Consistency Conditions on the Strong Interactions Implied by a Partially Conserved Axial-Vector Current

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It is shown that a partially conserved $\Delta S=0$ axial-vector current $(\partial_\lambda J_\lambda{}^A = C\varphi_\pi)$ implies consistency conditions involving the strong interactions alone. The most interesting of these is a relation among the symmetric isotopic-spin pion-nucleon scattering amplitude $A^{\pi N(+)}$, the pionic form factor of the nucleon $K^{NN\pi}$, and the rationalized, renormalized pion-nucleon coupling constant g_r:

$$g_r{}^2/M = A^{\pi N(+)}(\nu=0, \nu_B=0,\ k^2=0)/K^{NN\pi}(k^2=0).$$

[M is the nucleon mass and $-k^2$ the (mass)^2 of the initial pion. The final pion is on mass shell; the energy and momentum transfer variables ν and ν_B are defined in the text.] By using experimental pion-nucleon scattering data, we find that this relation is satisfied to within 10%. Consistency conditions involving the $\pi\pi$ and the $\pi\Lambda$ scattering amplitudes are stated.

IN 1958 Goldberger and Treiman[1] proposed a remarkable formula for the charged pion decay amplitude, which agrees with experiment to within 10%. Subsequently, Nambu, Gell-Mann and others[2] suggested that the success of the Goldberger-Treiman relation could be simply understood if it were postulated that the strangeness-conserving axial-vector current is partially conserved. The partial-conservation hypothesis leads to a number of relations connecting the weak and strong interactions, of which the Goldberger-Treiman relation is the simplest.[3] So far, only the relation for charged pion decay has been tested experimentally.

We wish to point out in this paper that, in addition to giving relations connecting the weak and strong interactions, the partially conserved axial-vector current hypothesis leads *to consistency conditions involving the strong interactions alone.*[4] This comes about, as will be explained below, because under special circumstances only the Born approximation contributes to matrix elements of the divergence of the axial-vector current. The most interesting consistency condition is a nontrivial relation among the symmetric isotopic spin pion-nucleon scattering amplitude $A^{\pi N(+)}$, the pionic form factor of the nucleon $K^{NN\pi}$, and the rationalized, renormalized pion-nucleon coupling constant g_r:

$$\frac{g_r{}^2}{M} = \frac{A^{\pi N(+)}(\nu=0,\ \nu_B=0,\ k^2=0)}{K^{NN\pi}(k^2=0)}. \tag{1}$$

[Here M is the nucleon mass and $-k^2$ is the (mass)^2 of the initial pion. The final pion is on mass shell; the energy and momentum transfer variables ν and ν_B are defined in Eq. (15) below.] By using experimental pion-nucleon scattering data, we find that this relation is satisfied to within 10%.

In Sec. I we define and discuss the concept of a partially conserved axial-vector current. In Sec. II, we derive the consistency condition relating the pion-nucleon scattering amplitude to the pion-nucleon coupling constant. In Sec. III, pion-nucleon dispersion relations and experimental pion-nucleon scattering data are used to test whether the consistency condition is satisfied. In Sec. IV, other consistency conditions on the strong interactions are stated.

I. DEFINITION OF PARTIALLY CONSERVED AXIAL-VECTOR CURRENT

We assume that the weak interactions between leptons and strongly interacting particles are described by a current-current effective Lagrangian of the form

$$-\mathcal{L}_{\rm eff} = J_\lambda(x) j_\lambda(x) + \text{adjoint}, \tag{2a}$$

where

$$j_\lambda(x) = (1/\sqrt{2})[\bar\psi_\mu \gamma_\lambda(1+\gamma_5)\psi_{\nu_\mu} + \bar\psi_e\gamma_\lambda(1+\gamma_5)\psi_{\nu_e}] \tag{2b}$$

is the weak current of the leptons and where J_λ is the weak current of the strongly interacting particles. Let $J_\lambda{}^V$ and $J_\lambda{}^A$ denote the vector and the axial-vector parts of the strangeness-conserving weak current

$$J_\lambda(\Delta S=0) \equiv J_\lambda{}^V + J_\lambda{}^A. \tag{2c}$$

Definition: By partially conserved axial-vector current (PCAC) we mean the hypothesis that

$$\partial_\lambda J_\lambda{}^A = -[i\sqrt{2} M M_\pi{}^2 g_A(0)/g_r K^{NN\pi}(0)]\varphi_\pi + R. \tag{3}$$

Here M is the nucleon mass, M_π is the pion mass, $g_A(0)$ is the β-decay axial-vector coupling constant [$g_A(0) \approx 1.2\cdot 10^{-5}/M^2$], g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r{}^2/4\pi \approx 14.6$), and φ_π

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1 M. L. Goldberger and S. B. Treiman, Phys. Rev. **109**, 193 (1958).
2 Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960); M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); J. Bernstein, M. Gell-Mann, and W. Thirring, Nuovo Cimento **16**, 560 (1960).
3 J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960); S. L. Adler, Phys. Rev. **135**, B963 (1964).
4 Related ideas have been discussed within the framework of a model calculation by K. Nishijima, Phys. Rev. **133**, B1092 (1964).

is the renormalized field operator which creates the π^+. The quantity $K^{NN\pi}(0)$ is the pionic form factor of the nucleon evaluated at zero virtual pion mass; $K^{NN\pi}$ is normalized so that $K^{NN\pi}(-M_\pi^2)=1$. It is explained below how the constant multiplying φ_π in Eq. (3) is chosen. In order to give content to the definition, we must specify properties of the residual operator R. We suppose that for states α and β for which $\langle\beta|\varphi_\pi|\alpha\rangle\neq 0$, and for momentum transfer near the one pion pole at $-M_\pi^2$ [say, for $-M_\pi^2<(p_\beta-p_\alpha)^2<M_\pi^2$], the matrix element of R is much smaller than the matrix element of the pion operator term. In other words, we postulate that if $\langle\beta|\varphi_\pi|\alpha\rangle\neq 0$ and if $|(p_\beta-p_\alpha)^2|<M_\pi^2$, then

$$\frac{|\langle\beta|R|\alpha\rangle|}{[\sqrt{2}MM_\pi^2 g_A(0)/g_rK^{NN\pi}(0)]|\langle\beta|\varphi_\pi|\alpha\rangle|}\ll 1. \quad (4)$$

In what follows, we derive equalities which hold rigorously if the residual operator R is zero. If R is not zero, but satisfies the inequality of Eq. (4), the "equals" signs should be replaced by "approximately equals" signs. The magnitude of the squared momentum transfer $|(p_\beta-p_\alpha)^2|$ is understood to be always less than M_π^2.

It is not actually necessary to specify the constant in front of φ_π in the definition of PCAC. If we simply postulate that

$$\partial_\lambda J_\lambda^A=C\varphi_\pi, \quad (5)$$

the constant C may be determined as follows: Let us consider the matrix element of $\partial_\lambda J_\lambda^A$ between nucleon states $\langle N|\partial_\lambda J_\lambda^A|N\rangle$. Let p_2 and p_1 be, respectively, the four-momenta of the final and the initial nucleon, and let us denote by k the momentum transfer p_2-p_1. According to the usual invariance arguments, $\langle N|J_\lambda^A|N\rangle$ has the form

$$\langle N|J_\lambda^A|N\rangle=\left(\frac{M}{p_{20}}\frac{M}{p_{10}}\right)^{1/2}\bar{u}(p_2)[g_A(k^2)\gamma_\lambda\gamma_5 - f_A(k^2)\sigma_{\lambda\eta}k_\eta\gamma_5-ih_A(k^2)k_\lambda\gamma_5]\tau^+u(p_1), \quad (6)$$

where $\tau^+=\frac{1}{2}(\tau_1+i\tau_2)$ is the isospin raising operator. From Eq. (6), we find that

$$\begin{aligned}\langle N|\partial_\lambda J_\lambda^A|N\rangle|_{k^2=0} &= -ik_\lambda\langle N|J_\lambda^A|N\rangle|_{k^2=0}\\ &=2Mg_A(0)\left(\frac{M}{p_{20}}\frac{M}{p_{10}}\right)^{1/2}\bar{u}(p_2)\gamma_5\tau^+u(p_1). \quad (7)\end{aligned}$$

