Finite Quantum Electrodynamics: A Field Theory Using an Indefinite Metric*

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A finite relativistic field theory of quantum electrodynamics is formulated; the theory involves as an essential element the use of an indefinite metric to remove the divergences. Two auxiliary fermion fields with anticommutation relations of opposite sign from the normal fermion field (electron) are coupled with the electron to the electromagnetic field in such a manner that any matrix element, calculated by a perturbation expansion, is finite. Although the new Lagrangian is not explicitly gauge invariant, the electromagnetic field is quantized by a method that yields a propagator in the true Landau gauge, and it is shown that this is sufficient to insure that the physical vector particles are zero-mass transverse quanta (i.e., photons). The physical electron mass and the fine-structure constant are put into the theory, leaving only the masses of the two auxiliary fields as parameters. The effects of mass and charge renormalization are calculated (the former using a technique involving a Taylor expansion in the mass), and are required to be small so that there is no explicit contradiction to the validity of the perturbation expansion. The anomalous magnetic moment of the electron and the differential and total cross sections for Compton scattering are calculated and compared with experiment. A range of the auxiliary mass parameters is found for which the predictions agree with experiment and for which the expansion criteria are satisfied. Thus, a finite quantum electrodynamics is accomplished.

I. INTRODUCTION

THE idea that it might be of interest to introduce an indefinite metric into the Hilbert space of quantum states was first suggested by Dirac[1] as early as 1942, and discussed in greater detail by Pauli.[2] Since that time, there have been various cases of field theories and models where quantization has involved the use of such a metric. The most well-known example, of course, is the quantization of the electromagnetic field by Gupta[3] and Bleuler.[4] Their procedure, which canonically quantizes all four components of the vector potential, yields a manifestly covariant local theory, but demands that the metric no longer be positive definite.[5] Another example, not always recognized as such, is the Pauli-Villars regularization.[6] This work is often referred to as just a mathematical technique to treat divergent integrals; however, in order to place the technique on a Lagrangian basis (and even to restore unitarity into the perturbation expansions), quantization requires the use of auxiliary fields which satisfy commutation relations with the "wrong" sign (i.e., sign opposite to those of ordinary fields).[7] As is well known, quantization with "wrong"-sign commutation relations implies a vector space with an indefinite metric.[8] In the realm of model field theories, the indefinite metric has appeared via the notorious ghost states[9] of the Lee model.[10] Field theories with higher order Lagrangians, such as those considered by Green[11] and by Pais and Uhlenbeck,[11] strive to remove infinities only at the expense of introducing an indefinite metric.

In spite of these persistent appearances of an indefinite metric, its use has been viewed with considerable skepticism. On one hand, such theories, when interpreted in the usual fashion, lead to difficulties connected with a probabilistic interpretation through the appearance of negative probabilities of quantum states.[12] On the other hand, it has been suggested that, if it were possible to construct a consistent indefinite metric theory, it should be possible to provide a reformulation without using such a metric.

These points have been discussed in detail in an earlier paper by one of the authors (E.C.G.S.).[13] The point of view presented there is that an indefinite metric with "local" interactions may be the most elegant

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A180**, 1 (1942).
[2] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).
[3] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).
[4] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).
[5] Of course, the Dirac-Schwinger method of quantizing only the transverse fields can be used without introducing an indefinite metric. In this case, the Hamiltonian contains a nonlocal interaction corresponding to the instantaneous Coulomb interaction between sources, and, while the theory is relativistically invariant, it is not manifestly so.
[6] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).
[7] S. N. Gupta, Proc. Phys. Soc. (London) **A66**, 129 (1962).
[8] For a comprehensive review of vector spaces with an indefinite metric, as well as a brief summary of physical examples, see L. K. Pandit, Nuovo Cimento Suppl. **11**, 157 (1959), and K. L. Nagy, Nuovo Cimento Suppl. **17**, 92 (1960).
[9] G. Källen and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955), and W. Heisenberg, Nucl. Phys. **4**, 532 (1957).
[10] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[11] A. E. S. Green, Phys. Rev. **73**, 26 (1948). A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
[12] This is sometimes referred to in relation to the so-called "pseudounitarity" of the S matrix. "Pseudounitarity" simply means unitarity with respect to the indefinite inner product. Since the concept of unitarity is undefined without specifying the inner product, the terminology is mathematically misleading, but common in the literature.
[13] E. C. G. Sudarshan, Phys. Rev. **123**, 2183 (1961); see also E. C. G. Sudarshan, in *1961 Brandeis Summer Lectures, Vol. 2*, (W. A. Benjamin, Inc., New York, 1962). For a comprehensive discussion, see G. Barton, *Introduction to Advanced Field Theory* (Interscience Publishers, Inc., New York, 1963), Chap. 12.

method of constructing a theory that is equivalent to a nonlocal relativistic theory (with a positive-definite metric). In particular, using the example of the Gupta-Bleuler and Dirac-Schwinger forms of treating the electromagnetic field (as well as some simple models[13,14]), it is suggested that if one requires (or prefers) manifest covariance, negative probabilities may be inevitable. Furthermore, this prospect is not viewed with alarm, but a general invariant method of introducing a subsidiary condition is outlined, which defines a 'physical' subspace capable of probabilistic interpretation. (All states in this subspace have positive norm.) A general scheme for the construction of a finite covariant theory of interacting fields is presented[15] in which the interaction involves a linear superposition of 'normal' fields (usual-sign commutation relations) and "auxiliary" fields ("wrong"-sign commutation relations) locally coupled in such a fashion as to eliminate divergent integrals from the perturbation expansion. In general, physical scattering states would be defined by a subspace consisting of those eigenstates of the S matrix with positive norm (determined exactly or approximately).

In this paper, we present such an indefinite metric Lagrangian formulation of quantum electrodynamics.[16] The purpose is to determine, by a detailed examination, if such a theory as discussed above can be carried out consistently. Quantum electrodynamics has been chosen since there already exists a generally successful conventional (Lagrangian) theory, whose solution by a perturbation expansion is marred by the appearance of divergent integrals. We must show that we can eliminate these divergences and still make predictions in agreement with experimental knowledge.

In addition to a normal fermion field (electron) and electromagnetic field, the formulation involves two auxiliary fermion fields with different masses. These fields are quantized using an indefinite metric, and are so coupled to the electromagnetic field that no ultraviolet divergences appear. The two extra mass parameters are then restricted in their values by comparison of predictions with experiment, and by the requirement of self-consistency for the perturbation expansion. Since the theory is finite, all renormalizations involve finite, analytic functions of bare parameters, and thus bare and "physical" quantities can be related to one another. In Secs. II and III, we construct the Lagrangian formalism and appropriate perturbation expansion, and discuss some new aspects of the fermion and electromagnetic fields. The mass and charge renormalizations are carried out in Sec. IV, in which we examine the allowed ranges for the two auxiliary masses in order that the perturbation expansion be self-consistent (i.e., that the higher order corrections be truly small). In Secs. V and VI, calculations are made for the anomalous magnetic moment of the electron and for the cross sections for Compton scattering. In Sec. VII, we summarize the basic conditions on our auxiliary masses, and make a few extra remarks. Some theoretical aspects of the present model are considered in Sec. VIII. Appendix A presents the method used for quantizing the electromagnetic field, and Appendix B exhibits a formal technique for mass renormalization.

II. THE FERMION FIELDS

The conventional theory of quantum electrodynamics contains a Lagrangian density for the free fermion field given by[17]

$$\mathcal{L}_F=-\tfrac{1}{2}\bar{\psi}(-i\gamma^\mu\overleftrightarrow{\partial}_\mu)\psi-m\bar{\psi}\psi\,, \qquad \text{(II.1)}$$

where $\bar{\psi}\overleftrightarrow{\partial}_\mu\psi=\bar{\psi}(\partial_\mu\psi)-(\partial_\mu\bar{\psi})\psi$, with the anticommutation relations for the free field as

$$\{\psi(x),\bar{\psi}(x')\}=-iS(x-x')\,, \qquad \text{(II.2)}$$

where $S(x-x')$ is the usual invariant function. To this are then added the Lagrangian density for the free electromagnetic field and an interaction Lagrangian density,

$$\mathcal{L}_I=e\bar{\psi}\gamma^\mu\psi A_\mu\,. \qquad \text{(II.3)}$$

The theory is then solved by a perturbation expansion of the S matrix in the interaction picture with the propagator for the fermions in momentum space given by[18]

$$S(p)=(\not{p}-m+i\epsilon)^{-1}\,. \qquad \text{(II.4)}$$

We now construct a scheme of coupled fermion fields to yield a manifestly covariant and convergent quantum electrodynamics. The basic program is as follows: We construct a simple manifestly covariant Lagrangian density in terms of a "normal" fermion field with a local coupling, i.e., the Lagrangian density give in (II.1) and (II.3); with the 'normal' fermion field we associate two "auxiliary" fermion fields with all the quantum numbers the same except the mass and satisfying the "wrong"-sign anticommutation relations; we then couple a linear superposition of the 'normal' fermion field with unit weight and the two 'auxiliary' fermion fields with arbitrary weights (to be determined later by the convergence conditions) to the electromagnetic field.

In particular, the free Lagrangian density (II.1) is

[14] H. J. Schnitzer and E. C. G. Sudarshan, Phys. Rev. **123**, 2193 (1961).

[15] Reference 13, Sec. 7.

[16] Such a theory has been outlined for the four-fermion coupling in weak interactions by one of us (E. C. G. S.) [Nuovo Cimento **21**, 7 (1961)].

[17] We use the natural unit system, $\hbar=c=1$, and the Lorentz metric $g_{00}=-g_{11}=-g_{22}=-g_{33}=+1$. Strictly, we should further antisymmetrize these bilinear expressions; we shall not explicitly indicate this antisymmetrization.

[18] This propagator differs from the usual expression [e.g., S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Inc., Evanston, Illinois, 1961)] by a factor of $i/(2\pi)^4$; we shall use this definition throughout the paper. [Note that now $S(p)$ is the Fourier transform $S(x)$ only to within a multiplicative constant.]

generalized to[19]

$$\mathcal{L}_F=\sum_{i=1}^{3} N_i[-\tfrac{1}{2}\bar{\psi}_i(-i\gamma^\mu\overleftrightarrow{\partial}_\mu)\psi_i-m_i\bar{\psi}_i\psi_i], \quad \text{(II.5)}$$

with $N_1=-N_2=-N_3=+1$, where ψ_1 is the normal fermion field, and ψ_2 and ψ_3 are the two auxiliary fermion fields. These three fields satisfy the anticommutation relations

$$\{\psi_i(x),\bar{\psi}_j(x')\}=-iN_i\delta_{ij}S(x-x';m_i). \quad \text{(II.6)}$$

The interaction Lagrangian density is generalized to

$$\mathcal{L}_I=e\bar{\Psi}\gamma^\mu\Psi A_\mu, \quad \text{(II.7)}$$

with $\Psi=\psi_1+c_2'\psi_2+c_3'\psi_3$.

The indefinite sign for the anticommutator shows that the metric is not positive definite, and thus an indefinite metric is required for proper quantization. The appropriate metric is given by[20]

$$\eta=\exp\left[i\pi\int d^3x(\psi_2^\dagger\psi_2+\psi_3^\dagger\psi_3)\right].$$

Here $\psi^\dagger$ is the adjoint[21] and $\bar{\psi}$ the Dirac adjoint of ψ. From the anticommutation relations (II.6), it follows that the propagators of the fermion fields in momentum space are given by

$$S_i(p)=N_i(p\!\!\!/-m_i+i\epsilon)^{-1}. \quad \text{(II.8)}$$

The rules for the perturbation expansion of the S matrix in the interaction picture appropriate to the new Lagrangian can then be constructed in the usual fashion.[22] We see that for every normal fermion line in a Feynman diagram, we have the same diagram with a fermion line corresponding to each auxiliary field in place of it. In particular, since the internal line comes from the contraction $:\psi_i\bar{\psi}_i:$ in the Wick expansion, we can absorb the factors $|c_2'|^2$ and $|c_3'|^2$ into the internal lines from fields ψ_2 and ψ_3, leaving the vertices identical. Then, we can write one diagram with an effective propagator for each internal fermion line to represent all diagrams with the three different kinds of internal lines. The effective fermion propagator is then[23]

$$S(p)=(p\!\!\!/-m_1+i\epsilon)^{-1}-|c_2'|^2(p\!\!\!/-m_2+i\epsilon)^{-1} -|c_3'|^2(p\!\!\!/-m_3+i\epsilon)^{-1}, \quad \text{(II.9)}$$

or defining

$$c_1\equiv 1, \quad c_2\equiv -|c_2'|^2 \quad \text{and} \quad c_3\equiv -|c_3'|^2,$$

$$S(p)=\sum_{i=1}^{3} c_i(p\!\!\!/-m_i+i\epsilon)^{-1}. \quad \text{(II.10)}$$

In order to remove the ultraviolet divergences in quantum electrodynamics, it is sufficient to make the effective propagator decrease as fast as p^{-3} for large values of the momentum. It is readily seen that this can be achieved by imposing two conditions on the c parameters. The two conditions are[24]

$$\textstyle\sum_i c_i=0 \quad \text{and} \quad \sum_i c_i m_i=0. \quad \text{(II.11)}$$

Thus, with $c_1=1$,[25]

$$c_2=\frac{m_1-m_3}{m_3-m_2}, \quad c_3=\frac{m_1-m_2}{m_2-m_3}. \quad \text{(II.12)}$$

Using (II.12), the effective fermion propagator can be alternatively written in a form which shows the desired asymptotic behavior, i.e.,

$$S(p)=\frac{(m_1-m_2)(m_1-m_3)}{\prod_i(p\!\!\!/-m_i+i\epsilon)}. \quad \text{(II.13)}$$

We now simply prove that the effective fermion propagator is sufficient to insure convergence of all integrals in the theory. As Dyson[26] has done for primitive divergences, an integral corresponding to a general diagram can be written as

$$M=\int\frac{N}{D}d^4k_1\cdots d^4k_r, \quad \text{(II.14)}$$

where r is the number of independent internal four-momenta, given by the relation

$$r=F_i+P_i-(n-1). \quad \text{(II.15)}$$

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[19] The negative sign of N_2 and N_3 then yields the desired "wrong"-sign anticommutation relations (II.6) for ψ_2 and ψ_3 after canonical quantization is carried out.

