_\alpha t_\alpha + 2qk_j t_j + \Delta_0,$$

and

$$\Delta_0 = \Delta(2M^*/\hbar^2).$$

We perform the integration over t_α,

$$-1 < t_\alpha < T_\alpha, \quad \text{where} \quad T_\alpha = (k_F^2 - k_\alpha^2 - q^2)/2qk_\alpha < 1,$$

to get

$$C(k_\alpha) = \frac{C}{k_\alpha} \int K^2(q) \left(\frac{1}{2q^2 - 2qk_\alpha T_\alpha + 2qk_j t_j + \Delta_0} - \frac{1}{2q^2 + 2qk_\alpha + 2qk_j t_j + \Delta_0} \right) qdqk_j^2 dk_j dt_j. \tag{B4}$$

Next the integration over t_j is performed,

$$T_j < t_j < 1, \quad T_j = (k_F^2 - k_j^2 - q^2)/2qk_j < -1,$$

to get

$$C(k_\alpha) \quad \frac{C}{k_\alpha} \int_{q=k_F-k_\alpha}^{\infty} \int_{k_j=k_F-q}^{k_F} K^2(q) \ln \left| \frac{(2q^2 - 2qk_\alpha T_\alpha + 2qk_j + \Delta_0)(2q^2 + 2qk_\alpha + 2qk_j T_j + \Delta_0)}{(2q^2 - 2qk_\alpha T_\alpha + 2qk_j T_j + \Delta_0)(2q^2 + 2qk_\alpha + 2qk_j + \Delta_0)} \right| qdqk_j dk_j. \tag{B5}$$

If $K(q)$ is given by (A8) statistical factors, etc., now gives $C = (3/2\pi^2)(2M^*/\hbar^2)$. The exchange term was omitted.

The numerical computations were performed on the CDC-1604 computer of the Computer Center at the University of California at San Diego in La Jolla.

The integrations were all made by Simpson's rule, successively dividing the integration intervals by two until the integral changed less than a specified amount.

We calculated, unless otherwise specified, with a Fermi momentum $k_F = 1.4$ F^{-1}.

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Spin, Hyperfine Structure, and Nuclear Magnetic Dipole Moment of 23-sec Na^{21}†

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The spin and hyperfine structure of 23-sec Na^{21}, the mirror nucleus to Ne^{21}, have been measured by the atomic-beam magnetic-resonance technique. The Na^{21} was produced by the reaction $Mg^{24}(p,\alpha)$ Na^{21}. The target material in the form of a powder was bombarded in an oven in the resonance apparatus. If the oven was heated to about 450°C, the Na^{21} escaped from the Mg in sufficient quantity to make a useful atomic beam. Both $\Delta F = 0$ and $\Delta F = 1$ resonances were observed, and the final values are $I = \frac{3}{2}$, $\Delta\nu = 1906.466 \pm 0.021$ Mc/sec. Comparison with Na^{23} yields $\mu_I = 2.386\,12 \pm 0.000\,10$ nm (diamagnetically corrected). The results are discussed in terms of the current nuclear theories.

I. INTRODUCTION

In the last several years much experimental effort has been made both at this laboratory and at others to measure the magnetic moments of radioactive mirror nuclei. The incentive for this effort lies partly with the hope that knowledge of the magnetic dipole moments of both members of a mirror pair will yield information on mesonic currents in these nuclei.[1,2] The success of such a program depends on our being able to use the sum of the moments and other relevant experimental data to obtain a good wave function for the nuclear ground state. By good we mean precise enough so that a discrepancy of the order of 0.1 nm between the individual experimental moments and the theoretical values obtained from this wave function can be attributed to mesonic effects. The magnetic-moment operator used to calculate the moments from the nuclear wave function is

$$\boldsymbol{\mu}_{\text{op}} = \sum_{\substack{\text{all} \\ \text{nucleons}}} (g_l \mathbf{l} + g_s \mathbf{s}),$$

where the g factors are those for the free nucleons. Thus the measurable mesonic effects will include quenching[3] of the g factors in addition to exchange currents. It

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