We also have

$$\begin{aligned}\langle N|C\varphi_\pi|N\rangle &=\frac{C}{k^2+M_\pi^2}\langle N|(-\Box+M_\pi^2)\varphi_\pi|N\rangle\\ &=\frac{C}{k^2+M_\pi^2}\langle N|j_\pi|N\rangle=\frac{C}{k^2+M_\pi^2}ig_r\sqrt{2}K^{NN\pi}(k^2)\\ &\quad\times\left(\frac{M}{p_{20}}\frac{M}{p_{10}}\right)^{1/2}\bar{u}(p_2)\gamma_5\tau^+u(p_1), \quad (8)\end{aligned}$$

where $K^{NN\pi}(k^2)$ is the pionic form factor of the nucleon. From Eq. (8), we find

$$\langle N|C\varphi_\pi|N\rangle|_{k^2=0}=\frac{C}{M_\pi^2}ig_r\sqrt{2}K^{NN\pi}(0)\times\left(\frac{M}{p_{20}}\frac{M}{p_{10}}\right)^{1/2}\bar{u}(p_2)\gamma_5\tau^+u(p_1), \quad (9)$$

and comparing this with Eq. (7) gives

$$C=-i\sqrt{2}MM_\pi^2g_A(0)/g_rK^{NN\pi}(0). \quad (10)$$

If we form the matrix element of $\partial_\lambda J_\lambda^A$ between the one pion state and the vacuum, we find that

$$(2k_0)^{1/2}\langle\pi^+|\partial_\lambda J_\lambda^A|0\rangle=-i\sqrt{2}MM_\pi^2g_A(0)/g_rK^{NN\pi}(0), \quad (11)$$

which is the Goldberger-Treiman relation for charged pion decay. For general states β and α, such that $\langle\beta|\varphi_\pi|\alpha\rangle\neq 0$, we find that

$$\langle\beta|\partial_\lambda J_\lambda^A|\alpha\rangle=-\frac{i\sqrt{2}MM_\pi^2g_A(0)}{g_rK^{NN\pi}(0)}\times\frac{1}{k^2+M_\pi^2}(2k_0)^{1/2}\mathcal{T}(\pi^++\alpha\to\beta). \quad (12)$$

Here $\mathcal{T}(\pi^++\alpha\to\beta)$ is the transition amplitude for the strong reaction $\pi^++\alpha\to\beta$, where the (mass)^2 of the initial π^+ is $-k^2=-(p_\beta-p_\alpha)^2$. Thus, we see that PCAC leads to a whole class of relations connecting the weak and the strong interactions.

The definition of PCAC which we have given is *not* the same as the definition which would be suggested by a polology approach. This would be to define PCAC as the hypothesis that the covariant amplitudes contributing to $\langle\beta|\partial_\lambda J_\lambda^A|\alpha\rangle$ satisfy unsubtracted dispersion relations in the variable k^2, and that these dispersion relations, for $|k^2|<M_\pi^2$ and for *all* values of the other invariants formed from four-momenta in α and β, are dominated by the one pion pole. It is easy to see that if $\langle\beta|\partial_\lambda J_\lambda^A|\alpha\rangle$ depends on invariants other than k^2, the polology version of PCAC is ambiguous. Suppose that A is a covariant amplitude contributing to $\langle\beta|\partial_\lambda J_\lambda^A|\alpha\rangle$, and that A depends on two invariants, s and k^2. Then the polology version of PCAC implies that

$$A(s,k^2)\approx\bar{A}(s)/(k^2+M_\pi^2), \quad (13)$$

where $\bar{A}$ is the residue of A at $k^2=-M_\pi^2$. Let us now define a new variable $s'=s-ak^2$ and treat A as a function of independent variables s' and k^2. To evaluate the residue we set every *explicit* k^2 equal to $-M_\pi^2$. We then find from the polology version of PCAC that

$$A(s',k^2)\approx\frac{\bar{A}[(s-ak^2)+a(-M_\pi^2)]}{k^2+M_\pi^2}=\frac{\bar{A}[s'-aM_\pi^2]}{k^2+M_\pi^2}. \quad (14)$$

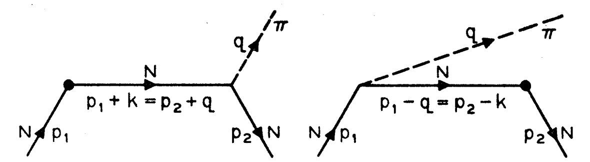

FIG. 1. Generalized Born approximation diagrams for $\langle\pi N|J_\lambda{}^A|N\rangle$. The heavy dot marks the vertex where the operator $J_\lambda{}^A$ acts.

Clearly, Eqs. (13) and (14) differ unless A has no dependence on the variable s to begin with. In other words, the polology definition of PCAC is inherently ambiguous, since the value of the residue at $k^2=-M_\pi{}^2$ depends on how the invariants other than k^2 are chosen.

This ambiguity is not present in the definition of PCAC given in Eqs. (3) and (4). The reason is that k^2 is at no point set equal to $-M_\pi{}^2$ but is kept at whatever value it has in the weak matrix element $\langle\beta|\partial_\lambda J_\lambda{}^A|\alpha\rangle$. We use the unambiguous version of PCAC in the remainder of this paper.[5]

II. CONSISTENCY CONDITION ON PION-NUCLEON SCATTERING

In the previous section we saw, in Eq. (12), that PCAC leads to relations between the strong and the weak interactions. These allow one to predict the weak interaction matrix element $\langle\beta|\partial_\lambda J_\lambda{}^A|\alpha\rangle$, if one knows the strong interaction transition amplitude $\mathcal{T}(\pi^++\alpha\rightarrow\beta)$. The principal point we wish to make in this paper is that there are cases in which only the Born approximation contributes to a covariant amplitude of $\langle\beta|\partial_\lambda J_\lambda{}^A|\alpha\rangle$, for appropriately chosen values of the energy, momentum transfer and other invariants on which the covariant amplitude depends. The Born approximation, in turn, is known in terms of weak and strong interaction coupling constants. Using PCAC to eliminate the weak interaction coupling constants leaves a consistency condition involving the strong interactions alone. In this section, we study the matrix element $\langle\pi N|\partial_\lambda J_\lambda{}^A|N\rangle$ and derive the consistency condition stated in Eq. (1). In Sec. IV, we discuss conditions obtained from other matrix elements of $\partial_\lambda J_\lambda{}^A$.

We begin by writing down the structure of the matrix element $\langle\pi N|J_\lambda{}^A|N\rangle$. Let p_1, p_2, and q be, respectively, the four-momenta of the initial nucleon, the final nucleon, and the final pion. The momentum transfer k is given by $k=p_2+q-p_1$. We define invariants ν and ν_B by

$$\begin{aligned}\nu&=-(p_1+p_2)\cdot k/(2M)\,,\\ \nu_B&=q\cdot k/(2M)\,.\end{aligned}\tag{15}$$

The matrix element can be decomposed into eight covariant amplitudes $A_j(\nu,\nu_B,k^2)$ according to

$$\left(\frac{p_{10}}{M}\frac{p_{20}}{M}2k_0\right)^{1/2}\langle\pi N|J_\lambda{}^A|N\rangle=\bar{u}(p_2)i\sum_{j=1}^{8}O_j{}^\lambda A_j(\nu,\nu_B,k^2)u(p_1)\,.\tag{16}$$

The quantities $O_j{}^\lambda$ are given by[6]

$$\begin{aligned}&O_1{}^\lambda=\tfrac{1}{2}(\boldsymbol{q}\gamma_\lambda-\gamma_\lambda\boldsymbol{q})\,, && O_5{}^\lambda=i\boldsymbol{k}(p_1+p_2)_\lambda\,,\\ &O_2{}^\lambda=(p_1+p_2)_\lambda\,, && O_6{}^\lambda=i\boldsymbol{k}q_\lambda\,,\\ &O_3{}^\lambda=q_\lambda\,, && O_7{}^\lambda=k_\lambda\,,\\ &O_4{}^\lambda=iM\gamma_\lambda\,, && O_8{}^\lambda=i\boldsymbol{k}k_\lambda\,.\end{aligned}\tag{17}$$

The amplitudes $A_j(\nu,\nu_B,k^2)$ have been chosen so that they have no kinematic singularities.[7]

The isotopic spin structure of the amplitudes $A_j(\nu,\nu_B,k^2)$ is specified by writing

$$\begin{aligned}A_j(\nu,\nu_B,k^2)&=\chi_f{}^*\psi_\alpha{}^*A_j(\nu,\nu_B,k^2)_{\alpha\beta}\psi_\beta{}^+\chi_i\,,\\ A_j(\nu,\nu_B,k^2)_{\alpha\beta}&=A_j{}^{(+)}(\nu,\nu_B,k^2)\delta_{\alpha\beta}\\ &\quad+A_j{}^{(-)}(\nu,\nu_B,k^2)\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\,.\end{aligned}\tag{18}$$

Here χ_i and χ_f are, respectively, the isospinors of the initial and final nucleon and ψ_α is the isotopic spin wave function of the final pion. [If the final pion is a $\pi^\pm$, $\psi_\alpha=2^{-1/2}(1,\pm i,0)_\alpha$, while if it is a π^0, $\psi_\alpha=(0,0,1)_\alpha$.] The quantity $\psi_\beta{}^+$ is defined by $\psi_\beta{}^+=\frac{1}{2}(1,i,0)_\beta$, so that $\psi_\beta{}^+\tau_\beta=\tau^+$. The presence of $\psi_\beta{}^+$ is just a reflection of the fact that the weak current $J_\lambda{}^A$ transforms like I_1+iI_2 under isotopic spin rotations.