[20] Compare, for example, S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).

[21] The term adjoint here is with respect to the indefinite metric; the term "pseudo-Hermitian" adjoint is frequently used in the literature.

[22] A similar procedure was first elaborated in a previous paper, [E. C. G. Sudarshan, Nuovo Cimento **21**, 7 (1961)], for a theory of leptons using an indefinite metric; a similar formulation has also been used by H. Hofer, doctoral dissertation, University of Bern, 1964 (unpublished), for computing higher order corrections to muon decay.

[23] Hereafter, unless otherwise specified, the $S(p)$ without index i will denote the *effective* fermion propagator. We note here the fact that as $m_2, m_3\to\infty$, $S(p)$ becomes just the usual propagator for the electron and the theory is identical with conventional quantum electrodynamics. This is a convenient check on many of our calculations.

[24] We may note that formally this procedure resembles the Pauli-Villars regularization (see Sec. I). Although we have pointed out that the indefinite metric is involved in that technique, the approach here is quite different. In the Pauli-Villars case, all divergent integrals are separately regularized and the masses tend to infinity at the end with no attempt to construct a physically interpretable theory.

[25] We note the constants c_2 and c_3 do not have to be negative (even though they have been set equal to $-|c_2'|^2$ and $-|c_3'|^2$). Depending on our choice of m_2 and m_3, we can go back and adjust the sign of the anticommutation relations for one of the auxiliary fields to avoid any inconsistency. In this case, only one of the auxiliary fields may involve an indefinite metric, but since we deal only with the effective propagators, this has no effect on the equations in this paper.

[26] F. Dyson, Phys. Rev. **75**, 1736 (1949).

Here F_i and F_e are the number of fermion internal and external lines, P_i and P_e are the number of photon internal and external lines and n is the number of corners. The numerator N has no powers of k and the denominator D has 2 powers for each photon propagator and 3 for each fermion propagator (as opposed to 1 in the usual theory). Thus

$$D=3F_i+2P_i. \qquad \text{(II.16)}$$

For the diagram to converge, we must have

$$4r+N<D,$$

or

$$F_i+2P_i+4<4n, \qquad \text{(II.17)}$$

which, using $n=F_i+\frac{1}{2}F_e=2P_i+P_e$, becomes

$$2n+P_e+\tfrac{1}{2}F_e>4. \qquad \text{(II.18)}$$

Since we must have at least two corners in order to have an integration, $n\geqslant 2$ gives

$$P_e+\tfrac{1}{2}F_e>0, \qquad \text{(II.19)}$$

which is true for all diagrams except vacuum diagrams (of no interest in the present context), so the elimination of the ultraviolet divergence for any Feynman diagram is proved.

III. THE ELECTROMAGNETIC FIELD

In this section, we shall discuss some aspects of the electromagnetic field, $A_\mu(x)$, which is coupled to the fermion fields via the interaction Lagrangian (II.7). The basic problem is to insure that the field, $A_\mu(x)$, interacting with the fermion fields will actually represent photons as the observable physical quanta, i.e., zero-mass particles of spin one, polarized perpendicular to their space momenta. In the conventional quantum electrodynamics, the free electromagnetic field is quantized as a vector field of zero mass and the use of gauge invariance, which implies a conserved current interacting with the electromagnetic field, is sufficient to insure the zero mass and transversality of the physical photons, provided perturbation theory is applicable.[27]

In the present version of the theory, however, the form of the interaction Lagrangian is such that it does not have the property of gauge invariance, and the conserved current is not the one that interacts with the electromagnetic field. It is easily seen that the total Lagrangian, $\mathcal{L}_F+\mathcal{L}_E+\mathcal{L}_I$,[28] is invariant under the gauge transformation of the first kind:

$$\psi_i\rightarrow e^{i\alpha}\psi_i, \quad i=1,2,3, \qquad \text{(III.1)}$$

which implies that there is a conserved current,

$$j^\mu=i\alpha\sum_{i=1}^{3}\left(\frac{\partial\mathcal{L}_F}{\partial(\partial_\mu\psi_i)}\psi_i-\bar{\psi}_i\frac{\partial\mathcal{L}_F}{\partial(\partial_\mu\bar{\psi}_i)}\right). \qquad \text{(III.2)}$$

Substituting (II.5) into (III.2), the conserved current is

$$j^\mu=-\alpha\textstyle\sum_i N_i\bar{\psi}_i\gamma^\mu\psi_i, \qquad \text{(III.3)}$$

which is clearly not the current j_μ' coupled to the electromagnetic field in (II.7):

$$j_\mu'=e\bar{\Psi}\gamma_\mu\Psi. \qquad \text{(III.4)}$$

The conserved current j^μ corresponds to 'fermion' conservation in the usual way. The fact that the interacting current is j_μ' implies the noninvariance of $\mathcal{L}_I$ under the gauge transformation of the second kind.

This means that, in general, the physical photon mass will be different from the bare mass, and there is no *a priori* reason why only the transverse polarizations will contribute in a physical scattering process. In contradistinction to the conventional quantum electrodynamics, therefore, the zero mass and transverse polarization properties of the photons must be separately incorporated into the theory. We now assert that we can obtain the desired physical quanta if the bare photon propagator can be written in the true Landau gauge[29] with a given bare mass μ chosen so that the physical mass (pole of the dressed photon propagator) is zero, i.e.,

$$D_{\mu\nu}(k)=(g_{\mu\nu}-k_\mu k_\nu k^{-2})(k^2-\mu^2+i\epsilon)^{-1}. \qquad \text{(III.5)}$$

The most important property of this propagator is that it is transverse in the covariant sense, that is, perpendicular to k_μ.[30] It is to be noted that the above propagator is different from the one corresponding to the usual transverse neutral vector field of mass μ given by

$$\bar{D}_{\mu\nu}=(g_{\mu\nu}-k_\mu k_\nu\mu^{-2})(k^2-\mu^2)^{-1}. \qquad \text{(III.6)}$$

Now (III.6) can be written as

$$\bar{D}_{\mu\nu}=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2-\mu^2}-\frac{k_\mu k_\nu}{\mu^2 k^2}, \qquad \text{(III.7)}$$

and the last term seems to correspond to the propagator of a longitudinal field of zero mass. Thus, if one can superimpose such a field on the usual massive vector field, it will lead to the $D_{\mu\nu}$ in (III.5). A method of quantization of the electromagnetic field leading to the true Landau gauge (III.5) and the proof of its covariant transversality will be given in Appendix A.

We now assume the bare photon propagator is given

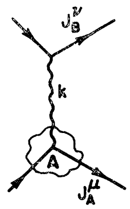
[27] See, for instance, F. C. Khanna and F. Rohrlich, Phys. Rev. **131**, 2721 (1963). For a further discussion of this point, see J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[28] $\mathcal{L}_E$, the free electromagnetic Lagrangian given in Appendix A, is, of course, independent of the ψ_i's and is irrelevant here.

[29] The term Landau gauge is used to refer to a propagator in momentum space proportional to $g_{\mu\nu}-k_\mu k_\nu/k^2$. The term Feynman gauge refers to a propagator proportional to $g_{\mu\nu}$.

[30] To avoid confusion in the ensuing work, we shall adopt the following notation. A vector a_μ is (a) transverse if $k^\mu a_\mu=0$; (b) longitudinal if $a_\mu\sim k_\mu$; (c) perpendicular if $a_0=0$, $\mathbf{a}\cdot\mathbf{k}=0$; (d) 3 longitudinal if $\mathbf{a}\sim\mathbf{k}$; and (e) scalar if $\mathbf{a}=0$.

FIG. 1. Photon detection.



by (III.5). The dressed photon propagator $D_{\mu\nu}'$ is as in the usual theory

$$D_{\mu\nu}'=D_{\mu\nu}+D_{\mu\lambda}\Pi^{\lambda\sigma}D_{\sigma\nu}', \qquad \text{(III.8)}$$

where $\Pi^{\lambda\sigma}$ is the sum of all proper photon self-energy diagrams having a general form

$$\Pi_{\mu\nu}=g_{\mu\nu}C(k^2)+k_\mu k_\nu C'(k^2). \qquad \text{(III.9)}$$

Here, $C(k^2)$ and $C'(k^2)$ are general functions of k^2 which, in the conventional theory, are related to each other by the requirement of gauge invariance. In this case, there is no particular relationship between them. From (III.5) and (III.9), we have

$$D_{\mu\nu}'=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)\frac{1}{k^2-\mu^2-C(k^2)+i\epsilon'}, \qquad \text{(III.10)}$$

i.e., the $k_\mu k_\nu$ part of $\Pi_{\mu\nu}$ drops out and the propagator has retained its transverse form. Now since the pole of the dressed propagator corresponds to the physical mass, we can insure that the physical photon has zero mass by requiring that

$$\mu^2=-C(k^2=0), \qquad \text{(III.11)}$$

which determines the parameter μ^2.

Next, we consider the detection of physical photons. Following Feynman,[31] we note that any photon that is detected will actually be absorbed in the process of detection. We consider a case in which a photon is emitted by some process A and is absorbed by an electron (fermion of type 1) as shown in Fig. 1.

The amplitude for the process is proportional to M where

$$M=J_A{}^\mu(k)D_{\mu\nu}'(j)J_B{}^\nu(k), \qquad \text{(III.12)}$$

and $J_A{}^\mu$ is the current that emits the photon of momentum k and $J_B{}^\nu$ is the electron current which absorbs it.[32] Due to the transversality of $D_{\mu\nu}'$, only the pure transverse projection of $J_A{}^\mu$ remains, i.e.,

$$J_A{}^\mu D_{\mu\nu}'=J_A{}^{\mu(\tau)}D_{\mu\nu}', \qquad \text{(III.13)}$$

where $J_A{}^{\mu(\tau)}$ is the transverse part of $J_A{}^\mu$ and using (III.10), (III.12) becomes

$$M=g_{\mu\nu}J_A{}^{\mu(\tau)}J_B{}^\nu(k^2-\mu^2-C(k^2)+i\epsilon')^{-1}. \qquad \text{(III.14)}$$

[31] R. P. Feynman, *The Theory of Fundamental Processes* (W. A. Benjamin, Inc., New York, 1961), p. 95.

[32] Note that $J_A{}^\mu$ which is the general interacting current in our theory is not conserved, but that $J_B{}^\mu$, which is just an electron without other interactions, is conserved (this fact is not necessary to the proof).

Now we can apply Feynman's argument directly. Using $k^2+i\epsilon$ for the denominator since we are interested in M as $k^2\to 0$ and since $\mu^2=-C(0)$, (III.14) can be written as

$$M=\frac{J_A{}^{0(\tau)}J_B{}^0}{k^2}-\frac{J_A{}^{1(\tau)}J_B{}^1}{k^2}-\frac{J_A{}^{2(\tau)}J_B{}^2}{k^2}-\frac{J_A{}^{3(\tau)}J_B{}^3}{k^2}. \qquad \text{(III.15)}$$

If we choose the coordinate system where $\mathbf{k}$ is along the 3 axis, we can write

$$J_A{}^{1(\tau)}J_B{}^1+J_A{}^{2(\tau)}J_B{}^2=\sum_{i=1}^{2}(\mathbf{J}_A{}^{(\tau)}\cdot\boldsymbol{\varepsilon}^{(i)})(\mathbf{J}_B\cdot\boldsymbol{\varepsilon}^{(i)}), \qquad \text{(III.16)}$$

where $\epsilon_\mu{}^{(1)}$ and $\epsilon_\mu{}^{(2)}$ are the two perpendicular polarization vectors referred to in Ref. 55. The transverse conditions of $J_A{}^{(\tau)}$ and J_B in the coordinate system chosen, i.e.,

$$k_0J_A{}^{(\tau)0}-k_3J_A{}^{(\tau)3}=0,\qquad k_0J_B{}^0-k_3J_B{}^3=0, \qquad \text{(III.17)}$$

gives the relation

$$J_A{}^{3(\tau)}J_B{}^3=(k_0{}^2/k_3{}^2)J_A{}^{0(\tau)}J_B{}^0. \qquad \text{(III.18)}$$

Using (III.16) and (III.18), (III.15) becomes

$$M=-\frac{J_A{}^{0(\tau)}J_B{}^0}{k_3{}^2}-\sum_{i=1}^{2}\frac{(\mathbf{J}_A{}^{(\tau)}\cdot\boldsymbol{\varepsilon}^{(i)})(\mathbf{J}_B\cdot\boldsymbol{\varepsilon}^{(i)})}{k^2}, \qquad \text{(III.19)}$$

which shows clearly that, as we approach the mass shell (as $k^2\to 0$), only the transverse part of (III.19) has a pole and contributes to the amplitude, so that the only real photons are those with perpendicular polarization. Thus, by what has been stated in this section and in Appendix A, we have demonstrated that the physical photons in the present theory have the desired properties, namely, zero physical mass and transverse polarization.

IV. RENORMALIZATIONS AND THE CONSISTENCY OF PERTURBATION EXPANSION

The seven parameters that originally enter the present formulation are the bare coupling constant, photon bare mass, three fermion masses and two weight factors for the auxiliary fermion fields. The last two, namely, c_2 and c_3, are determined by the two regulating conditions as functions of the fermion masses as given in (II.12), and the photon bare mass μ^2 is fixed by (III.11) to give zero mass for physical photons, leaving four parameters e, m_1, m_2, and m_3 thus far undetermined. The observed mass and charge of a physical fermion particle (electron) supply two conditions which may be considered to determine the corresponding bare parameters m_1 and e via the mass and charge renormalizations. In conventional quantum electrodynamics such renormalization procedures are inseparably associated with the subtraction of infinites and, as such, do not "determine" bare parameters, but simply "re-express" them in terms of the observed value. In the present theory, all renormalizations are finite functions of bare parameters and will involve only finite shifts between bare and renormalized parameters.

We have then two parameters, m_2 and m_3, at our disposal. Allowed values (or ranges) of these parameters can then be determined by comparing the predictions of the theory with experimental data, such as the anomalous magnetic moment of the electron and Compton scattering. However, since all radiative corrections are now finite functions of bare parameters, we can first calculate some of these higher order corrections and look for possible ranges of values for m_2 and m_3 for which the higher order corrections are small. This test of the self-consistency of the perturbation expansion will then lead to some preliminary determination of m_2 and m_3.

A. Mass Renormalization

Introducing a slightly more general notation, let us write the effective fermion propagator (II.10) as the following:

$$S(p)=\sum_{i,j=1}^{3} S_{ij}(p)\,, \tag{IV.1}$$

where S_{ij} stands for the bare propagator between two vertices where particle i leaves one vertex and particle j arrives at the other and is defined by

$$S_{ij}=\delta_{ij}S(i) \tag{IV.2}$$

and

$$S(i)=c_i(p-m_i+i\epsilon)^{-1}\,. \tag{IV.3}$$

Similarly, the effective dressed fermion propagator is defined by

$$S'=\sum_{i,j=1}^{3} S_{ij}'\,, \tag{IV.4}$$

where S_{ij}' is the dressed propagator consisting of particle i leaving and of particle j arriving and is given by

$$S_{ij}'=S_{ij}+\sum_{l,k=1}^{3} S_{il}\Sigma_{lk}S_{kj}' \tag{IV.5}$$

and Σ_{lk} is the sum of proper fermion self-energy diagrams connecting a propagator for a fermion of the type l to one of type k, as shown in Fig. 2. Since, however, Σ_{lk} is actually independent of k and l, (IV.4) becomes

$$S'=\sum_{i,j} S_{ij}+\sum_{i,j,k,l} S_{il}\Sigma S_{kj}'$$
$$=S+S\Sigma S'\,, \tag{IV.6}$$

or

$$S'^{-1}=S^{-1}-\Sigma\,. \tag{IV.7}$$

A formal method of mass renormalization is described in Appendix B for the simple case of a single fermion field. A technique of Taylor expansion in the mass parameter is used to show the complete equivalence between a perturbation expansion using bare masses (without the counter term) and one using renormalized masses (with the counter term). We may then effect counter-term mass renormalization in our case by writing the perturbation expansion in terms of bare masses and then making a Taylor expansion of all diagrams in the mass about the physical mass. For generality, we shall outline the expansion of the dressed propagator in a Taylor series in all three masses simultaneously, although for the purpose of calculation only the expansion in m_1 will be used.

As in Appendix B, the dressed propagator (IV.7) is expanded in Taylor series about the renormalized masses m_{i0} defined as zeros of the inverse dressed propagator:

$$S'^{-1}=0 \quad \text{for} \quad p=m_{10},\, m_{20},\, m_{30}\,, \tag{IV.8}$$

where m_{10} corresponds to physical electron mass and m_{20} and m_{30} have no direct physical significance and can have complex values. We have

$$S'^{-1}=S^{-1}(p;\, m_1,m_2,m_3)-\Sigma(p;\, m_1,m_2,m_3)$$
$$=S^{-1}(p;\, m_{10},m_{20},m_{30})+\sum_{i=1}^{3}(m_i-m_{i0})\frac{\partial S^{-1}}{\partial m_{i0}}+\cdots$$
$$-\Sigma(p;\, m_{10},m_{20},m_{30})-\sum_{i=1}^{3}(m_i-m_{i0})\frac{\partial\Sigma}{\partial m_{i0}}+\cdots. \tag{IV.9}$$

Since $S^{-1}(p;\, m_{i0})=0$ for $p=m_{10},\, m_{20},\, m_{30}$, (IV.8) becomes

$$\sum_{i=1}^{3}(-\delta m_i)\frac{\partial S^{-1}}{\partial m_{i0}}+\frac{1}{2!}\sum_{i,j=1}^{3}\delta m_i\delta m_j\frac{\partial^2 S^{-1}}{\partial m_{i0}\partial m_{j0}}+\cdots$$
$$-\Sigma(p;\, m_{10},m_{20},m_{30})+\sum_{i=1}^{3}\delta m_i\frac{\partial\Sigma}{\partial m_{i0}}+\cdots=0\,, \tag{IV.10}$$

for $p=m_{10},\, m_{20},\, m_{30}$, where $\delta m_i=m_{i0}-m_i$. From (IV.10) and (B17) we find in the lowest order:

$$\delta m_1^{(1)}=\Sigma^{(1)}(p=m_{10};\, m_{10},\, m_{20},\, m_{30})\,, \tag{IV.11a}$$

$$\delta m_2^{(1)}=-\frac{(m_{10}-m_{30})}{(m_{20}-m_{30})}\times\Sigma^{(1)}(p=m_{20};\, m_{10},\, m_{20},\, m_{30})\,, \tag{IV.11b}$$

$$\delta m_3^{(1)}=-\frac{(m_{10}-m_{20})}{(m_{30}-m_{20})}\times\Sigma^{(1)}(p=m_{30};\, m_{10},\, m_{20},\, m_{30})\,. \tag{IV.11c}$$

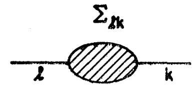


FIG. 2. Σ_{lk}.

Since only m_{10} (the electron physical mass) is input, it will be convenient to use the above procedure only for the normal fermion field (electron) and use the bare masses m_2 and m_3 for the auxiliary fields. Thus, we shall use the physical electron mass which we now call m_0 (rather than m_{10}, for convenience) in all subsequent expressions, and all terms dictated by the Taylor expansion in m_1 must be used to the order (in the coupling constant) desired. To the lowest order, we calculate (IV.11a) with

$$\delta m_1 = \sum(p=m_0;\, m_0,\, m_2,\, m_3)\,, \qquad \text{(IV.12)}$$

where the superscript (1), signifying lowest order in α, has been suppressed.

The second-order electron self-energy function is given by

$$\Sigma(p) = -\frac{i\alpha_0}{4\pi^3}\int d^4k\, \gamma^\mu S(p-k)\gamma^\nu D_{\mu\nu}(k)\,, \qquad \text{(IV.13)}$$

where $\alpha_0 = e_0{}^2/4\pi$ (we use the renormalized value of 1/137; see charge renormalization below). In $S(p-k)$, m_1 is replaced by m_0, but m_2 and m_3 are retained, and $D_{\mu\nu}(k)$ is as given in (III.5) but to this order we use the renormalized photon mass of zero. Evaluating on the mass shell, we have a finite expression for $\Sigma(p)$, namely,

$$\Sigma(p=m_0) = (3\alpha_0/4\pi)m_0 \sum_i c_i \Sigma_i\,, \qquad \text{(IV.14a)}$$

with

$$\Sigma_1 = 0\,,$$

$$\Sigma_2 = \rho^3 \ln\left(\frac{\rho^2-1}{\rho^2}\right) - \rho\ln(\rho^2-1)\,, \qquad \text{(IV.14b)}$$

$$\Sigma_3 = \Sigma_2(\rho\to\sigma)\,,$$

where

$$\rho = m_2/m_0 \quad \text{and} \quad \sigma = m_3/m_0\,. \qquad \text{(IV.15)}$$

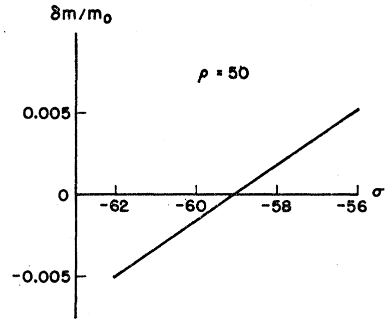


FIG. 3. $\delta m/m_0$ near $\rho=50$, $\sigma=-50$.

From (IV.12), we have

$$\delta m_1/m_0 = (m_0 - m_1)/m_0 = \Sigma(p=m_0)/m_0\,, \qquad \text{(IV.16)}$$

and

$$m_1/m_0 = 1 - (\Sigma(p=m_0)/m_0)\,. \qquad \text{(IV.17)}$$

Computation of $\delta m_1/m_0$ and m_1/m_0 for various values of ρ and σ is carried out with an IBM 7074. Since all expressions are symmetric under the interchange of ρ and σ, we confine ourselves to values for which $\rho-\sigma \geq 0$. Some values of $\delta m_1/m_0$ are shown in Table I. The table is just presented to give an idea of how $\delta m_1/m_0$ varies in the ρ-σ plane.

We see that along the $\rho=-\sigma$ line, it grows less fast than elsewhere and it changes sign as we cross the $\rho=-\sigma$ line, passing through zero somewhere below that line. This may be seen more explicitly if we write the expression for $\delta m_1/m_0$ for $\rho=+\sigma$ and $\rho=-\sigma$ when $|\rho|$ is large (>20):

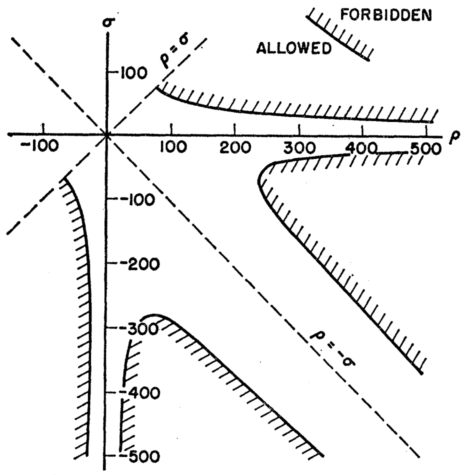
$$\rho=+\sigma,\ |\rho|>20:\quad \delta m_1/m_0 = (3\alpha_0/4\pi) \times(3-2\rho-1/\rho+\ln\rho^2) \qquad \text{(IV.16a)}$$

$$\rho=-\sigma,\ |\rho|>20:\quad \delta m_1/m_0 = (3\alpha_0/4\pi)(1+\ln\rho^2)\,.$$

We see that for $\rho=+\sigma$ the dominant term is linear in ρ, but for $\rho=-\sigma$ it is only logarithmic in ρ. Figure 3 gives a closer view of $\delta m_1/m_0$ in a typical neighborhood of this $\rho=-\sigma$ line.

TABLE I. $\delta m/m_0$ in ρ-σ plane. (Note: Because of ρ-σ symmetry, the values in the upper left have been left out as they would only duplicate those given above.)

σ \ ρ	−1000	−500	−200	−100	−40	−5	5	40	100	200	500	1000
1000												−3.46
500											−1.72	−2.39
200										−0.673	−1.04	−1.38
100									−0.327	−0.461	−0.676	−0.865
40								−0.121	−0.193	−0.259	−0.386	−0.441
5							−0.007	−0.026	−0.037	−0.046	−0.058	−0.067
−5						0.029	0.007	0.046	0.067	0.083	0.103	0.118
−40					0.158	0.057	−0.019	0.015	0.108	0.206	0.349	0.457
−100				0.370	0.233	0.073	−0.033	−0.074	0.018	0.181	0.490	0.755
−200			0.721	0.506	0.302	0.087	−0.043	−0.167	−0.142	0.020	0.479	0.960
−500		1.77	1.09	0.726	0.407	0.105	−0.053	−0.303	−0.445	−0.434	0.023	0.830
−1000	3.51	2.44	1.43	0.918	0.494	0.119	−0.066	−0.406	−0.704	−0.910	−0.780	0.026



(NOTE: THERE IS SYMMETRY ABOUT THE ρ=σ LINE)

FIG. 4. Allowed region for $\delta m/m_0$ in ρ-σ plane.

We can apply various criteria for $\delta m_1/m_0$. First, we cannot allow it to be greater than unity if we are to have any confidence in our perturbation expansion.[33] In fact, if we require $\delta m_1/m_0<\frac{1}{4}$, the expansion is somewhat more acceptable. In a heuristic way, we would hope that if the lowest order term in $\delta m_1/m_0$ is of this magnitude or less, the succeeding terms should contribute less than 10% to m_0. (We realize, of course, that it is possible to make $\delta m_1/m_0$ much smaller than $\frac{1}{4}$.) In Fig. 4 we show the approximate region in the ρ-σ plane for which $|\delta m_1/m_0|<\frac{1}{4}$. It still allows considerable variation of ρ and σ.

We may consider more strict criteria. If we unrealistically assume that the proton-neutron mass difference is such an electromagnetic self-energy effect,[34] then

$$\delta m_1/m_0=\frac{m_0-m_1}{m_0}=\frac{m_p-m_n}{m_p}=-0.0014. \quad \text{(IV.18)}$$

From the Table I we see that there is a whole line (just below $\rho=-\sigma$ line) for which this is satisfied. Some typical values are given in Table II.

[33] Of course, in the usual theory, δm is infinite and the series is considered to be somehow legitimate as a rearranged asymptotic expansion. [See S. Schweber, *Introduction to Relative Quantum Field Theory* (Row Peterson Inc., Evanston, Illinois, 1961), p. 644, and F. Dyson, Phys. Rev. **85**, 631 (1952).] We could allow $\delta m_1/m_0$ to be greater than unity and hope that the formal expansion is still useful. However, since the primary purpose of this model is to remove the divergences, our spirit is to avoid these arguments, if possible. We still have no proof of the convergence of the perturbation expansion, but we can avoid any explicit contradiction to its summability.

[34] Of course, both the neutron and the proton have effectively smeared-out charge distributions, and their mass difference will depend on strong interaction effects nontrivially through the form factors. Thus (IV.18) and Table II are not to be taken very seriously.

TABLE II. Typical values (approximate) of ρ and σ for which $m/m_0=m_{\text{neutron}}/m_{\text{proton}}$.