Let us split each amplitude $A_j(\nu,\nu_B,k^2)_{\alpha\beta}$ into two parts,

$$A_j(\nu,\nu_B,k^2)_{\alpha\beta}=A_j{}^P(\nu,\nu_B,k^2)_{\alpha\beta}+\bar{A}_j(\nu,\nu_B,k^2)_{\alpha\beta}\,.\tag{19}$$

The part $A_j{}^P$ is defined as the sum of all pole terms contributing to A_j, while $\bar{A}_j$ is simply everything that is left over when the pole terms are removed from A_j. The amplitudes $A_j{}^P$ are calculated from the generalized Born approximation diagrams shown in Fig. 1. In each diagram, the heavy dot marks the vertex where the operator $J_\lambda{}^A$ acts. The nucleon vertex of $J_\lambda{}^A$ is given by

$$\tau^+[g_A(k^2)\gamma_\lambda\gamma_5-f_A(k^2)\sigma_{\lambda\eta}k_\eta\gamma_5-ih_A(k^2)k_\lambda\gamma_5]\,.\tag{20}$$

Evaluation of the Born diagrams gives

$$\begin{aligned}&\bar{u}(p_2)i\sum_{j=1}^{8}O_j{}^\lambda\chi_f{}^*\psi_\alpha{}^*A_{j\alpha\beta}{}^P\psi_\beta{}^+\chi_iu(p_1)\\ &=\bar{u}(p_2)\chi_f{}^*\psi_\alpha{}^*\{i\tau_\alpha\gamma_5g_r[1/(\boldsymbol{p}_2+\boldsymbol{q}-iM)]\\ &\quad\times\tau^+[g_A(k^2)\gamma_\lambda\gamma_5-f_A(k^2)\sigma_{\lambda\eta}k_\eta\gamma_5-ih_A(k^2)k_\lambda\gamma_5]\\ &\quad+\tau^+[g_A(k^2)\gamma_\lambda\gamma_5-f_A(k^2)\sigma_{\lambda\eta}k_\eta\gamma_5-ih_A(k^2)k_\lambda\gamma_5]\\ &\quad\times[1/(\boldsymbol{p}_1-\boldsymbol{q}-iM)]i\tau_\alpha\gamma_5g_r\}\chi_iu(p_1)\,,\end{aligned}\tag{21}$$

from which the $A_j{}^P$ are easily obtained. Since the divergence of the terms proportional to $f_A(k^2)$ vanishes

[5] In a previous paper [S. L. Adler, Phys. Rev. **135**, B963 (1964)] we used the polology version of PCAC. If, instead, the definition of Eqs. (3) and (4) had been used, $\mathfrak{M}(\pi^++\alpha\rightarrow\beta)$ in Theorem 2 of the paper would simply have been the invariant matrix element for $\pi^++\alpha\rightarrow\beta$, with the initial π^+ of (mass)$^2=-k^2$.

[6] The kinematic structure of the matrix element $\langle\pi N|J_\lambda{}^A|N\rangle$ has been discussed by N. Dombey, Phys. Rev. **127**, 653 (1962) and by P. Dennery, Phys. Rev. **127**, 664 (1962).

[7] A simple modification of the argument used by Ball [J. S. Ball, Phys. Rev. **124**, 2014 (1961)] can be used to show that the amplitudes A_j have no kinematical singularities.

identically and since the divergence of the terms proportional to $h_A(k^2)$ vanishes when $k^2=0$, we write down only the pole contributions proportional to $g_A(k^2)$:

$$
\begin{aligned}
A_1{}^P&=\frac{g_r g_A(k^2)}{2M}\left[\delta_{\alpha\beta}\left(\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)\right],\\
A_3{}^P&=\frac{g_r g_A(k^2)}{2M}\left[\delta_{\alpha\beta}\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\left(\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right)\right].
\end{aligned}\tag{22}
$$

The amplitudes A_2 and A_4, $\cdots$, A_8 have no pole contributions proportional to $g_A(k^2)$.

Let us now evaluate

$$\langle\pi N|\partial_\lambda J_\lambda{}^A|N\rangle=-ik_\lambda\langle\pi N|J_\lambda{}^A|N\rangle$$

at $k^2=0$. Using the decomposition of $\langle\pi N|J_\lambda{}^A|N\rangle$ into covariants A_j, splitting each A_j into parts $A_j{}^P$ and $\bar{A}_j$, and evaluating the $A_j{}^P$ from Eq. (22), leads to the result that

$$
\begin{aligned}
&[(p_{10}/M)(p_{20}/M)2k_0]^{1/2}\langle\pi N|\partial_\lambda J_\lambda{}^A|N\rangle|_{k^2=0}\\
&\qquad=\bar{u}(p_2)\chi_f{}^*\psi_\alpha{}^*M_{\alpha\beta}(\sqrt{2}\psi_\beta{}^+)\chi_i u(p_1),
\end{aligned}\tag{23}
$$

with

$$
\begin{aligned}
M_{\alpha\beta}&=A(\nu,\nu_B)_{\alpha\beta}-i\boldsymbol{k}B(\nu,\nu_B)_{\alpha\beta},\\
A(\nu,\nu_B)_{\alpha\beta}&=\frac{1}{\sqrt{2}}\{-2M\nu(\bar{A}_1+\bar{A}_2)_{\alpha\beta}+2M\nu_B\bar{A}_{3\alpha\beta}+2g_r g_A(0)\delta_{\alpha\beta}\},\\
B(\nu,\nu_B)_{\alpha\beta}&=\frac{1}{\sqrt{2}}\Big\{2M\bar{A}_{1\alpha\beta}-M\bar{A}_{4\alpha\beta}+2M\nu\bar{A}_{5\alpha\beta}-2M\nu_B\bar{A}_{6\alpha\beta}+g_r g_A(0)\\
&\quad\times\left[\delta_{\alpha\beta}\left(\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)\right]\Big\}.
\end{aligned}\tag{24}
$$

According to the PCAC hypothesis, we can also evaluate $\langle\pi N|\partial_\lambda J_\lambda{}^A|N\rangle$ as $\langle\pi N|C\varphi_\pi|N\rangle$. This gives

$$
\begin{aligned}
M_{\alpha\beta}&=\frac{\sqrt{2}Mg_A(0)}{g_r K^{NN\pi}(0)}[A^{\pi N}(\nu,\nu_B,k^2=0)_{\alpha\beta}-i\boldsymbol{k}B^{\pi N}(\nu,\nu_B,k^2=0)_{\alpha\beta}]\\
&=\frac{\sqrt{2}Mg_A(0)}{g_r K^{NN\pi}(0)}\Big\{A^{\pi N}(\nu,\nu_B,k^2=0)_{\alpha\beta}-i\boldsymbol{k}\bar{B}^{\pi N}(\nu,\nu_B,k^2=0)_{\alpha\beta}-i\boldsymbol{k}\frac{g_r{}^2}{2M}K^{NN\pi}(0)\\
&\quad\times\left[\delta_{\alpha\beta}\left(\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)\right]\Big\}.
\end{aligned}\tag{25}
$$

The amplitudes $A^{\pi N}(\nu,\nu_B,k^2=0)$ and $B^{\pi N}(\nu,\nu_B,k^2=0)$ describe pion-nucleon scattering with the initial pion a virtual pion of $(\text{mass})^2=-k^2=0$ and with the final pion a real pion of $(\text{mass})^2=M_\pi{}^2$.[8] We have separated off the pole terms of B (A has no pole terms); $\bar{B}$ denotes everything which is left over after this separation is made.

Comparing Eqs. (24) and (25), we see that the pole terms proportional to

$$\delta_{\alpha\beta}\left(\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)\tag{26}$$

are identical. This is consistent with the requirements of PCAC. A remarkable fact emerges when we consider the equation for the A amplitudes,

$$
\begin{aligned}
&(1/\sqrt{2})[-2M\nu(\bar{A}_1+\bar{A}_2)_{\alpha\beta}+2M\nu_B\bar{A}_{3\alpha\beta}+2g_r g_A(0)\delta_{\alpha\beta}]\\
&\quad=[\sqrt{2}Mg_A(0)/g_r K^{NN\pi}(0)]A^{\pi N}(\nu,\nu_B,k^2=0)_{\alpha\beta}.
\end{aligned}\tag{27}
$$

Let us set $\nu=\nu_B=0$. Since the $\bar{A}_j$ have all pole terms removed, and since they have no kinematic singularities,

$$\lim_{\nu\to 0}\nu(\bar{A}_1+\bar{A}_2)=\lim_{\nu_B\to 0}\nu_B\bar{A}_3=0.\tag{28}$$

Hence at $\nu=\nu_B=k^2=0$, all the unknown amplitudes drop out. Equation (27) then becomes

$$\delta_{\alpha\beta}\frac{g_r{}^2}{M}=\frac{A^{\pi N}(\nu=0,\nu_B=0,k^2=0)_{\alpha\beta}}{K^{NN\pi}(0)}.\tag{29}$$

Decomposing $A_{\alpha\beta}{}^{\pi N}$ into symmetric and antisymmetric isotopic spin parts,

$$A_{\alpha\beta}{}^{\pi N}=A^{\pi N(+)}\delta_{\alpha\beta}+A^{\pi N(-)}\tfrac{1}{2}[\tau_\alpha,\tau_\beta]\tag{30}$$

gives

$$\frac{g_r{}^2}{M}=\frac{A^{\pi N(+)}(\nu=0,\nu_B=0,k^2=0)}{K^{NN\pi}(0)},\tag{31}$$

$$0=A^{\pi N(-)}(\nu=0,\nu_B=0,k^2=0).\tag{32}$$

Equation (32) is automatically satisfied by virtue of the odd crossing symmetry of $A^{\pi N(-)}$. Equation (31) is a nontrivial consistency condition which must be satisfied if PCAC is true.

We saw above that the pole terms, which are the only pion-nucleon scattering terms of second order in the coupling constant g_r, do not contribute to the amplitude $A^{\pi N(+)}$. The leading term in the perturbation series for $K^{NN\pi}$ is 1. Consequently, if $A^{\pi N(+)}/K^{NN\pi}$ is expanded in a renormalized perturbation series, no term of order $g_r{}^2$ will be present. Thus it is clear that the consistency condition is not an identity in the coupling constant. This makes it fundamentally different from relations obtained from unitarity or from crossing symmetry, which are always true order by order in perturbation theory.

[8] Pion-nucleon scattering with the initial pion virtual has been discussed by E. Ferrari and F. Selleri, Nuovo Cimento **21**, 1028 (1961) and by J. Iizuka and A. Klein, Progr. Theoret. Phys. (Kyoto) **25**, 1017 (1961).

Note that a similar consistency condition cannot be derived for the B amplitudes, since the presence of the terms $2M\bar{A}_1 - M\bar{A}_4$ in Eq. (24) prevents the elimination of the unknown amplitudes $\bar{A}_1$ and $\bar{A}_4$.

In the next section, the condition of Eq. (31) is compared with experiment. Before going on to do this, let us summarize the properties of $J_\lambda{}^A$ that were actually used in the derivation. Nowhere did we use the fact that $J_\lambda{}^A$ is the weak axial-vector current which couples to the leptons. Clearly, the consistency condition may be derived if the following two conditions are met:

(i) There exists a local axial-vector current J_λ, the divergence of which is proportional to the pion field,

$$\partial_\lambda J_\lambda = C\varphi_\pi ; \tag{33}$$

(ii) In the nucleon vertex of J_λ, which apart from isospin is

$$\langle N|J_\lambda|N\rangle = \bar{u}(p_2)[G(k^2)\gamma_\lambda\gamma_5 - F(k^2)\sigma_{\lambda\eta}k_\eta\gamma_5 - iH(k^2)k_\lambda\gamma_5]u(p_1) , \tag{34}$$

the form factors G, F, and H are finite at $k^2=0$, and furthermore, $G(0)$ is nonvanishing. In the matrix element $\langle\pi N|J_\lambda|N\rangle$, the covariant amplitudes $A_{1,\ldots,8}(\nu,\nu_B,k^2)$ are finite at $\nu=\nu_B=k^2=0$ once the poles which arise from the Born-approximation (one-particle intermediate state) diagrams are subtracted off. [Except for the requirement that $G(0)$ be nonvanishing, these conditions are necessarily satisfied if the form factors and covariant amplitudes in the two matrix elements of J_λ satisfy the usual spectral conditions, that is, if their singularities as functions of the complex variables k^2, ν and ν_B arise only from allowed intermediate states.]

Condition (ii) and the requirement of locality are essential for the derivation to go through. They are very restrictive conditions, and it is easy to find axial-vector currents which do not satisfy them but which obey Eq. (33). For instance, the current J_λ' defined by

$$J_\lambda' = C\partial_\lambda \int D(x-x')\varphi_\pi(x')d^4x' ,$$
$$D(x) = -\int \frac{e^{ik\cdot x}}{(2\pi)^4 k^2} d^4k , \tag{35}$$

satisfies $\partial_\lambda J_\lambda' = C\varphi_\pi$, by construction. But J_λ' is not local, and in the nucleon vertex of J_λ', $G(k^2)\equiv 0$ and $H(k^2)\alpha 1/k^2$, so that (ii) is violated.

III. DISPERSION RELATIONS TEST OF CONSISTENCY CONDITION

In this section, we use pion-nucleon dispersion relations and experimental pion-nucleon scattering data to test whether Eq. (31) is satisfied in nature. By using dispersion relations, the on-mass-shell amplitude $A^{\pi N(+)}(\nu=0, \nu_B=0, k^2=-M_\pi{}^2)$ may be calculated from scattering data. However, Eq. (31) involves the off-mass-shell combination $A^{\pi N(+)}(\nu=0, \nu_B=0, k^2=0)/K^{NN\pi}(k^2=0)$, requiring us to use a model to calculate the difference,

$$\frac{A^{\pi N(+)}(\nu=0, \nu_B=0, k^2=0)}{K^{NN\pi}(k^2=0)} - A^{\pi N(+)}(\nu=0, \nu_B=0, k^2=-M_\pi{}^2) . \tag{36}$$

We first give several alternative ways of using pion-nucleon dispersion relations to calculate the on-mass-shell amplitude. We then discuss a model for going off mass shell in k^2, and summarize the final results. In the remainder of this section, we take the charged pion mass to be unity. In these units the nucleon mass is $M=6.72$ and[9]

$$g_r{}^2/M = 27.4\pm 0.7 . \tag{37}$$

The equations used in making the calculations described in this section are derived in the Appendix.

A. Evaluation of $A^{\pi N(+)}$ ($\nu=0$, $\nu_B=0$, $k^2=-1$)

We wish to evaluate the on-mass-shell amplitude $A^{\pi N(+)}(0,0,-1)$. Since the point $\nu=\nu_B=0$ is not a physical one, we must use pion-nucleon dispersion relations to compute $A^{\pi N(+)}(0,0,-1)$ from scattering data. The fixed momentum transfer dispersion relation satisfied by $A^{\pi N(+)}(\nu,\nu_B,-1)$ is[10]

$$A^{\pi N(+)}(\nu,\nu_B,-1) = \frac{1}{\pi}\int_{\nu_0+\nu_B}^{\infty} d\nu' \, \mathrm{Im}A^{\pi N(+)}(\nu',\nu_B,-1) \times\left[\frac{1}{\nu'-\nu}+\frac{1}{\nu'+\nu}\right], \quad \nu_0 = 1+1/(2M) . \tag{38}$$

Since the integral in Eq. (38) probably does not converge, it is necessary to introduce a subtraction.

1. *Threshold Subtraction*

The usual procedure is to make a subtraction at threshold. This gives

$$A^{\pi N(+)}(0,0,-1) = A^{\pi N(+)}(\nu_0,0,-1) - \frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'} \times \frac{\mathrm{Im}A^{\pi N(+)}(\nu',0,-1)\nu_0{}^2}{(\nu'^2-\nu_0{}^2)} , \tag{39}$$

which has a strongly convergent integral. The integrand can be calculated in terms of phase shifts. The integral

[9] The coupling constant $g_r{}^2$ is related to the coupling constant f^2 by $g_r{}^2 = 4\pi\cdot 4M^2 f^2$. We use the value $f^2 = 0.081\pm 0.002$ quoted by W. S. Woolcock, *Proceedings of the Aix-en-Provence International Conference on Elementary Particles* (Centre d'Etudes Nucléaires de Saclay, Seine et Oise, 1961), Vol. I, p. 459.

[10] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

was evaluated using the phase shift analysis of Roper[11] up to a pion laboratory kinetic energy of $T_\pi=700$ MeV, where the integral was truncated. A convergence check indicated that the truncation error is small. The result is

$$\frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'}\frac{\mathrm{Im}A^{\pi N(+)}(\nu',0,-1)\nu_0^2}{\nu'^2-\nu_0^2}=7.4. \tag{40}$$

We make no error estimate here since Roper gives no error estimate for his phase shifts.

The threshold subtraction constant can be expressed in terms of scattering lengths by

$$\frac{A^{\pi N(+)}(\nu_0,0,-1)}{4\pi}=\left(1+\frac{1}{2M}\right)\sum_{l=0}^{\infty}\left[\tfrac{2}{3}a_{l+}^{(3/2)}+\tfrac{1}{3}a_{l+}^{(1/2)}\right]\frac{(l+1)[2(l+1)]!}{2^{l+1}[(l+1)!]^2}$$
$$-2M\sum_{l=1}^{\infty}\{\tfrac{2}{3}[a_{l-}^{(3/2)}-a_{l+}^{(3/2)}]+\tfrac{1}{3}[a_{l-}^{(1/2)}-a_{l+}^{(1/2)}]\}\frac{l(2l)!}{2^l(l!)^2}, \tag{41}$$

where $a_{l\pm}^{(I)}$ is the scattering length in the channel with isospin I, orbital angular momentum l, and total angular momentum $J=l\pm\frac{1}{2}$. Equation (41) is rapidly convergent and it suffices to keep only the S-, P-, D-, and F-wave scattering lengths. Using the S- and P-wave scattering lengths quoted by Woolcock[12] and obtaining D- and F-wave scattering lengths from Roper's polynomial and resonance fits to the phase shifts, gives

$$A^{\pi N(+)}(\nu_0,0,-1)=37.3\pm0.7,\quad A^{\pi N(+)}(0,0,-1)=29.9\pm0.7. \tag{42}$$

The threshold subtraction constant arises almost entirely from the P-wave scattering lengths. The error estimates take into account only the errors in the S- and P-wave scattering lengths quoted by Woolcock.