ρ	σ
+50	−60
100	−111.1
500	−514.3
1000	−1015.6

B. Charge Renormalization

In the usual theory, the bare coupling constant is a meaningless quantity, owing to the infinities encountered in renormalization, and the perturbation expansion *must* be in terms of the renormalized coupling constant. In the present theory, since all renormalizations are finite, it is our approach that the perturbation expansion is originally in the bare coupling constant, with the option of readjusting the series so as to use the renormalized coupling constant. We consider the renormalized coupling constant (=1/137) to be input from which the bare coupling constant can be determined. For any process calculated to a particular order, we may expand in the bare charge and include all diagrams that contribute to that order, or alternatively, we may expand in the renormalized constant, leaving out those terms that have already contributed to charge renormalization. Because we insist that both approaches must lead to compatible results (i.e., equal to within the order of α involved), a condition on our model is that the bare charge must also be small, and, in fact, close in value to the renormalized charge. To determine the effect of this condition, we briefly summarize the different contributions to charge renormalization (to lowest order).

The contribution of the fermion self-energy to charge renormalization is defined, as in the usual theory, as the residue of the pole in the dressed propagator at the physical mass, i.e.,

$$S'=\frac{Z_2}{p-m_0}+\text{(terms which are regular at } p=m_0). \quad \text{(IV.19)}$$

Thus, near $p=m_0$,

$$S'^{-1}\cong Z_2^{-1}(p-m_0). \quad \text{(IV.20)}$$

Expanding S^{-1} and Σ of (IV.7) about $p=m_0$ and comparing with (IV.20), the expression of Z_2 to lowest order is shown to be[35]

$$Z_2=1+B+2\delta m_1\left(\frac{1}{m_0-m_2}+\frac{1}{m_0-m_3}\right), \quad \text{(IV.21)}$$

[35] M. E. Arons, thesis, University of Rochester, 1964 (unpublished).

where δm_1 is as given in (IV.16) and B is given by[36]

$$B=(3\alpha_0/2\pi)\sum_{i=1}^{3} c_iB_i \tag{IV.22}$$

with

$$B_1=\int_0^1 \frac{dx}{x},$$

$$B_2=-\rho^3 \ln[(\rho^2-1)/\rho^2],$$

$$B_3=B_2(\rho\to\sigma),$$

or, for large values of ρ and σ (leaving out the infrared term), we find[37]

$$B=(3\alpha_0/2\pi)[-1-\tfrac{1}{2}(1/\rho+1/\sigma-1/\rho\sigma)]. \tag{IV.23}$$

The vertex function is

$$\Gamma_\mu(p_2,p_1)=\gamma_\mu+\Lambda_\mu(p_2,p_1), \tag{IV.24}$$

where the Λ_μ function is given (to second order) by

$$\Lambda_\mu(p_2,p_1)=-\frac{i\alpha_0}{4\pi^3}\int d^4k\, \gamma^\rho S(p_2-k) \times\gamma_\mu S(p_1-k)\gamma^\sigma D_{\rho\sigma}(k). \tag{IV.25}$$

If we write

$$\Lambda_\mu(p_2,p_1)=\gamma_\mu L+\Lambda_{\mu c}(p_2,p_1), \tag{IV.26}$$

where $\Lambda_{\mu c}\to 0$ as $p_2=p_1=m_0$, then, on the mass shell,

$$\Gamma_\mu=Z_1^{-1}\gamma_\mu, \tag{IV.27}$$

with

$$Z_1^{-1}=1+L. \tag{IV.28}$$

The expression of L is then shown to be[35]

$$L=\frac{-\alpha}{4\pi}\sum_{i,j=1}^{3} c_ic_j\int_0^1 dx\int_0^x dy \times\left[\frac{L_1}{A^2}+\int_0^y dz\left(\frac{L_2}{A'^4}+\frac{L_3}{A'^2}\right)\right] \tag{IV.29}$$

with

$$L_1=-2(m_0-m_i)(m_0-m_j) +3m_0x[m_0(2-x)-(m_i+m_j)],$$

$$L_2=m_0^2y^2(m_0+m_i)(m_0+m_j),$$

$$L_3=(m_0-m_i)(m_0-m_j),$$

$$A^2=m_0^2x^2+(m_j^2-m_0^2)x+(m_i^2-m_j^2)y,$$

$$A'^2=m_0^2y^2+(m_j^2-m_0^2)y+(m_i^2-m_j^2)z.$$

[36] The quantity $\int_0^1(dx/x)$ is the usual infrared term which is cancelled by the contribution from the vertex function and is discussed in Schweber (Ref. 18) and Jauch and Rohrlich, Ref. 40. Although it is not exhibited explicitly here, expression (IV.29) for L can be shown to contain the same infrared term as in B.

[37] We see that B does not diverge as ρ and $\sigma\to\infty$. This is a consequence of the fact that, in the conventional theory, the contributions of the electron self energy and vertex diagrams are finite to second order because of the Landau gauge [Schweber (Ref. 18), p. 539; H. M. Fried and D. R. Yennie, Phys. Rev. **112**, 1391 (1958)].

This quantity is, as previously expected, finite as m_i, $m_j\to\infty$, because of the Landau gauge.

Expanding $C(k^2)$ of (III.9) about the physical photon mass of $k^2=0$, the denominator of $D_{\mu\nu}'$ of (III.10) becomes

$$k^2-\mu^2-C(0)-C'(0)k^2+\tfrac{1}{2}(k^2)^2C''(0)+\cdots, \tag{IV.30}$$

where the primes denote the derivatives with respect to k^2. Near $k^2=0$, writing $D_{\mu\nu}'$ in the form

$$D_{\mu\nu}'=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{Z_3}{k^2}, \tag{IV.31}$$

and using (IV.30) and (III.11), we have

$$Z_3^{-1}=1-C'(0). \tag{IV.32}$$

It can be shown that $C'(0)$ is given by[38]

$$C'(0)=(\alpha\theta^2/\pi)\int_0^\infty dt(N/\Lambda^2), \tag{IV.33}$$

where

$$\theta=(\rho-1)(\sigma-1),$$

$$\Lambda=(t+1)(t+\rho^2)(t+\sigma^2),$$

$$N=t\Gamma[a-\Gamma e+t(e\Delta-a\Gamma)] +\frac{t^2d}{3}[3(1-d\Gamma)+2t(d\Delta-\Gamma)],$$

with

$$a=1+\rho+\sigma,$$

$$b=\rho+\sigma+\rho\sigma,$$

$$c=\rho\sigma,$$

$$d=t-b,$$

$$e=at-c,$$

$$\Gamma=\frac{1}{t+1}+\frac{1}{t+\rho^2}+\frac{1}{t+\sigma^2},$$

$$\Delta=\frac{1}{(t+1)^2}+\frac{1}{(t+\rho^2)^2}+\frac{1}{(t+\sigma^2)^2} +\frac{1}{(t+1)(t+\rho^2)}+\frac{1}{(t+1)(t+\sigma^2)}+\frac{1}{(t+\rho^2)(t+\sigma^2)}.$$

The renormalized coupling constant $\alpha_0(=e_0^2/4\pi)$ is then

$$\alpha_0=Z_1^{-2}Z_2^2Z_3\alpha. \tag{IV.34}$$

If the perturbation expansion is really good and self-consistent, all of the Z's should be quite close to unity, and the charges e_0 and e should be almost equal. We

[38] M. Y. Han, thesis, University of Rochester, 1963 (unpublished).

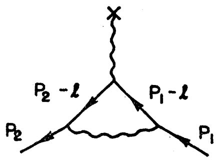


FIG. 5. Second-order correction to electron scattering from an external field (vertex contribution).

could actually calculate the bare coupling constant for various values of ρ and σ using (IV.34) and the Z's given in (IV.22), (IV.28), and (IV.32), but, since we are only interested in restrictions on ρ and σ, we shall not, in general, exhibit values of α for various ρ and σ. With the value of $\alpha_0=1/137$, computations have been made on the IBM 7074 and the results show that over the range of ρ and σ from 10^{-6} to 10^6, α differs from α_0 by no more than 5%. Thus, the charge renormalizations (at least to second order)[39] give essentially no new restriction on ρ and σ, and we can expand equally well in the bare or renormalized charge.

As a result of this section, we have the allowed region of ρ and σ as shown in Fig. 4. For the computation of the succeeding sections, we shall use the value of $\alpha_0=1/137$ for the coupling constant (making sure to exclude diagrams that have already contributed to charge renormalization).

V. THE ANOMALOUS MAGNETIC MOMENT

One of the most significant and precise numerical predictions of the conventional quantum electrodynamics is the magnetic moment of the electron. We now examine the prediction of the present formulation to determine whether it is consistent with experimental evidence for some choice of the auxiliary mass parameters. To this end, we shall consider the lowest order correction to the electron scattering from an external field, in particular, the second-order anomalous magnetic moment. As in the usual theory, it can be shown[35] that the only contribution to the magnetic moment comes from the second-order vertex diagram shown in Fig. 5. The vertex function is given by

$$\Lambda_\mu(p_2,p_1)=-\frac{i\alpha_0}{4\pi^3}\int d^4l\,\gamma^\rho S(p_2-l)\times\gamma_\mu S(p_1-l)\gamma^\sigma D_{\rho\sigma}(l)\,. \quad \text{(V.1)}$$

The extraction of the magnetic moment term from (V.1) follows the standard technique[40] of evaluating (V.1) on the mass shell, i.e., $p_2{}^2$ and $p_1{}^2=m_0{}^2$, and retaining only terms proportional to $\sigma_{\mu\nu}k^\nu$, where

$$k=p_1-p_2 \quad \text{(V.2)}$$

and

$$\sigma_{\mu\nu}=\tfrac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,. \quad \text{(V.3)}$$

A straightforward, but quite lengthy, calculation yields the magnetic moment part of $\Lambda_\mu(\not p_2=\not p_1=m_0)$, denoted by M_μ:

$$M_\mu=\sigma_{\mu\nu}k^\nu\frac{1}{2m_0}\frac{\alpha_0}{2\pi}J\,, \quad \text{(V.4)}$$

with

$$J=\sum_{i,j=1}^{3}c_ic_jJ_{ij}\,, \quad \text{(V.5)}$$

where

$$J_{11}=1\,,\quad J_{22}=6-5\rho+\rho(5\rho-1)(\rho-1)\ln\left(\frac{\rho^2-1}{\rho^2}\right),\quad J_{33}=J_{22}(\rho\to\sigma)\,,$$

$$J_{12}(=J_{21})=\tfrac{2}{3}\rho^2-2(\rho-1)-\frac{1}{(\rho^2-1)}\left[\left(\frac{5}{3}-2\rho\right)\ln(\rho^2-1)-\frac{\rho^4}{3}(2\rho^2-6\rho+1)\ln\left(\frac{\rho^2-1}{\rho^2}\right)\right],\quad J_{13}(=J_{31})=J_{12}(\rho\to\sigma)\,,$$

$$\begin{aligned}J_{23}(=J_{32})=&\frac{17}{6}-\rho-\sigma-\frac{1+\rho\sigma}{2(\rho+\sigma)}+\tfrac{2}{3}(\rho^2+\sigma^2)+\frac{1}{\rho^2-\sigma^2}\left[2(\rho+\sigma)-\frac{11}{6}-\frac{1+\rho\sigma}{2(\rho+\sigma)}\right]\ln\left(\frac{\rho^2-1}{\sigma^2-1}\right)\\&-\frac{1}{\rho^2-\sigma^2}\left\{\rho^3\left[1-\tfrac{5}{2}\rho+\frac{\rho^2}{2}-\tfrac{2}{3}\rho^3-\frac{\rho(1-\rho^2)}{2(\rho+\sigma)}+\sigma(\rho^2+2\rho-1)\right]\ln\left(\frac{\rho^2-1}{\rho^2}\right)\right.\\&\left.-\sigma^3\left[1-\tfrac{5}{2}\sigma+\tfrac{1}{2}\sigma^2-\tfrac{2}{3}\sigma^3-\frac{\sigma(1-\sigma^2)}{2(\rho+\sigma)}+\rho(\sigma^2+2\sigma-1)\right]\ln\left(\frac{\sigma^2-1}{\sigma^2}\right)\right\}.\end{aligned}$$

[39] While higher (than second) order contributions to the charge renormalizations may become large for large ρ and σ (the Landau gauge no longer helps), there should be no significant change in the allowed region for ρ and σ because of the factors of α^2 or higher.

[40] J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 342.

TABLE III. ΔJ in ρ-σ plane. (Note: Because of ρ-σ symmetry, the values in the upper left have been omitted.)

σ \ ρ	−100 000	−10 000	−1000	−100	+100	+1000	+10 000	+100 000
+100 000								−0.0018
+10 000							−0.014	−0.0083
+1000						−0.11	−0.063	−0.058
+100					−0.69	−0.42	−0.39	−0.38
−100				0.71	0.0055	0.36	0.40	0.40
−1000			0.11	0.44	−0.35	0.000018	0.052	0.057
−10 000		0.014	0.063	0.41	−0.38	−0.052	0.000001	0.0068
−100 000	0.0018	0.0083	0.058	0.40	−0.38	−0.057	−0.0069	0.0000001

For the special case of $\rho=\pm\sigma$, and $|\rho|=|\sigma|\gg1$, the expression for J simplifies to

$$J=1-\frac{4}{\rho}[2\ln(\rho^2-1)-1]+\frac{1}{\rho^2}\left\{14\ln(\rho^2-1)-98/9\right.$$
$$\left.-\frac{1}{3\rho}[68\ln(\rho^2-1)-65]\right\}, \quad \text{(V.6)}$$

for $\rho=\sigma$, and

$$J=1+\frac{1}{\rho^2}\left\{6\ln(\rho^2-1)-2/9\right.$$
$$\left.+\frac{1}{\rho^2}[2\ln(\rho^2-1)-17/6]\right\} \quad \text{(V.7)}$$

for $\rho=-\sigma$.