Alternatively, we can obtain all scattering lengths from the threshold behavior of Roper's fits to the phase shifts, giving

$$A^{\pi N(+)}(\nu_0,0,-1)=40.7,\quad A^{\pi N(+)}(0,0,-1)=33.3. \tag{43}$$

2. *Broad Area Subtraction Method*

There is a fairly large discrepancy between Woolcock's scattering lengths and the threshold behavior of Roper's phase shifts. This suggests that it would be desirable to perform the subtraction in a manner which does not weight threshold behavior so heavily. We give a method which effectively smears the subtraction over a finite segment of the real axis and has the additional advantage of containing a built-in consistency check on the phase shift data used. Let us consider the function

$$F(\nu)=\frac{A^{\pi N(+)}(\nu,\nu_B=0,k^2=-1)}{[(\nu-\nu_0)(\nu+\nu_0)(\nu-\nu_m)(\nu+\nu_m)]^{1/2}}, \tag{44}$$

where $\nu_m>\nu_0$ lies on the physical cut. Since $F(\nu)$ approaches zero at $\nu=\infty$, we can write an unsubtracted dispersion relation

$$F(\nu)=\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\frac{\Delta F(\nu')}{2i}\left(\frac{1}{\nu'-\nu}+\frac{1}{\nu'+\nu}\right), \tag{45}$$

where $\Delta F(\nu')=F(\nu'+i\epsilon)-F(\nu'-i\epsilon)$ is the discontinuity of F across the cut from ν_0 to ∞. The square root in the denominator has opposite signs on the opposite sides of its cut from ν_0 to ν_m and has no cut from ν_m to ∞. Consequently,

$$\frac{\Delta F(\nu')}{2i}=-\frac{\mathrm{Re}A^{\pi N(+)}(\nu',0,-1)}{[(\nu'-\nu_0)(\nu'+\nu_0)(\nu_m-\nu')(\nu_m+\nu')]^{1/2}},\quad \nu_0<\nu'<\nu_m,$$
$$\frac{\Delta F(\nu')}{2i}=\frac{\mathrm{Im}A^{\pi N(+)}(\nu',0,-1)}{[(\nu'-\nu_0)(\nu'+\nu_0)(\nu'-\nu_m)(\nu'+\nu_m)]^{1/2}},\quad \nu_m<\nu'<\infty, \tag{46}$$

giving

$$A^{\pi N(+)}(0,0,-1)=\frac{2}{\pi}\int_{\nu_0}^{\nu_m}\frac{d\nu'}{\nu'}\frac{\mathrm{Re}A^{\pi N(+)}(\nu',0,-1)\nu_0\nu_m}{[(\nu'-\nu_0)(\nu'+\nu_0)(\nu_m-\nu')(\nu_m+\nu')]^{1/2}}$$
$$-\frac{2}{\pi}\int_{\nu_m}^{\infty}\frac{d\nu'}{\nu'}\frac{\mathrm{Im}A^{\pi N(+)}(\nu',0,-1)\nu_0\nu_m}{[(\nu'-\nu_0)(\nu'+\nu_0)(\nu'-\nu_m)(\nu'+\nu_m)]^{1/2}}. \tag{47}$$

This equation involves $\mathrm{Re}A^{\pi N(+)}$ over a segment of finite length, not just at threshold. In the limit as ν_m approaches ν_0, Eq. (47) becomes identical with Eq. (39) for the threshold subtraction. The fact that Eq. (47) involves no principal value integrals makes numerical evaluation easy.

If the exact values of $\mathrm{Re}A^{\pi N(+)}$ and $\mathrm{Im}A^{\pi N(+)}$ were used to evaluate the integrals, Eq. (47) would clearly give the same answer for all values of ν_m between ν_0 and ∞. Thus, by varying ν_m we can check the consistency of the phase shifts used to evaluate $A^{\pi N(+)}(\nu',0,-1)$.

Using the phase shift data of Roper and integrating up to a pion laboratory kinetic energy of 700 MeV gives

[11] L. D. Roper, Phys. Rev. Letters **12**, 340 (1964) and private communication. We use Roper's $l_m=3$ phase shift fit for pion laboratory kinetic energy T_π in the range $0\leq T_\pi\leq 700$ MeV. In terms of ν and ν_B, $T_\pi=\nu-\nu_B-\nu_0$.

[12] W. S. Woolcock, Ref. 9. Woolcock's results are quoted in J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962). The slightly different scattering lengths proposed by Hamilton *et al.* give the same result for $A^{\pi N(+)}(\nu_0,0,-1)$ as do Woolcock's.

TABLE I. $A^{\pi N(+)}$ versus length of the square-root cut, as calculated by using fixed momentum transfer dispersion relations. The upper end of the square-root cut lies at pion-nucleon center-of-mass energy $W_m = M+\omega_m$. At threshold, $\omega_m=1$, and at the peak of the (3,3) resonance, $\omega_m=2.1$. In terms of ω_m, ν_m is given by $\nu_m=\nu_B+\omega_m+\omega_m^2/2M$.

ω_m	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
$A^{\pi N(+)}(0,0,-1)$ $(\nu_B=0)$	29.70	29.43	29.16	28.90	28.66	28.44	28.23	28.02	27.83
$A^{\pi N(+)}(0,-1/2M,-1)$ $(\nu_B=-1/2M)$	26.73	26.54	26.36	26.17	26.00	25.83	25.67	25.51	25.36
D	2.97	2.89	2.80	2.73	2.66	2.61	2.56	2.51	2.47

the results shown in Table I. It is convenient to introduce a parameter ω_m, such that the upper end of the square-root cut lies at pion-nucleon center-of-mass energy $W_m=M+\omega_m$. In terms of ω_m, the parameter ν_m is given by $\nu_m=\nu_B+\omega_m+\omega_m^2/2M$. In changing ω_m from 1.7 to 2.5, we move the upper end of the square-root cut across the peak of the (3,3) resonance, thus considerably altering the distribution of the integral between the two terms in Eq. (47). Still, the total varies by less than 10%, indicating that Roper's phase shifts are reasonably consistent with dispersion relations in the (3,3) resonance region. The end of the cut was not taken greater than $\omega_m=2.5$ to avoid introducing a large truncation error from extending the integrals only to 700 MeV. A convergence check indicated that in all cases shown in Table I the truncation error is small.

The result of this analysis may be stated as

$$A^{\pi N(+)}(0,0,-1)=28.7\pm0.9, \tag{48}$$

where we have taken as the error estimate the variation of $A^{\pi N(+)}$ as ω_m is moved across the peak of the (3,3) resonance.

3. *Alternative Broad-Area Subtraction Method*

As a further check, we have used an alternative method to evaluate $A^{\pi N(+)}(0,0,-1)$. Let us write

$$\begin{aligned} A^{\pi N(+)}(0,0,-1)&=D+A^{\pi N(+)}(\nu=0,\nu_B=-1/2M,-1),\\ D&=A^{\pi N(+)}(\nu=0,\nu_B=0,-1)\\ &\quad -A^{\pi N(+)}(\nu=0,\nu_B=-1/2M,-1). \end{aligned} \tag{49}$$

In other words, we add and subtract the quantity $A^{\pi N(+)}(0,-1/2M,-1)$. In the difference term D, we evaluate $A^{\pi N(+)}(0,-1/2M,-1)$ by using the fixed momentum transfer dispersion relation for $A^{\pi N(+)}$. [See Eq. (38)] with a broad area subtraction. This is just the method used above to evaluate $A^{\pi N(+)}(0,0,-1)$. The results are shown in Table I. Clearly, in forming the difference D of the amplitudes for different values of momentum transfer ν_B, much of the variation of the result with ω_m cancels out. This is probably not accidental. If we take as error estimate the variation of D as ω_m is moved across the (3,3) resonance peak, we find

$$D=2.65\pm0.3. \tag{50}$$

We now add back $A^{\pi N(+)}(0,-1/2M,-1)$ evaluated by an independent method. Let us recall that $\nu_B=-1/(2M)$ corresponds to forward pion-nucleon scattering. Since the even isotopic spin forward scattering amplitude is given by

$$\begin{aligned} F^{(+)}(\nu)=A^{\pi N(+)}&(\nu,-1/2M,-1)\\ &+\nu B^{\pi N(+)}(\nu,-1/2M,-1), \end{aligned} \tag{51}$$

we have

$$F^{(+)}(0)=A^{\pi N(+)}(0,-1/2M,-1). \tag{52}$$

Thus, we can use ordinary forward dispersion relations[13] to evaluate $A^{\pi N(+)}(0,-1/2M,-1)$. Making a broad area subtraction gives

$$\begin{aligned} &A^{\pi N(+)}(0,-1/2M,-1)\\ &=\frac{g_r^2}{M}\frac{\nu_m}{[(\nu_m^2-1/4M^2)(1-1/4M^2)]^{1/2}}\\ &+\frac{2}{\pi}\int_1^{\nu_m}\frac{d\nu'}{\nu'}\frac{\mathrm{Re}F^{(+)}(\nu')\nu_m}{[(\nu'-1)(\nu'+1)(\nu_m-\nu')(\nu_m+\nu')]^{1/2}}\\ &-\frac{2}{\pi}\int_{\nu_m}^{\infty}\frac{d\nu'}{\nu'}\frac{\mathrm{Im}F^{(+)}(\nu')\nu_m}{[(\nu'-1)(\nu'+1)(\nu'-\nu_m)(\nu'+\nu_m)]^{1/2}}. \end{aligned} \tag{53}$$

We recall that

$$\mathrm{Re}F^{(+)}(\nu')=\frac{4\pi}{M}[2M\nu'+M^2+1]^{1/2}\,\mathrm{Re}(f_1^{(+)}+f_2^{(+)})', \tag{54}$$

where $f_1^{(+)}$ and $f_2^{(+)}$ are the usual center-of-mass [isospin (+)] pion-nucleon scattering amplitudes. Furthermore,[13]

$$\mathrm{Im}F^{(+)}(\nu')=\tfrac{1}{2}(\nu'^2-1)^{1/2}[\sigma_+(\nu')+\sigma_-(\nu')], \tag{55}$$

where $\sigma_+(\nu')$ and $\sigma_-(\nu')$ are, respectively, the total π^+p and π^-p cross sections. To evaluate the integrals, we used Roper's phase shifts for laboratory pion kinetic energies below 700 MeV. Above 700 MeV, we used the tabulation of σ_+ and σ_- given by Amblard *et al.*[14] and the

[13] For example, see the article by J. D. Jackson in *Dispersion Relations*, edited by G. R. Screaton (Interscience Publishers, Inc., New York, 1961), p. 38.

[14] B. Amblard *et al.*, Phys. Letters **10**, 138 (1964).

TABLE II. $A^{\pi N(+)}$ versus length of the square root cut, as calculated by using forward scattering dispersion relations.

ω_m	1.5	2.1	2.7	3.3	3.9
$A^{\pi N(+)}(0, -1/2M, -1)$	26.33	26.23	26.15	26.09	26.07

asymptotic region fit of Von Dardel *et al.*[15] The results, shown in Table II, give

$$\begin{aligned} A^{\pi N(+)}(0, -1/2M, -1) &= 26.15\pm0.2\,, \\ A^{\pi N(+)}(0, 0, -1) &= 28.8\pm0.4\,, \end{aligned} \tag{56}$$

where we have taken as the error estimate the variation of $A^{\pi N(+)}(0, -1/2M, -1)$ as ω_m is varied from 1.5 to 3.9. We have not included in the error estimate the error in the factor g_r^2/M appearing in Eq. (53), since when we divide by g_r^2/M to compare the left- and right-hand sides of Eq. (31) this error drops out.

The values of $A^{\pi N(+)}(0, -1/2M, -1)$ obtained by using fixed momentum transfer dispersion relations (Table I) and forward scattering dispersion relations (Table II) are in excellent agreement. When fixed momentum transfer dispersion relations are used, the total result for $A^{\pi N(+)}(0, -1/2M, -1)$ comes from the integration over the physical cut. By contrast, when forward scattering dispersion relations are used, nearly all of the total comes from the pole term in the dispersion relations, which leads to the term $(g_r^2/M)\nu_m[(\nu_m^2-1/4M^2)\times(1-1/4M^2)]^{-1/2}$ in Eq. (53). Thus, the two methods "sample" pion-nucleon scattering in very different ways. Their agreement gives us confidence that the numbers obtained from the dispersion relations calculations are reliable.

B. Model for Going Off Mass Shell in k^2

In order to compare the consistency condition with experiment we must calculate the difference

$$[A^{\pi N(+)}(0,0,0)/K^{NN\pi}(0)]-A^{\pi N(+)}(0, 0, -1). \tag{57}$$

To motivate the model which we use, let us return for a moment to the fixed momentum transfer dispersion relation for $A^{\pi N(+)}(\nu, 0, -1)$,

$$A^{\pi N(+)}(\nu, 0, -1)=\frac{1}{\pi}\int_{\nu_0}^{\infty} d\nu' \,\mathrm{Im}A^{\pi N(+)}(\nu', 0, -1) \times\left[\frac{1}{\nu'-\nu}+\frac{1}{\nu'+\nu}\right]. \tag{58}$$

Let us proceed as if no subtractions were necessary. We evaluate the integral by keeping only the resonant (3,3) state in the integrand and going to the static limit. This gives

$$A^{\pi N(+)}(0, 0, -1)\approx\frac{32}{3}M\pi\cdot\frac{1}{\pi}\int_1^{\infty} d\omega\frac{\mathrm{Im}f_{3,3}}{|\mathbf{q}|^2}, \tag{59}$$

where $|\mathbf{q}|$ is the pion center-of-mass momentum and where $f_{3,3}$ is the resonant (3,3) partial wave amplitude. According to Chew *et al.*,[10] in the narrow resonance approximation one finds that

$$\frac{1}{\pi}\int_1^{\infty} d\omega\frac{\mathrm{Im}f_{3,3}}{|\mathbf{q}|^2}\approx\frac{g_r^2}{12\pi M^2}, \tag{60}$$

giving

$$A^{\pi N(+)}(0, 0, -1)\approx(8/9)(g_r^2/M)\approx24.4. \tag{61}$$

This number is in good agreement with those obtained above by the proper procedure of using subtracted dispersion relations. The fact that a (3,3) dominant, unsubtracted dispersion relation calculation gives a reasonable result for $A^{\pi N(+)}(0, 0, -1)$ suggests that such a calculation may also give a reasonable estimate of the change in $A^{\pi N(+)}$ produced by going off mass shell. Thus, as our model for going off mass shell in k^2, we take

$$\begin{aligned} \Delta &\equiv [A^{\pi N(+)}(0,0,0)/K^{NN\pi}(0)]-A^{\pi N(+)}(0, 0, -1)\,, \\ &= \frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'}\mathrm{Im}\left[\frac{A_{3,3}{}^{\pi N(+)}(\nu',0,0)}{K^{NN\pi}(0)} - A_{3,3}{}^{\pi N(+)}(\nu', 0, -1)\right], \end{aligned} \tag{62}$$

where the subscript 3, 3 indicates that only the resonant partial wave is to be retained.[16]

The integral in Eq. (62) can be evaluated once the off-mass-shell partial wave amplitude $f_{3,3}(\nu', k^2=0)$ is known. It turns out that in the (3,3) resonance region, a very good estimate of $f_{3,3}(\nu', k^2=0)$ is given by

$$f_{3,3}(\nu', k^2=0)\approx f_{3,3}(\nu', k^2=-1)\frac{f_{3,3}{}^B(\nu', k^2=0)}{f_{3,3}{}^B(\nu', k^2=-1)}, \tag{63}$$

where $f_{3,3}{}^B$ denotes the (3,3) projection of the Born approximation.[17] Roughly speaking, the reasons for the validity of Eq. (63) are:

(i) Equation (63) gives $f_{3,3}(\nu', k^2=0)$ the phase of the (3,3) on-mass-shell amplitude, as is required by unitarity.

(ii) The left hand, or "potential" singularity of $f_{3,3}(\nu',k^2)$ nearest to the physical cut is determined entirely by $f_{3,3}{}^B(\nu',k^2)$. Multiplying $f_{3,3}(\nu', -1)$ by $f_{3,3}{}^B(\nu',0)/f_{3,3}{}^B(\nu', -1)$ gives the right-hand side of Eq. (63) approximately the correct nearly potential singularity structure for $f_{3,3}(\nu',0)$. A detailed numerical analysis[18] indicates that the error involved in using Eq.

[15] G. von Dardel *et al.*, Phys. Rev. Letters **8**, 173 (1962).

[16] A justification for this model would be provided if one could prove that $\Delta(\nu)\equiv A^{\pi N(+)}(\nu,0,0)/K^{NN\pi}(0)-A^{\pi N(+)}(\nu, 0, -1)$ satisfies an unsubtracted dispersion relation in the variable ν. Then $\Delta(0)$ could be expressed as an integral of $\mathrm{Im}\Delta(\nu)$ over the physical cut. Since only the (3,3) phase shift is appreciable at low energy, it would be reasonable to keep only the (3,3) partial wave in $\mathrm{Im}\Delta(\nu)$.