We see immediately that the dominant term for the case $\rho=\sigma$ is $\ln\rho/\rho$ and for $\rho=-\sigma$ is $\ln\rho/\rho^2$, thus again illustrating the greater convergence for $\rho=-\sigma$.

Let us write

$$J=1+\Delta J \quad \text{(V.8)}$$

so that ΔJ is the term added by the present model which goes to zero as the auxiliary masses go to infinity. What condition must be put on ΔJ? We know that the second- and fourth-order contributions to the anomalous magnetic moment in the usual theory agree with the experimental value within the experimental uncertainty. In order that our model not contradict experiment, we insist that, to second order, the deviation between our calculation and that of the usual theory be not greater than the experimental uncertainty. A recent experimental result[41] gives

$$(\alpha_0/2\pi)J=0.0011609\pm0.0000024.$$

Thus, we require that

$$(\alpha_0/2\pi)|\Delta J|<2.4\times10^{-6}, \quad \text{(V.9)}$$

or, using $\alpha_0=1/137$,

$$|\Delta J|<0.0021. \quad \text{(V.10)}$$

The expression for ΔJ as a function of ρ and σ was computed on an IBM 7074 computer and some typical values are given in Table III. For the case where $\rho=\sigma$, we find that we must have $|\rho|>9\times10^4$, but for the case where $\rho=-\sigma$, $|\rho|>200$ is sufficient to satisfy (V.10). Actually, the value of ΔJ changes sign below the $\rho=-\sigma$ line. For $\rho=50$, this is shown in Fig. 6. The general regions of the ρ-σ plane for which ΔJ satisfies (V.10) is shown in Fig. 7. An enlarged view of the region around the $\rho=-\sigma$ line is shown in Fig. 8.

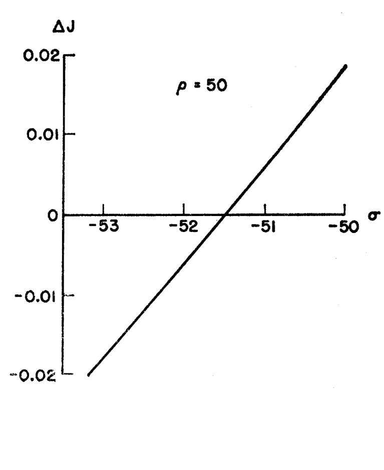


FIG. 6. ΔJ near $\rho=50$, $\sigma=-50$.

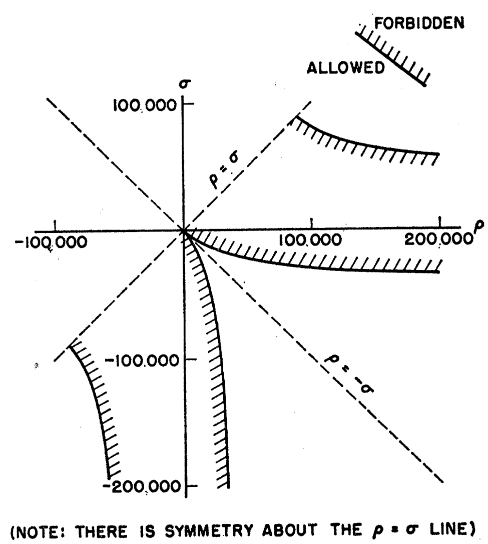


FIG. 7. Allowed region for ΔJ in ρ-σ plane.

[41] A. A. Schupp, R. W. Pidd, and H. R. Crane, Phys. Rev. **121**, 1 (1961).

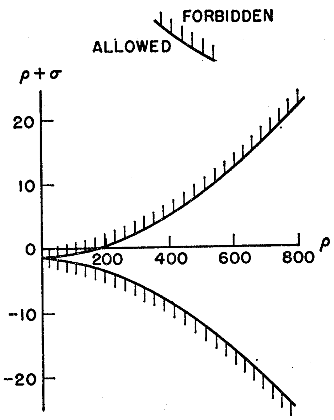


FIG. 8. Allowed region for ΔJ near $\rho=-\sigma$.

VI. COMPTON SCATTERING

Now, we examine the predictions of our model for Compton scattering. To the lowest order, the two diagrams that contribute to the matrix element are shown in Fig. 9. The initial state consists of an electron of momentum p_1 and spin s_1 and a photon of momentum k_1 and polarization λ_1, whereas p_2, s_2, k_2, and λ_2 are the corresponding labels for the final state. The matrix element corresponding to these diagrams is given by

$$R_{fi}=-\frac{ie_0^2}{2(2\pi)^3}\delta(p_2+k_2-p_1-k_1)\times\frac{m_0}{[E(p_1)E(p_2)]^{1/2}}w^{s_2}(p_2)\mathfrak{M}w^{s_1}(p_1)\,, \quad \text{(VI.1)}$$

where

$$\mathfrak{M}=\boldsymbol{\varepsilon}_2 S(p_1+k_1)\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_1 S(p_1-k_2)\boldsymbol{\varepsilon}_2\,, \quad \text{(VI.2)}$$

the w's are the spinors for the incoming and outgoing electrons, $\epsilon_{1\mu}$ and $\epsilon_{2\mu}$ are the photon polarizations corresponding to $\epsilon_\mu^{(\lambda_1)}$ and $\epsilon_\mu^{(\lambda_2)}$, respectively, and the δ function represents over-all energy-momentum conservation. The only difference between these expressions and the conventional ones is that $S(p)$ in (VI.2) represents the effective fermion propagator. As in the usual calculation, we choose the rest frame of p_1 as our coordinate system, so that $p_1=(m_0,0,0,0)$. Since we are only interested in perpendicular polarizations for the photon, we have

$$\begin{aligned} p_1\cdot\epsilon_1=p_2\cdot\epsilon_2=k_1\cdot\epsilon_1=k_2\cdot\epsilon_2=0\,,\\ \boldsymbol{p}_1\boldsymbol{\varepsilon}_1=-\boldsymbol{\varepsilon}_1\boldsymbol{p}_1\,,\\ \boldsymbol{p}_1\boldsymbol{\varepsilon}_2=-\boldsymbol{\varepsilon}_2\boldsymbol{p}_1\,. \end{aligned} \quad \text{(VI.3)}$$

Using (VI.3), the vanishing of the physical photon mass, and the fact that $\boldsymbol{p}_1 w^{s_1}(p_1)=m_0 w^{s_1}(p_1)$, we find

$$\mathfrak{M}=\sum_i c_i\left[\frac{\boldsymbol{\varepsilon}_2(m_i-m_0+\boldsymbol{k}_1)\boldsymbol{\varepsilon}_1}{2m_0\omega_1+m_0^2-m_i^2}+\frac{\boldsymbol{\varepsilon}_1(m_i-m_0-\boldsymbol{k}_2)\boldsymbol{\varepsilon}_2}{-2m_0\omega_1+m_0^2-m_i^2}\right], \quad \text{(VI.4)}$$

where $\omega_1(=k_{10})$ and $\omega_2(=k_{20})$ are the photon energies. Following the usual procedure,[42] we sum and average, respectively, over the final and the initial spins and polarizations, and arrive at the following expression for the differential cross section:

$$d\sigma/d\Omega=r_0^2(\omega_2/\omega_1)^2m_0^2F\,, \quad \text{(VI.5)}$$

where $r_0=e_0^2/4\pi m_0$ (the classical electron radius) and

$$F=\frac{1}{8m_0^2}\,\mathrm{tr}[(\boldsymbol{p}_2+m_0)\mathfrak{M}(\boldsymbol{p}_1+m_0)\mathfrak{M}'] \quad \text{(VI.6)}$$

with $\mathfrak{M}'=\gamma^0\mathfrak{M}^\dagger\gamma^0$.

The function F is given by the rather lengthy expression

$$m_0^2F=\tfrac{1}{2}\sum_{i,j}c_ic_j\sum_{k=1}^{4}F_{ij}^{(k)}\,, \quad \text{(VI.7)}$$

where

$$F_{ij}^{(1)}=2\omega_1\omega_2[2(a_ia_j+b_ib_j)+(1+\omega_1-\omega_2)\sin^2\phi(a_ib_j+a_jb_i)]+4\sin^2\phi(\omega_1^3a_ia_j-\omega_2^3b_ib_j)$$

$$F_{ij}^{(2)}=(2-m_i-m_j)[2\sin^2\phi(\omega_1^2a_ia_j+\omega_2^2b_ib_j)+2(\omega_1+\omega_2)(a_ia_j-b_ib_j)+\omega_1\omega_2\cos\phi,\ (1+\cos^2\phi)(a_ib_j+a_jb_i)]$$

$$F_{ij}^{(3)}=\sin^2\phi\{(m_i-1)[(2\omega_1^2-\omega_1-\omega_2)a_ib_j+(2\omega_1^2+\omega_1+\omega_2)b_ia_j]+(m_j-1)[(2\omega_1^2+\omega_1+\omega_2),\ a_ib_j+(2\omega_2^2-\omega_1-\omega_2)b_ia_j]\}\,,$$

$$F_{ij}^{(4)}=(m_i-1)(m_j-1)(2+\omega_1-\omega_2)[2a_ia_j+2b_ib_j-\sin^2\phi(a_ib_j-a_jb_i)]\,,$$

with

$$\omega_2=\omega_1[1+\omega_1(1-\cos\phi)]^{-1},\quad a_i=(2\omega_1+1-m_i^2)^{-1},\quad\text{and}\quad b_i=(-2\omega_2+1-m_i^2)^{-1}.$$

(In the above expressions, ω_1, ω_2, and m_i are all in units of m_0.) This is to be contrasted to the usual result (Klein-Nishina formula)[43]

$$m_0^2F=\tfrac{1}{2}(\omega_1/\omega_2+\omega_2/\omega_1-\sin^2\phi)\,, \quad \text{(VI.8)}$$

which is the limit of (VI.7) as m_2 and $m_3\to\infty$. The total cross section is defined as

$$\sigma_T=2\pi\int_0^\pi\frac{d\sigma}{d\Omega}\sin\phi d\phi\,. \quad \text{(VI.9)}$$

Actually it will be convenient to calculate σ_T in units

[42] Schweber, Ref. 18, p. 487.

[43] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1954), 3rd ed., p. 219.

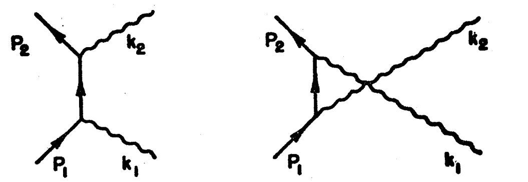


FIG. 9. Lowest order contributions to Compton scattering.

of the Thompson cross section $\sigma_0=(8\pi/3)r_0^2$,[43] which is the limit of the usual result for $\omega_1/m_0 \ll 1$. For the purposes of calculation, the differential cross section was computed on an IBM 7074 digital computer and the total cross section was calculated by Simpson's rule numerical integration using points at every five degrees between 0° and 180°. In all cases, both the conventional formula (VI.8) and the result of our model (VI.7) were calculated and compared for particular values of ρ and σ and ω_1/m_0 (incident photon energy). It is, however, only required that the predictions of our model correspond to the Klein-Nishina predictions to the extent that the latter is in accord with experiment. In practice, there seems to be only a limited range of photon energies for which experimental data for Compton scattering are available. The basic difficulty seems to be the fact that the experiments are done with atomic electrons in such elements as carbon, copper, aluminum, and lead. For high energies, pair production dominates the photon scattering and effectively overwhelms the Compton contributions. In the case of total cross section, the experimentally measured quantity is the absorption or attenuation coefficient in various metals. The absorption is essentially due to the combined effect of photoelectric absorption, Compton scattering and pair production. Although there is considerable variation from metal to metal, the photoelectric effect is important below about 0.05–0.5 MeV (0.1–1 electron masses); Compton scattering dominates from there until about 5–15 MeV (10–30 m_0), and pair production above that range.[44]

We shall only calculate Compton scattering for those values of the parameters ρ and σ for which the previous criteria concerning δm and the magnetic moment have been satisfied. Examination of Figs. 4, 7, and 8 reveals that the allowed region is in the neighborhood of the $\rho=-\sigma$ line extending out farther than $\rho=250\,000$. For small values of ρ, the allowed region corresponds to that shown in Fig. 8. We shall choose for calculations a typical point in the narrowest part of the allowed region (small ρ), where the allowed region is essentially a line ($\rho=+10$, $\sigma=-10.1$) and two typical points in the wider area ($\rho=+300$, $\sigma=-300$ and $\rho=+1000$, $\sigma=-1015$).

The differential cross sections for these values of ρ and σ were calculated as a function of ϕ for an energy of $\omega_1/m_0=0.173$; an energy for which experimental data is given in Heitler.[45] For $\rho=+10$, $\sigma=-10.1$ there are small deviations from the conventional result but these are less than 1% and smaller than the experimental error given. For the other two cases, there is no observable difference between our prediction and the conventional one.

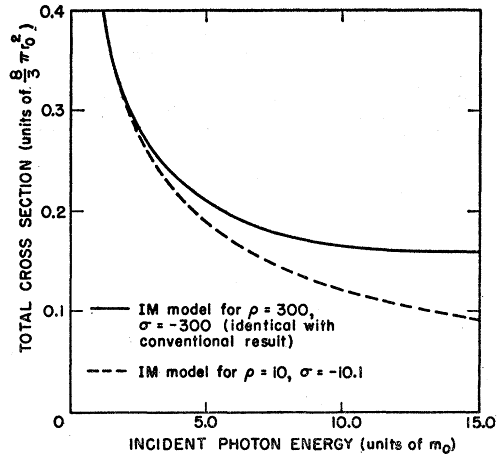


FIG. 10. Total cross section for Compton scattering.

The total cross section as a function of incident photon energy up to $20m_0$ (10 MeV) was plotted by the computer for the values of ρ and σ mentioned above and these results are shown in Fig. 10 (the cases for $\rho=300$, $\sigma=-300$, and $\rho=1000$, $\sigma=-1015$ gave identical results). Both the prediction of our indefinite metric (IM) model and that of the conventional theory are shown.