[17] E. Ferrari and F. Selleri, Nuovo Cimento **21**, 1028 (1961).

[18] S. L. Adler (to be published).

(63) for $f_{3,3}(\nu',0)$, in the (3,3) resonance region, may be as small as half a percent.

Since $f_{3,3}{}^B(\nu',0)$ is proportional to $K^{NN\pi}(0)$, the pionic form factor of the nucleon drops out of the calculation. Substituting Eq. (63) into Eq. (62) and doing the integration numerically gives the result

$$\Delta \approx -0.5. \tag{64}$$

Hence the model we have used indicates that extrapolation off mass shell has only a small effect, of order 2% of $g_r{}^2/M$. This figure corresponds to the fact that the two terms in the integrand of Eq. (62) cancel up to small terms of order $M_\pi{}^2/M^2$, which is about 2%. The need to use a model is unfortunate, and the extrapolation off mass shell is the least certain aspect of the comparison of Eq. (31) with experiment. However, the apparent smallness of Δ indicates that the model would have to fail very badly for there to be an appreciable effect on the numerical results.

C. Summary

Adding the -0.5 obtained from going off mass shell to the results of Subsection A gives the final results shown in Table III. They indicate that unless the model

TABLE III. Final results for $A^{\pi N(+)}(0,0,0)M/K^{NN\pi}(0)g_r{}^2$. The error estimates are obtained as indicated in the text.

Method	Result	Error estimate
Threshold subtraction, using Woolcock's S- and P-wave scattering lengths.	1.07	± 0.04
Threshold subtraction, using Roper's phase shift fits for all scattering lengths.	1.20	$\cdots$
Broad area subtraction, using fixed momentum transfer dispersion relations.	1.03	± 0.04
Alternative broad area subtraction method, using forward scattering dispersion relations.	1.03	± 0.015

used for going off mass shell is badly in error, the consistency condition of Eq. (31) is satisfied to within 10%, and quite possibly to within 5%. This fact, together with the success of the Goldberger-Treiman relation, suggests that the PCAC hypothesis deserves further study.

IV. OTHER CONSISTENCY CONDITIONS

The consistency condition on pion-nucleon scattering is not the only condition on the strong interactions which is implied by PCAC. In this section, we discuss briefly the conditions connected with several other scattering amplitudes.

A. Condition on Pion-Pion Scattering

Let us consider the pion-pion scattering reaction illustrated in Fig. 2.[19] Let (p_1,α), (p_2,β) be the four-momenta and isospin indices of the initial pions, and (p_3,α'), (p_4,β') the four-momenta and isospin indices of the final pions. We take all four-momenta to be incoming, so that the condition of energy-momentum conservation reads

$$p_1+p_2+p_3+p_4=0. \tag{65}$$

We introduce the standard Mandelstam variables s, t, u by

$$\begin{aligned} s&=(p_1+p_2)^2=(p_3+p_4)^2,\\ t&=(p_1+p_4)^2=(p_2+p_3)^2,\\ u&=(p_1+p_3)^2=(p_2+p_4)^2,\\ s+t+u&=p_1{}^2+p_2{}^2+p_3{}^2+p_4{}^2. \end{aligned} \tag{66}$$

The isospin structure of the pion-pion scattering matrix element is

$$\begin{aligned}(16p_{10}p_{20}p_{30}p_{40})^{1/2}&\langle\pi\pi^{\text{out}}|\pi\pi^{\text{in}}\rangle\\ &=\psi_{\alpha'}{}^*\psi_{\beta'}{}^*M^{\pi\pi}(s,t,u)_{\alpha\beta,\alpha'\beta'}\psi_\alpha\psi_\beta.\end{aligned} \tag{67}$$

From the requirement that the scattering amplitude be symmetric under interchange of the pions, we find that

$$\begin{aligned}M^{\pi\pi}(s,t,u)_{\alpha\beta,\alpha'\beta'}=A^{\pi\pi}(s|t,u)\delta_{\alpha\beta}\delta_{\alpha'\beta'}+A^{\pi\pi}(t|u,s)\delta_{\alpha\alpha'}\delta_{\beta\beta'}\\ +A^{\pi\pi}(u|t,s)\delta_{\alpha\beta'}\delta_{\beta\alpha'},\end{aligned} \tag{68}$$

where $A^{\pi\pi}(s|t,u)$ is a symmetric function of t and u. $A^{\pi\pi}$ also depends on $p_1{}^2$, $p_2{}^2$, $p_3{}^2$, and $p_4{}^2$. It is easy to see that at the symmetric point $s=t=u=(p_1{}^2+p_2{}^2+p_3{}^2+p_4{}^2)/3$, $A^{\pi\pi}$ is left invariant by the interchange $p_1{}^2\leftrightarrow p_2{}^2$, by the interchange $p_3{}^2\leftrightarrow p_4{}^2$, and by the simultaneous interchanges $p_1{}^2\leftrightarrow p_3{}^2$, $p_2{}^2\leftrightarrow p_4{}^2$.

Let us now consider the axial-vector matrix element $\langle\pi\pi|J_\lambda{}^A|\pi\rangle$. Let $p_2\equiv k$ be the momentum transfer and β the isospin index associated with the current $J_\lambda{}^A$, while we take (p_1,α), (p_3,α'), and (p_4,β') to be the four-momenta and isospin indices of the three pions. The isospin structure of the axial-vector matrix element is

$$\begin{aligned}(8p_{10}p_{30}p_{40})^{1/2}&\langle\pi\pi|J_\lambda{}^A|\pi\rangle\\ &=\psi_{\alpha'}{}^*\psi_{\beta'}{}^*M(s,t,u)_{\alpha\beta,\alpha'\beta'}{}^\lambda\psi_\alpha\psi_\beta{}^+.\end{aligned} \tag{69}$$

Defining Mandelstam variables as above, we find that the amplitude $M(s,t,u)_{\alpha\beta,\alpha'\beta'}{}^\lambda$ is given by

$$\begin{aligned}M(s,t,u)_{\alpha\beta,\alpha'\beta'}{}^\lambda\\ =[A_1(s|t,u)(p_3+p_4)_\lambda+A_2(s|t,u)(p_3-p_4)_\lambda\\ +A_3(s|t,u)p_{1\lambda}]\delta_{\alpha\beta}\delta_{\alpha'\beta'}+[A_1(t|u,s)(p_1+p_3)_\lambda\\ +A_2(t|u,s)(p_1-p_3)_\lambda+A_3(t|u,s)p_{4\lambda}]\delta_{\alpha\alpha'}\delta_{\beta\beta'}\\ +[A_1(u|t,s)(p_1+p_4)_\lambda+A_2(u|t,s)(p_1-p_4)_\lambda\\ +A_3(u|t,s)p_{3\lambda}]\delta_{\alpha\beta'}\delta_{\beta\alpha'},\end{aligned} \tag{70}$$

where $A_1(s|t,u)$ and $A_3(s|t,u)$ are symmetric functions and $A_2(s|t,u)$ is an antisymmetric function of the variables t and u.

There are no pole terms which contribute to the amplitudes A_1, A_2, and A_3 of Eq. (70). Thus, when

$$p_2\cdot p_1=p_2\cdot p_3=p_2\cdot p_4=0, \tag{71}$$

[19] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

we have $p_{2\lambda}M(s,t,u)_{\alpha\beta,\alpha'\beta'}{}^{\lambda}=0$, in other words,

$$\langle\pi\pi|\partial_\lambda J_\lambda{}^A|\pi\rangle=0. \tag{72}$$

Equation (71) implies that we are at the symmetric point

$$s=t=u=k^2-M_\pi{}^2. \tag{73}$$

Since $s+t+u=-3M_\pi{}^2+k^2$, we see that Eq. (73) can be satisfied when $k^2=0$, giving the result that $\langle\pi\pi|\partial_\lambda J_\lambda{}^A|\pi\rangle$ vanishes when $k^2=0$ and $s=t=u=-M_\pi{}^2$. The PCAC hypothesis allows us to write

$$\langle\pi\pi|\partial_\lambda J_\lambda{}^A|\pi\rangle=C\langle\pi\pi|\varphi_\pi|\pi\rangle. \tag{74}$$

Consequently, PCAC implies that

$$A^{\pi\pi}(s=-M_\pi{}^2|t=-M_\pi{}^2,\, u=-M_\pi{}^2|k^2=0)=0\,, \tag{75}$$

where $-k^2$ is the (mass)^2 of one of the four pions and where the other three pions are on mass shell.

Comparison of Eq. (75) with experiment will be difficult, since the effect of one of the pions being off mass shell is very likely to be important. In particular, the negative of the pion-pion amplitude at the on-mass-shell symmetric point,

$$-A^{\pi\pi}(s=-\tfrac{4}{3}M_\pi{}^2|t=-\tfrac{4}{3}M_\pi{}^2,\, u=-\tfrac{4}{3}M_\pi{}^2|k^2=-M_\pi{}^2) \tag{76}$$

is just the effective pion-pion coupling constant[19] and is not zero.