It is clear that the deviation from the conventional result for $\rho=+10$, $\sigma=-10.1$ is significant. Since in this energy region, the conventional result is considered to have excellent agreement with experiment, our prediction seems to clearly violate the known situation. On the other hand, the other two choices of parameters again yield predictions that coincide with the conventional theory. It might be fruitful to consider comparisons with experimental data in greater detail. The absorption coefficient $\tau(\text{cm}^{-1})$ is given[46]:

$$\tau=NZ\sigma_T, \tag{VI.10}$$

where N is the number of atoms per cm^3, Z is the number of electrons per atom, and σ_T is the total scattering cross section. If we consider that the only contribution to attenuation is from Compton scattering in aluminum at 1.076 MeV ($\omega_1/m_0=2.11$) and copper at 1.51 MeV ($\omega_1/m_0=2.96$), we can make a comparison with experiment. Table IV lists the experimental value, the conventional prediction from the Klein-Nishina formula, and the prediction of the indefinite metric model (IM

[44] Heitler, Ref. 43, p. 363. See also National Bureau of Standards Circular 583, 1957 (unpublished).

[45] See also W. Friedrich and G. Goldhaber, Z. Physik 44, 700 (1927), and G. E. M. Jauncy and G. G. Harvey, Phys. Rev. 37, 698 (1931). The Klein-Nishina formula agrees exactly with the experimental data within experimental error.

[46] Heitler, Ref. 43, p. 222.

TABLE IV. Absorption coefficient τ (per cm) for aluminum and copper.

Metal	Photon energy (MeV)	Experimental[a]	Conventional	$\rho=10$ $\sigma=-10.1$	$\rho=300$ $\sigma=-300$	$\rho=1000$ $\sigma=-1015$
Al	1.076	0.1606	0.1594	0.1636	0.1594	0.1594
Cu	1.51	0.422	0.417	0.438	0.417	0.417

[a] J. J. Wyard, Phys. Rev. **87**, 165 (1952); and Proc. Phys. Soc. (London) **A66**, 382 (1953).

model) for several values of ρ and σ.[47] In both cases, the conventional result (and IM model for $\rho=300$, $\sigma=-300$, $\rho=1000$, and $\sigma=-1015$) agreed with the experimental value within about 1% and is slightly small, perhaps due to a small amount of pair production. (The experimental accuracy of the above figures is ~0.5–1%.) For the case $\rho=+10$, $\sigma=-10.1$, the prediction is 2% too high for Al and 4% too high for Cu, clearly in contradiction with experiment.

We see then that values of our auxiliary masses that are small in magnitude seem to be ruled out by the Compton data. Values such as $\rho=+300$ and $\sigma=-300$ give excellent agreement with the usual prediction and experiment. Further calculation and comparison with more experiments could fix this transition more precisely.

VII. REVIEW AND FURTHER REMARKS ON THE AUXILIARY MASSES

In the preceding sections, we have presented an indefinite metric theory (IM theory) of quantum electrodynamics for which all quantities are finite, and which allows us to calculate various scattering processes. In general, we have imposed two conditions on the theory.

The first is that we wish the theory to correspond to the actual interactions of electrons and the electromagnetic field, i.e., it must be able to reproduce the existing experimental data in this area. The "photon existence" has been built in by the method of quantization used. The remaining experimental data may be considered to be of two kinds. On one hand, there are two very precisely known constants, namely the magnetic moment and the Lamb shift, and, on the other hand, there are a great variety of scattering processes for which cross sections and angular distributions are known to some degree of accuracy, e.g., Compton scattering, Møller scattering, and pair production. A typical case of each of these kinds of data has been calculated in Secs. V and VI.

The second condition is one relating, in a sense, to the spirit of our approach. Since the model has been specifically introduced to avoid the ultraviolet divergences of the conventional theory, we should insist that the renormalization effects be truly small. In other words, while we cannot explicitly prove the convergence of the perturbation expansion, we will try to insist that any term to a particular order in the coupling constant be smaller than lower order terms. Thus, the renormalization constants, the Z's should all be close to unity and the perturbation expansion is equally good in α or α_0. Also the mass corrections should be small (since they are at least first order in α). These conditions have been considered in Sec. IV.

Since the physical photon mass, physical electron mass, and the renormalized coupling constant are considered to be input information in the theory, we have determined allowed ranges of the two auxiliary masses for which the theory satisfies these two conditions.

From Fig. 4 and Fig. 8 we see that there is an area around the $\rho=-\sigma$ line for which the magnetic moment and mass renormalization criteria can be both satisfied. As we have seen in Sec. IV, the charge renormalization does not put any restriction on this area. The criterion for the mass renormalization is not as precise as for the magnetic moment. In fact, the condition that the second-order mass correction be less than about $\frac{1}{4}$ is, in a sense, arbitrary, but seems reasonable and allows us to proceed in a quantitative manner. Adjustments in this value would slightly alter the boundaries of the allowed region.

For the magnetic moment case, the allowed region near the line $\rho=-\sigma$ is narrow (essentially a line) for small values of ρ and broadens indefinitely as ρ becomes large. On the other hand, for the mass renormalization condition, the allowed region is broader for small values of ρ, and computations show that it narrows for large values. Thus, in the neighborhood of the $\rho=-\sigma$ line, the 'net' allowed region is lower bounded by the magnetic moment and upperbounded by the mass renormalization (for example, at $\rho=10^5$, σ is limited from $-\rho+120$ to $-\rho-170$). As we have seen, the lower part of this 'net' area is further restricted by the Compton scattering data.

We have observed that the various criteria seem to be more easily satisfied in the neighborhood of the $\rho=-\sigma$ line.

Rewriting the effective fermion propagator as

$$S(p)=\frac{1}{p-m_0}-\frac{(p+m_0)-(m_2+m_3)}{(p-m_2)(p-m_3)}, \quad \text{(VII.1)}$$

we see that for large m_2 and m_3, the second term goes as $1/m_{2,3}$ for $m_2+m_3\neq 0$, and as $1/m_{2,3}^2$ for $m_2+m_3=0$. Thus, the greater "convergence" with regard to the auxiliary masses near the $\rho=-\sigma$ line could have been anticipated. The form of the second term also makes plausible the change in sign of the auxiliary mass contributions near the $\rho=-\sigma$ line, since m_2+m_3 in the numerator changes sign as that line is crossed.

Even though we have not exhausted experimental tests of the present theory, the foregoing analysis leads us to believe that for any further calculations, such as the Lamb shift, and other scattering processes, the allowed region will not undegro any startling changes,

[47] For aluminum, according to Heitler (Ref. 43, p. 422), $NZ\sigma_0=0.521$, where σ_0 is the Thompson cross section. For copper, $NZ\sigma_0=1.63$.

i.e., there will remain a neighborhood of the $\rho=-\sigma$ line for which the predictions of the IM theory will be consistent with experiment.

VIII. DISCUSSION

We have described, in some detail, a Lagrangian formulation of quantum electrodynamics which is local and manifestly covariant, yet finite, with the aid of auxiliary fields with an underlying indefinite metric. Here we shall make some theoretical remarks concerning the model.

One question concerns the arbitrariness of the approach. It may appear that certain features of the theory seem somewhat *ad hoc*. Actually, we feel that, given the goal of eliminating the ultraviolet divergences in a consistent manner and reproducing quantum electrodynamics using an indefinite metric, many aspects of the model have developed within a spirit of "minimal arbitrariness." The number of auxiliary fields is just sufficient to insure that all diagrams are finite. The method chosen for the quantization of the electromagnetic field is necessary to insure that the physical spin-1 quanta are photons, in the absence of gauge invariance. The criteria of a meaningful perturbation expansion and agreement with experiment have provided some restrictions of the auxiliary mass parameters. Even within these criteria, there seems to be a certain amount of nonuniqueness in the choice of these masses, i.e., a range of these masses discussed previously seems to achieve the desired results. While this nonuniqueness is unsatisfying in a certain sense, in another way it is quite useful. The fact that there is, so far, this range of permissible values for the auxiliary masses, helps convince us that further comparisons with experiment will not lead to difficulty.[48]

Another theoretical question is that of the quantization of the electromagnetic field and the question of gauge invariance. We do not have gauge invariance explicitly in the theory, and the current interacting with the electromagnetic field is not locally conserved. On the other hand, gauge invariance is designed to insure that the electromagnetic field is truly represented by the theory, i.e., that the physical quanta are photons (zero-mass transverse quanta). The existence of photons can be insured, as we have shown, by quantizing the vector field in a fashion so that the propagator has the form of the Landau gauge. This propagator acts as a projection operator that leaves only the transverse (conserved) part of the interacting current. A difficulty with this procedure is the Lagrangian prescription for quantization. We have shown in Appendix A that we can write a series of Lagrangians as a function of a continuous parameter λ that yields the desired result uniquely as $\lambda\to 0$; yet, at $\lambda=0$ a local Lagrangian cannot be written down. It would be preferable to avoid this procedure, but it seems necessary to insure the desired gauge.[49]

[48] See Sec. VII.

Another theoretical point concerns the problem, mentioned in Sec. I, of constructing a subsidiary condition to rule out negative norm states due to the auxiliary fields. Such a construction is difficult and far from trivial for theories which cannot be solved exactly. The general procedure, involving diagonalization of the S matrix in order to project out positive norm states in an invariant way, strictly speaking, requires a knowledge of the complete solutions of the theory. Whereas solutions are readily obtainable in the case of some simple models, they are practically inaccessable in a realistic theory which has to rely on perturbation approximations. However, as discussed elsewhere,[13] if the coupling constant is small enough, the physical states would be identifiable (after approximate diagonalization of the S matrix) by their containing only a "small" admixture of auxiliary field. This procedure can be avoided in our case, as long as the values of the auxiliary masses are high enough so that the processes considered are below threshold for these particles. Under that condition, the 'physical' state is pure electron (type 1 fermion). If processes involving energies significantly above threshold are to be calculated, some sort of diagonalization should be attempted.

Finally, after all the discussion, we wish to emphasize that we have demonstrated that there exist values for the parameters in our theory which can completely reproduce the successful predictions of conventional quantum electrodynamics, with a finite theory. This statement, we feel, is a nontrivial one. It was not at all certain, *a priori*, that such a model could be constructed in a self-consistent fashion. Although this result would not necessarily lead to a replacement of the usual theory by the present one, it has shown that such a model can work for the most well known and successful application of field theory.[50] For other theories, such as four-fermion interactions, where no successful theory exists, this approach may be promising.

[49] In fact, an attempt to use the usual Feynman gauge method of quantizing the electromagnetic field in this model has been carried out in great detail (Ref. 38). There were two major difficulties. If one started with a zero mass bare propagator in that gauge, and attempted to maintain the same mass and gauge in the dressed propagator, it turned out that the conditions on the parameters of the theory were impossible to satisfy without a coupling constant so large that the perturbation expansion was meaningless. Secondly, it was not possible to prove that this gauge propagator would guarantee photons in the theory.

[50] Actually, one could take a different point of view, and try to push the IM model as far as possible. While existing data can be explained with the ranges of the parameters indicated, we could attempt to calculate processes for which the experimental evidence is unknown or unclear. In these processes, there may be considerable deviation of our model from the usual one for various choices of parameters (allowed by the previous criteria). It is conceivable that the conventional model will not give good predictions for some of these processes, in which case, if an experiment can be done, we could, perhaps, make a test between the conventional theory and the IM model. We concede that this possibility seems remote, however.

The appearance of infinities inherent in many of the existing field theories may stem from the perhaps dubious assumption of point interactions. If this is the case, one may be led to attempt formulations of elementary particle theories that may involve nonlocal interactions. We have already mentioned the point of view[13] that a local indefinite-metric theory may be equivalent to a nonlocal positive-definite theory, i.e., the former may be considered a manifestly covariant canonical way of introducing the latter into a field theoretic formalism. Thus, it may be interesting to determine if there is such a nonlocal theory, equivalent to the one presented here, and to investigate its form.[51] This could lead to further insight into the physical foundations and structure of the indefinite metric theory, and perhaps aid in extensions to systems other than the electron and photon.

ACKNOWLEDGMENTS

We would like to express our thanks to Professor I. Birula, Professor K. Johnson, Professor S. Okubo, Professor F. Rohrlich, and Professor Y. Takahashi for many valuable and stimulating discussions. One of us (M. E. A.) wishes to thank Dr. N. Mukunda and S. Pepper for interesting discussions.

APPENDIX A: QUANTIZATION OF THE FREE ELECTROMAGNETIC FIELD

In this appendix, we shall present the quantization of the free electromagnetic field as a massive vector field with a true Landau gauge propagator.[29] The covariant transversality of this propagator will be demonstrated.

Although the notation is not identical, we are dealing with a special case of a technique developed in a recent paper by Feldman and Matthews.[52] As mentioned in Sec. III, we intend to superimpose a neutral vector field of mass μ with a zero mass longitudinal field. We shall first take the mass of the longitudinal field to be λ and then investigate the limit as $\lambda\to 0$.

We consider the field A_μ to be split into a transverse and a longitudinal part as follows[53]:

$$A_\mu(x)=A_\mu{}^\tau(x)+A_\mu{}^l(x)\,, \tag{A1}$$

where

$$\begin{aligned} A_\mu{}^\tau(x)&=\tau_\mu{}^\nu(x,y)A_\nu(y) \quad &&\text{(transverse part)}\,,\\ A_\mu{}^l(x)&=A_\mu(x)-A_\mu{}^\tau(x) \quad &&\text{(longitudinal part)}\,. \end{aligned} \tag{A2}$$

The transverse projection operator $\tau_{\mu\nu}$ is defined as

$$\tau_{\mu\nu}(x,y)=g_{\mu\nu}\delta(x-y)-\partial_\mu\partial_\nu D(x-y)\,, \tag{A3}$$

where

$$\partial^2 D(x-y)=\delta(x-y)\,,$$

so that

$$\partial^\mu\tau_{\mu\nu}(x,y)=0$$
$$\tau^n=\tau\,.$$

By virtue of these defining relations, the fields $A_\mu{}^\tau(x)$ and $A_\mu{}^l(x)$ satisfy the equations

$$\partial^\mu A_\mu{}^\tau(x)=0\,, \tag{A4a}$$

$$\tau_\mu{}^\nu(x,y)A_\nu{}^l(y)=0\,, \tag{A4b}$$

$$\partial_\mu A_\nu{}^l(x)=\partial_\nu A_\mu{}^l(x)\,. \tag{A4c}$$

As Feldman and Matthews show, a Lagrangian can be written down.

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu A_\nu\partial^\mu A^\nu-\partial_\nu A_\mu\partial^\mu A^\nu)+\frac{\mu^2}{2}(A_\mu A^\mu-\lambda^{-2}\partial_\mu A^\mu\partial_\nu A^\nu)\,, \tag{A5}$$

which yields the following equation of motion for A_μ:

$$\partial^\nu(\partial_\nu A_\mu-\partial_\mu A_\nu)+\mu^2(g_\mu{}^\nu+\lambda^{-2}\partial_\mu\partial^\nu)A_\nu=0\,. \tag{A6}$$

It can be easily seen that using Eqs. (A4), the above gives the desired equations of motion for $A_\mu{}^\tau$ and $A_\mu{}^l$.

$$(\partial^2+\mu^2)A_\mu{}^\tau(x)=0 \tag{A7a}$$

$$(\partial^2+\lambda^2)A_\mu{}^l(x)=0\,. \tag{A7b}$$

Now, we can proceed to examine the solutions of (A7) in momentum space. Writing the Fourier transform of $A_\mu{}^\tau(x)$,

$$A_\mu{}^\tau(x)=(2\pi)^{-3/2}\int d^4k\, e^{-ikx}f_\mu(k)\,, \tag{A8}$$

by (A7a) and (A4a), we have

$$\begin{aligned}(k^2-\mu^2)f_\mu(k)&=0\,,\\ k^\mu f_\mu(k)&=0\,.\end{aligned} \tag{A9}$$

For a fixed value of $\mathbf{k}$, there are three independent solutions of (A9) which can be written in the form[54]

$$f_\mu{}^{(r)}(k)=\sqrt{2}\epsilon_\mu{}^{(r)}\delta(k^2-\mu^2)a_r(k)\,,\quad r=1,2,3\,, \tag{A10}$$

where the $a_r(k)$ are arbitrary (the $\sqrt{2}$ is just for convenience in later equations), and the properties of the

[51] A version of nonlocal finite quantum electrodynamics by M. Levy (to be published) has come to the attention of the authors. Also, an asymptotic finite theory of quantum electrodynamics has recently appeared [R. E. Pugh, Ann. Phys. (N. Y.) **23**, 335 (1963)].

[52] G. Feldman and P. T. Matthews, Phys. Rev. **130**, 1633 (1963).

[53] Integration over the repeated variable y is assumed.

[54] These equations are, of course, the usual equations for the massive neutral vector meson field with its three polarizations. We deviate at this point from Feldman and Matthews (Ref. 52) who define four $f_\mu{}^{(r)}$ proportional to $g_{\mu\nu}-k^{-2}k_\mu k_\nu$ for $r=0, 1, 2, 3$, of which three are independent. Their method is, of course, covariant but obscures the situation in our case and results in nondiagonal commutation relations and Hamiltonian. The transformation from their set of $f_\mu{}^{(r)}$'s to our set of $f_\mu{}^{(r)}$'s can easily be written as $f_\mu{}^{(r)}=\Sigma_r c_{(r)}{}^\nu f_\mu{}^{(r)}$, where $c_{(r)}{}^\nu=-\epsilon^{\nu(r)}$. All equations can then be transformed from one set to another.

$\epsilon_\mu{}^{(r)}$ are

$$k_\mu \epsilon^{\mu(r)} = 0\,,\quad \epsilon_\mu{}^{(r)}\epsilon^{\mu(s)} = -\delta_{rs}\,,\quad \sum_r \epsilon_\mu{}^{(r)}\epsilon_\nu{}^{(r)} = -(g_{\mu\nu} - k_\mu k_\nu k^{-2})\,. \tag{A11}$$

The latter relation is, of course, the closure relation for this set of polarizations.[55]

Following the same procedure for $A_\mu{}^l(x)$, we have [using (A4b) and (A7b)],

$$A_\mu{}^l(x) = (2\pi)^{-3/2}\int d^4k\, e^{-ikx} d_\mu(k)\,, \tag{A12}$$

where $d_\mu(k)$ must satisfy

$$(k^2-\lambda^2)d_\mu(k) = 0\,,\quad (g_{\mu\nu} - k_\mu k_\nu k^{-2})d_\nu(k) = 0\,. \tag{A13}$$

This has the solution

$$d_\mu(k) = \sqrt{2}k_\mu \delta(k^2-\lambda^2)a(k)\,, \tag{A14}$$

with $a(k)$ arbitrary.

Combining these results, we can write $A_\mu(x)$ in terms of these plane-wave solutions as[56]

$$A_\mu(x) = (2)^{1/2}(2\pi)^{-3/2}\int d^4k\;\Theta(k_0)\{\sum_r \epsilon_\mu{}^{(r)}(k)\delta(k^2-\mu^2)[a_r(k)e^{-ikx} + a_r{}^\dagger(k)e^{ikx}] + k_\mu\delta(k^2-\lambda^2)[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}]\}\,, \tag{A15}$$

where we have used the fact that $a_r{}^\dagger(k) = a_r(-k)$ and $a^\dagger(k) = -a(-k)$. It will also be convenient to have the above expression for $A_\mu(x)$ integrated over k_0:

$$A_\mu(x) = (2)^{-1/2}(2\pi)^{-3/2}\left\{\int_{k_0=+(\mathbf{k}^2+\mu^2)^{1/2}} \frac{d^3k}{k_0}\sum_r \epsilon_\mu{}^{(r)}(k)[a_r(\mathbf{k})e^{-ikx} + a_r{}^\dagger(\mathbf{k})e^{ikx}] + \int_{k_0=+(\mathbf{k}^2+\lambda^2)^{1/2}} \frac{d^3k}{k_0} k_\mu[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}]\right\}\,, \tag{A16}$$

where

$$a_r(\mathbf{k}) = a_r(k) \quad \text{for} \quad k_0 = +(\mathbf{k}^2+\mu^2)^{1/2}\,,\qquad a(\mathbf{k}) = a(k) \quad \text{for} \quad k_0 = +(\mathbf{k}^2+\lambda^2)^{1/2}\,. \tag{A17}$$

We can now proceed to the quantum field theory by applying the equal-time canonical commutation relations:

$$[A_\mu(x), \pi_\nu(x')]_{x_0=x_0'} = ig_{\mu\nu}\delta(x-x')\,, \tag{A18}$$

where $\pi_\mu(x) \equiv \partial\mathcal{L}/\partial A^{\mu,0}$. From (A5),

$$\pi_\mu(x) = -[\partial_0 A_\mu(x) - \partial_\mu A_0(x) + \mu^2\lambda^{-2}g_{\mu 0}\partial_\nu A^\nu(x)]\,, \tag{A19}$$

or, in terms of the plane-wave solutions,

$$\pi_\mu(x) = i(2)^{-1/2}(2\pi)^{-3/2}\left\{\int_{k_0=+(\mathbf{k}^2+\mu^2)^{1/2}} \frac{d^3k}{k_0}\sum_r (k_0\epsilon_\mu{}^{(r)} - k_\mu\epsilon_0{}^{(r)})(a_r e^{-ikx} - a_r{}^\dagger e^{ikx}) + g_{\mu 0}\mu^2\int_{k_0=+(\mathbf{k}^2+\lambda^2)^{1/2}} \frac{d^3k}{k_0}(ae^{-ikx} - a^\dagger e^{ikx})\right\}\,. \tag{A20}$$

Using (A20) and (A18), we derive the commutation relations

$$[a_r(\mathbf{k}), a_{r'}{}^\dagger(\mathbf{k}')] = k_0\delta_{rr'}\delta(\mathbf{k}-\mathbf{k}')\,,\quad r, r' = 1, 2, 3\,,\qquad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = -k_0\mu^{-2}\delta(\mathbf{k}-\mathbf{k}')\,, \tag{A21}$$

and all the other commutators vanish.

The Hamiltonian is given by

$$P^0 = \int d^3x\, T^{00}\,, \tag{A22}$$

where[57]

$$T^{00} = \partial_0 A^\nu \pi_\nu - \mathcal{L} = -\tfrac{1}{2}\sum_{\mu=0}^{3}[\partial_\mu A_\nu(x)\partial_\mu A^\nu(x) - \partial_\mu A_\nu(x)\partial^\nu A_\mu(x) + \mu^2\lambda^{-2}\partial_\mu A_\mu(x)\partial^\nu A_\nu(x)] - \tfrac{1}{2}\mu^2 A_\mu(x)A^\mu(x)\,, \tag{A23}$$

[55] It may be instructive to look at a particular set of $\epsilon_\mu{}^{(r)}$. If we let $k = (k_0, \mathbf{k})$, and choose

$$\epsilon_\mu{}^{(r)} \text{ such that } \begin{cases} \epsilon_0{}^{(r)} = 0,\quad \boldsymbol{\varepsilon}^{(r)}\cdot\mathbf{k} = 0,\quad \boldsymbol{\varepsilon}^{(r)}\cdot\boldsymbol{\varepsilon}^{(s)} = \delta_{rs} & r,s = 1,2 \\ \epsilon_\mu{}^{(3)} = 1/|\mathbf{k}|\,(|\mathbf{k}|, k_0\mathbf{k}/|\mathbf{k}|) \end{cases}$$

we find that relations (A11) are satisfied for $\epsilon_\mu{}^{(r)}$, $r = 1, 2, 3$. Actually, if we define $\epsilon_\mu{}^{(0)} = k_\mu/|\mathbf{k}|$, and let $\epsilon_\mu{}^{(\nu)} = \epsilon_\mu{}^{(0)}$, $\nu = 0$, then we complete an orthonormal set in Lorentz space and the set has the properties:

$$\epsilon_\mu{}^{(\nu)}\epsilon^{\mu(\nu')} = g^{\nu\nu'} \quad \text{and} \quad \sum_{\rho,\sigma=0}^{3}\epsilon_\mu{}^{(\rho)}\epsilon_\nu{}^{(\sigma)}g^{\rho\sigma} = g_{\mu\nu}.$$

[56] The $a^\dagger$ will represent complex conjugate for the classical c-number case, and adjoint in the operator case.

[57] $\sum_{\mu=0}^{3}(A_\mu A_\mu) = A_0A_0 + \mathbf{A}\cdot\mathbf{A}$ as opposed to $A_\mu A^\mu = A_0A_0 - \mathbf{A}\cdot\mathbf{A}$.

which yields (with appropriate symmetrization)

$$P^0=\int d^3k \sum_{r=1}^{3} a_r^\dagger(\mathbf{k})a_r(\mathbf{k})-\int d^3k\, \mu^2 a^\dagger(\mathbf{k})a(\mathbf{k}). \quad \text{(A24)}$$

Now, we shall examine the special limiting case as $\lambda\to 0$, the case we shall need. The only equations above that exhibit any difficulty when $\lambda\to 0$ are the Lagrangian (A5), the equation of motion (A6), the canonical momentum density (A19), and the energy tensor (A23). All other expressions, particularly the solutions $A_\mu(x)$ to the field equations, the commutation relations, and the Hamiltonian, written in terms of creation and destruction operators, exhibit no difficulty in the case when $\lambda=0$. In all of the former expressions, λ enters in the form

$$\lambda^{-2}\partial^\nu A_\nu(x). \quad \text{(A25)}$$

For the transverse part $A_\mu{}^T(x)$, $\partial^\nu A_\nu{}^T$ vanishes for all λ so (A25) is zero for all $\lambda\neq 0$ and the limit as $\lambda\to 0$ is also zero. For the case of the longitudinal field $A_\nu{}^l(x)$, (A25) is equivalent to taking $i\lambda^{-2}k^\nu$ times $k_\nu\delta(k^2-\lambda^2)$ in momentum space, which gives a finite result at $\lambda=0$. This fact, that expression (A25) is unique and well defined at $\lambda=0$ when evaluated for solutions $A_\mu(x)$ of the equations of motion, is what allows us to proceed and is the reason that all expressions that were written in terms of creation and destruction operators exhibited no difficulty as λ went to 0.

Actually, at $\lambda=0$, the Lagrangian is nonlocal ($\lambda^{-2}\partial^\nu\to\partial^{-2}\partial^\nu$, which is a nonlocal operator) and should not be used, but since $\mathcal{L}$ is local for any small but finite value of λ, and all results exist and are unique at $\lambda=0$, we conclude that in the limiting case as λ tends to zero, we have a uniquely defined theory from a Lagrangian prescription. In all further work, we will take $\lambda=0$.

It can then be shown that the propagator can be written

$$\langle T(A_\mu(x)A_\nu(y))\rangle_0 = -i2(2\pi)^{-4}\int d^4k\, e^{-ik(x-y)}D_{\mu\nu}(k), \quad \text{(A26)}$$

where, in complex k_0 space, the integration is along the real axis from $-\infty$ to $+\infty$, and where

$$D_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)\frac{1}{k^2-\mu^2+i\epsilon'} \quad \text{(A27)}$$

with the limits ϵ, $\epsilon'\to 0$ taken after the k_0 integration. This is the desired Landau gauge propagator and we shall examine its transversality.

The question we ask is whether $\int f(k^2)k^\mu D_{\mu\nu}(k)d^4k$ will always vanish [assuming $f(k^2)$ is a well-behaved function]. If it does, the (A27) is covariantly transverse.

We can write

$$\int d^4k\, f(k^2)k^\mu D_{\mu\nu}(k) = \int d^4k\, f(k^2)k_\nu\left(\frac{i\epsilon}{k^2+i\epsilon}\right)\frac{1}{k^2-\mu^2-i\epsilon'}. \quad \text{(A28)}$$

Completing the contour in the lower half of the k_0 plane (ignoring numerical factors), we get

$$\int_{k_0=+(\mathbf{k}^2+i\epsilon)^{1/2}} d^3k \lim_{\epsilon\to 0}\frac{k_\nu\epsilon f(k^2)}{(\mathbf{k}^2)^{1/2}-i\epsilon} +\int_{k_0=+(\mathbf{k}^2+\mu^2)^{1/2}} d^3k \lim_{\epsilon'\to 0}\frac{k_\nu\epsilon' f(k^2)}{(\mathbf{k}^2+\mu^2)^{1/2}-i\epsilon'}. \quad \text{(A29)}$$

The second term clearly gives no contribution since its integrand vanishes as $\epsilon'\to 0$ for any value of $\mathbf{k}$. For the first term, the integrand is zero for any $\mathbf{k}\neq 0$ as $\epsilon\to 0$, and cannot become infinitely large as $\mathbf{k}\to 0$ and $\epsilon\to 0$, so it will not contribute[58] and expression (A28) vanishes as desired.

APPENDIX B: A FORMAL PRESCRIPTION FOR MASS RENORMALIZATION

In this Appendix, we shall discuss a formal method for treating mass renormalization by utilizing a Taylor expansion in the mass. For simplicity, the method is illustrated for the case of fermions in the conventional quantum electrodynamics. Its generalization to the present theory is then straightforward and is presented in Sec. IV.

Dealing with internal lines, we shall show that the method of Taylor expansion in the mass can be used to provide a "formal" proof of the equivalence of the dressed fermion propagator when the perturbation expansion is in terms of the bare mass, and when it is in terms of the physical mass (with a counter term in the Lagrangian).

In the first case, of the expansion in terms of the bare mass, the relevant Lagrangian densities are (II.1) and

[58] This can be seen in another way. For simplicity, we can consider the first term of (A28) to be equivalent to the one-dimensional integral

$$\int_{-\infty}^{\infty} dx \lim_{\epsilon\to 0}[\epsilon x/(x-i\epsilon)],$$

which can be rewritten as

$$\int_{-\infty}^{\infty} dx \lim_{\epsilon\to 0}[\epsilon x^2/(x^2+\epsilon^2)]+i\int_{-\infty}^{\infty} dx \lim_{\epsilon\to 0}[\epsilon^2 x/(x^2+\epsilon^2)].$$

Now

$$\lim_{\epsilon\to 0}[\epsilon/(x^2+\epsilon^2)]\sim\delta(x)$$

so that the first term is $\int x^2\delta(x)dx$ which clearly vanishes. Also the integrand for the second term contains $\lim[\epsilon^2/(x^2+\epsilon^2)]$ which will be smaller than $\delta(x)$, so the second term will always be smaller in magnitude than $\int x\delta(x)dx$ which is zero.

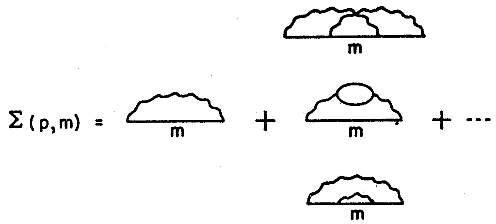


FIG. 11. Proper self-energy $\Sigma(p,m)$ showing explicit dependence of fermion propagators on m.

(II.3), namely

$$\mathcal{L}_F = -\tfrac{1}{2}\bar{\psi}(-i\gamma^\mu \overleftrightarrow{\partial}_\mu)\psi - m\bar{\psi}\psi\,, \tag{II.1}$$

$$\mathcal{L}_I = e\bar{\psi}\gamma_\mu\psi A^\mu\,, \tag{II.3}$$

where m is the mechanical (or bare) mass of the fermion. The bare propagator (II.4) will then be written as

$$S(p,m) = (p - m + i\epsilon)^{-1}\,, \tag{II.4}$$

where the explicit dependence of S on the mechanical mass m is written for clarity and the dressed fermion propagator is then

$$S'(p)^{-1} = p - m - \Sigma(p,m) \tag{B1}$$

where $\Sigma(p,m)$ is the sum of all proper fermion self-energy diagrams with all internal fermion lines in terms of the bare mass m. $\Sigma(p,m)$ is, of course, a series in the coupling constant, the lowest diagrams of which are shown in Fig. 11.

The physical mass, m_0, is considered to correspond to a pole of the dressed propagator $S'(p)$, i.e.,

$$p - m - \Sigma(p,m) = 0 \quad \text{for} \quad p = m_0\,. \tag{B2}$$

In the second case, where the expansion involves a counterterm, the Lagrangians are rearranged to be

$$\mathcal{L}_F' = -\tfrac{1}{2}\bar{\psi}(-i\gamma^\mu\partial_\mu)\psi - m_0\bar{\psi}\psi\,, \tag{B3}$$

$$\mathcal{L}_I' = e\bar{\psi}\gamma_\mu\psi A^\mu + \delta m\bar{\psi}\psi\,, \tag{B4}$$

where $\delta m = m_0 - m$. The dressed propagator is now ex-

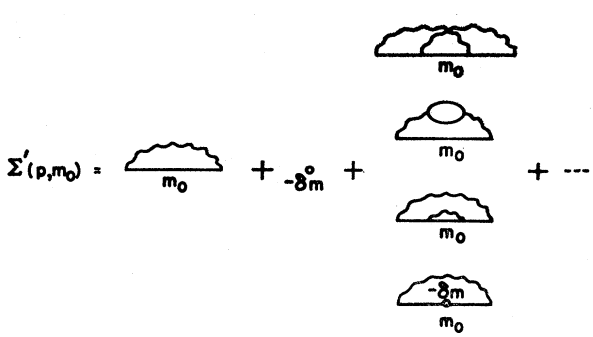
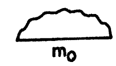


FIG. 12. Self-energy $\Sigma'(p,m_0)$ with counterterms, showing explicit dependence of fermion propagators on m_0.

FIG. 13. $\Sigma^{(2)}(p,m_0)$ lowest order self-energy term.



pressed as

$$\begin{aligned} S'(p)^{-1} &= S(p,m_0)^{-1} - \Sigma'(p,m_0) \\ &= p - m_0 - \Sigma'(p,m_0)\,, \end{aligned} \tag{B5}$$

where $S(p,m_0)$ is the bare propagator (II.4) with the bare mass m replaced by the physical mass m_0, and $\Sigma'(p,m_0)$ is the new sum of fermion proper self-energy diagrams (with all fermion propagators in terms of m_0), the lowest order terms of which are shown in Fig. 12.

The equation for the pole of $S'(p)$ is then

$$p - m_0 - \Sigma'(p,m_0) = 0 \quad \text{for} \quad p = m_0\,. \tag{B6}$$

We shall now proceed to show the equivalence of the two expansions. In particular, we shall demonstrate that expressions (B1) and (B5) for the dressed propagator, $S'(p)$, are truly equivalent.

Let us examine $\Sigma'(p,m_0)$ as shown in Fig. 12 more carefully. We see that this consists of the same terms as in $\Sigma(p,m)$ with m_0 replacing m (called $\Sigma(p,m_0)$) with the addition of terms where $-\delta m$ has replaced every second-order self-energy term in $\Sigma(p,m_0)$. In other words, if, in any diagram in $\Sigma(p,m_0)$, we have $\Sigma^{(2)}(p,m_0)$, shown in Fig. 13, then we add the same diagram with $\Sigma^{(2)}$ replaced by $-\delta m$. Now, it is also true that for any diagram to a particular order (of α) in $\Sigma(p,m_0)$, there will be a higher order diagram in $\Sigma(p,m_0)$ where an internal fermion line has had $\Sigma^{(2)}(p,m_0)$ inserted in the line. This means that for any diagram in $\sum(p,m_0)$, $\Sigma'(p,m_0)$ contains also all diagrams with $-\delta m$'s inserted into internal fermion lines in all possible ways.

We note that, formally, the term,

$$\underset{m_0}{\overset{-\delta m}{\text{——}\!\circ\!\text{——}}} \tag{B7}$$

corresponds to

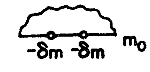
$$\frac{1}{p-m_0}(-\delta m)\frac{1}{p-m_0} = -\delta m\frac{\partial}{\partial m_0}\left(\frac{1}{p-m_0}\right) \tag{B8}$$

or, symbolically,

$$\overset{-\delta m}{\text{——}\!\circ\!\text{——}} = -\delta m\frac{\partial}{\partial m_0}\left(\text{——}\right).$$

This means, for example, that the term in $\Sigma'(p,m_0)$,

FIG. 14. Typical term in $\Sigma'(p,m_0)$.



shown in Fig. 14, can be written as

$$\tfrac{1}{2}(-\delta m)^2\frac{\partial^2}{\partial m_0{}^2} \quad \text{(Fig. 13)}. \tag{B9}$$

Now looking carefully at Fig. 12, we can write

$$\Sigma'(p,m_0)=\Sigma(p,m_0)-\delta m-\delta m\frac{\partial}{\partial m_0}\Sigma(p,m_0)+\cdots+\frac{(-\delta m)^n}{n!}\frac{\partial^n}{\partial m_0{}^n}\Sigma(p,m_0)+\cdots. \tag{B10}$$

If we write $\Sigma(p,m)$ in a Taylor series around the mass m_0, we have

$$\Sigma(p,m)=\Sigma(p,m_0)+(m-m_0)\frac{\partial}{\partial m}\Sigma(p,m)\bigg|_{m=m_0}+\cdots+\frac{(m-m_0)^n}{n!}\frac{\partial^n}{\partial m^n}\Sigma(p,m)\bigg|_{m=m_0}+\cdots. \tag{B11}$$

But $m-m_0=-\delta m$, and

$$\frac{\partial}{\partial m}\Sigma(p,m)\bigg|_{m=m_0}=\frac{\partial}{\partial m_0}\Sigma(p,m_0) \tag{B12}$$

so (B11) becomes

$$\Sigma(p,m)=\Sigma(p,m_0)-\delta m\frac{\partial}{\partial m_0}\Sigma(p,m_0)+\cdots+\frac{(-\delta m)^n}{n!}\frac{\partial^n}{\partial m_0{}^n}\Sigma(p,m_0)+\cdots. \tag{B13}$$

If we now compare (B10) and (B13), we find that

$$\begin{aligned}\Sigma'(p,m_0)&=\Sigma(p,m)-\delta m\\&=\Sigma(p,m)+m-m_0.\end{aligned} \tag{B14}$$

This equation is the final one needed to prove our result as it relates the fermion proper self-energy including all the counterterms (in terms of the physical mass) to the self-energy without counterterms (in terms of the bare mass). Substituting (B14) into the expression for $S'(p)$ in (B5) yields (B1), thus proving the formal equivalence of the two procedures.

From our viewpoint, the important thing about this equivalence is that we can use the Taylor expansion procedure rather than the counterterm procedure which is cumbersome in our case. We then write all diagrams as in the perturbation series with the bare mass, but use the physical mass in the fermion propagators and add all diagrams involved in the Taylor expansion. Of course, a Taylor expansion in m_0 of any diagram becomes just a Taylor expansion of all its fermion propagators. We can also write an unambiguous perturbation procedure for evaluating δm in any required order. From Eq. (B2) we have

$$m_0-m-\Sigma(p=m_0,\,m)=0$$

or

$$\delta m=m_0-m=\Sigma(p=m_0,\,m). \tag{B15}$$

But, using the Taylor expansion of $\Sigma(p,m)$ in the form of Eq. (B13), we get an expression involving only the physical mass.

$$\delta m=\Sigma(p=m_0,\,m_0)-\delta m\frac{\partial}{\partial m_0}\Sigma(p=m_0,\,m_0)+\cdots+\frac{(-\delta m)^n}{n!}\frac{\partial^n}{\partial m_0{}^n}\Sigma(p=m_0,\,m_0)+\cdots. \tag{B16}$$

Now this is not in closed form for δm; however, it is ideal for a power series solution. If we write δm and $\Sigma(p,m_0)$ as power series in α,

$$\begin{aligned}\delta m&=\delta m^{(1)}+\delta m^{(2)}+\cdots,\\ \Sigma(p,m_0)&=\Sigma^{(1)}(p,m_0)+\Sigma^{(2)}(p,m_0)+\cdots,\end{aligned} \tag{B17}$$

where the superscripts refer to the power of α, we get the following set of equations:

$$\begin{aligned}\delta m^{(1)}&=\Sigma^{(1)}(p=m_0,\,m_0),\\ \delta m^{(2)}&=\Sigma^{(2)}(p=m_0,\,m_0)-\delta m^{(1)}\frac{\partial}{\partial m_0}\Sigma^{(1)}(p=m_0,\,m_0),\\ \delta m^{(3)}&=\Sigma^{(3)}(p=m_0,\,m_0)-\delta m^{(2)}\frac{\partial}{\partial m_0}\Sigma^{(1)}(p=m_0,\,m_0),\\ &\quad-\delta m^{(1)}\frac{\partial}{\partial m_0}\Sigma^{(2)}(p=m_0,\,m_0)\\ &\quad+\frac{(-\delta m^{(1)})^2}{2}\frac{\partial^2}{\partial m_0{}^2}\Sigma^{(1)}(p=m_0,\,m_0),\ \text{etc.}\end{aligned} \tag{B18}$$

These equations (B18) provide a consistent approximation procedure for δm to any order in α.

A similar procedure for carrying out mass renormalizations for the external lines can be written down. This procedure involving squared matrix elements and a similar Taylor expansion was used to eliminate all but the vertex contribution to the second order anomalous magnetic moment. The reader is referred to the thesis of one of the authors (M. E. A.) for further details.[59]

[59] See Ref. 35. The procedure of writing down squared matrix elements diagrammatically is similar to that outlined by Thirring [W. E. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958), p. 147].