B. Condition on Pion-Lambda Scattering[20]

The derivation in this case closely parallels the derivation given in Sec. II for the condition on πN scattering. The generalized Born approximation diagrams for $\langle\pi\Lambda|J_\lambda{}^A|\Lambda\rangle$ are shown in Fig. 3. In the derivation of Sec. II, we make the replacements

$$ig_r\bar{\psi}_N\gamma_5\boldsymbol{\tau}\psi_N\cdot\boldsymbol{\varphi}_\pi\rightarrow ig_{\Lambda\Sigma}(\bar{\psi}_{\Sigma^-}\gamma_5\psi_\Lambda+\bar{\psi}_\Lambda\gamma_5\psi_{\Sigma^+})\varphi_{\pi^+}+\cdots \tag{77}$$

to define the $\Lambda\Sigma\pi$ strong vertex[21];

$$g_A\bar{\psi}_N\gamma_\lambda\gamma_5\tau^+\psi_N\rightarrow g_A{}^{\Lambda\Sigma}(\bar{\psi}_{\Sigma^+}\gamma_\lambda\gamma_5\psi_\Lambda+\bar{\psi}_\Lambda\gamma_\lambda\gamma_5\psi_{\Sigma^-})+\cdots \tag{78}$$

to define the $\Lambda\Sigma$ weak vertex; and

$$A_{\alpha\beta}{}^{\pi N}-ikB_{\alpha\beta}{}^{\pi N}\rightarrow(A^{\pi\Lambda}-ikB^{\pi\Lambda})\delta_{\alpha\beta} \tag{79}$$

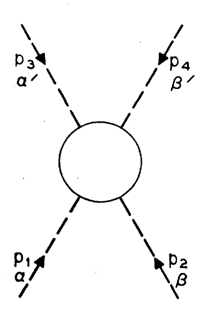

FIG. 2. Four-momenta and isospin indices for $\pi\pi$ scattering.

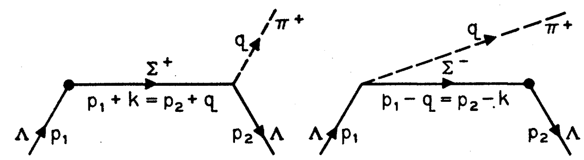

FIG. 3. Generalized Born approximation diagrams for $\langle\pi\Lambda|J_\lambda{}^A|\Lambda\rangle$.

to define the $\pi\Lambda$ scattering amplitudes. Equation (27) becomes

$$-2M\nu(\bar{A}_1+\bar{A}_2)+2M\nu_B\bar{A}_3+2g_{\Lambda\Sigma}g_A{}^{\Lambda\Sigma}(0)\times\left\{1-\frac{\sigma}{2}\left[\frac{1}{\nu_B-\nu+\sigma}+\frac{1}{\nu_B+\nu+\sigma}\right]\right\}=\frac{g_A{}^{\Lambda\Sigma}(0)(M_\Lambda+M_\Sigma)}{g_{\Lambda\Sigma}K^{\Lambda\Sigma}(0)}A^{\pi\Lambda}(\nu,\,\nu_B,\,k^2=0)\,, \tag{80}$$

where $\sigma=(M_\Sigma{}^2-M_\Lambda{}^2)/(2M_\Lambda)$, $\nu=-(p_1+p_2)\cdot k/(2M_\Lambda)$, $\nu_B=q\cdot k/(2M_\Lambda)$, and where $K^{\Lambda\Sigma}$ is the form factor of the $\Lambda\Sigma\pi$ vertex, normalized so that $K^{\Lambda\Sigma}(-M_\pi{}^2)=1$. Setting $\nu=\nu_B=0$ gives the consistency condition

$$0=A^{\pi\Lambda}(\nu=0,\,\nu_B=0,\,k^2=0)\,. \tag{81}$$

This is a null condition and thus differs greatly from the condition derived for πN scattering. The difference arises from the fact that the intermediate state baryon in the generalized Born approximation for $\langle\pi\Lambda|J_\lambda{}^A|\Lambda\rangle$ is a Σ, which has a mass unequal to that of the external Λ. This makes the quantity σ in Eq. (80) different from zero, with the result that the coefficient of $2g_{\Lambda\Sigma}g_A{}^{\Lambda\Sigma}(0)$ vanishes when ν and ν_B are set equal to zero. In the case of πN scattering, σ is zero, and a nonnull condition on $A^{\pi N}$ is obtained. It would be an interesting problem to try to determine from a study of $\pi\Lambda$ scattering whether Eq. (81) is satisfied.

C. Other Reactions

The space-spin structures of $\langle\pi K|J_\lambda{}^A|K\rangle$ and $\langle\pi(\Sigma,\Xi)|J_\lambda{}^A|(\Sigma,\Xi)\rangle$ are similar to the space-spin structures of $\langle\pi\pi|J_\lambda{}^A|\pi\rangle$ and $\langle\pi N|J_\lambda{}^A|N\rangle$, respectively. Consequently, there will be consistency conditions on the πK, the $\pi\Sigma$, and the $\pi\Xi$ scattering amplitudes. Since $\langle\pi(\Sigma,\Xi)|J_\lambda{}^A|(\Sigma,\Xi)\rangle$ has a generalized Born approximation diagram with an intermediate (Σ,Ξ), the consistency condition will be a nonnull condition, like Eq. (31) for πN scattering, rather than a null condition, like Eq. (81) for $\pi\Lambda$ scattering.

We have not studied reactions with more than two particles in the final state. It would be interesting, for example, to determine from a study of $\langle\pi\pi N|J_\lambda{}^A|N\rangle$ whether PCAC implies a consistency condition involving the amplitudes for $\pi+N\rightarrow\pi+\pi+N$.

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I wish to thank Professor S. B. Treiman for discussions which greatly helped to clarify the ideas presented

[20] References dealing with $\pi\Lambda$ scattering are given by T. L. Trueman, Phys. Rev. **127**, 2240 (1962).

[21] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

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APPENDIX

We derive here the equations used in the numerical calculations described in Sec. III. Let us consider the reaction $\pi(k)+N(p_1)\to\pi(q)+N(p_2)$, where the four-momenta of the particles are indicated in parentheses. We take the nucleons and the final pion to be on mass shell,

$$p_1^2=p_2^2=-M^2\,,\quad q^2=-M_\pi^2\,,\tag{A1}$$

but keep k^2 arbitrary. Let $\mathbf{k}=-\mathbf{p}_1$ and $\mathbf{q}=-\mathbf{p}_2$ be, respectively, the momenta of the initial and final pion in the center-of-mass frame of the reaction, and let k_0, p_{10}, q_0, p_{20} be the center-of-mass particle energies. We denote by W the total center-of-mass energy

$$\begin{aligned} W&=k_0+p_{10}=q_0+p_{20}\,,\\ k_0&=(W^2-M^2-k^2)/2W\,,\\ q_0&=(W^2-M^2+M_\pi^2)/2W\,,\\ p_{10}&=(W^2+M^2+k^2)/2W\,,\\ p_{20}&=(W^2+M^2-M_\pi^2)/2W\,.\end{aligned}\tag{A2}$$

We denote by φ the center-of-mass scattering angle between the final and initial pion, so that

$$y\equiv\cos\varphi=\hat{q}\cdot\hat{k}\,,\tag{A3}$$

where $\hat{q}$ and $\hat{k}$ are unit vectors along the directions of the final and initial pion, respectively. The magnitudes $|\mathbf{q}|$ and $|\mathbf{k}|$ are clearly given by

$$|\mathbf{q}|=(q_0^2-M_\pi^2)^{1/2}\,,\quad |\mathbf{k}|=(k_0^2+k^2)^{1/2}\,.\tag{A4}$$

The quantities ν and ν_B are related to W and $\cos\varphi$ by

$$\begin{aligned}\nu-\nu_B&=(W^2-M^2)/2M\,,\\ \nu_B&=(1/2M)[|\mathbf{q}||\mathbf{k}|\cos\varphi-q_0k_0]\,.\end{aligned}\tag{A5}$$

The variable ω, frequently used in going to the static limit, is defined by

$$\omega=W-M\,.\tag{A6}$$

Let us introduce center-of-mass amplitudes f_1 and f_2 by writing

$$\bar{u}(p_2)(A^{\pi N}-ikB^{\pi N})u(p_1)=\frac{4\pi W}{M}\chi_{sf}^\dagger[f_1+f_2\boldsymbol{\sigma}\cdot\hat{q}\boldsymbol{\sigma}\cdot\hat{k}]\chi_{si}\,,\tag{A7}$$

where $A^{\pi N}$ and $B^{\pi N}$ are the covariant amplitudes used in the text and where χ_{sf} and χ_{si} are the nucleon spinors. (We suppress isospin structure.) The transformation relating the amplitudes f_1, f_2 to the amplitudes $A^{\pi N}$, $B^{\pi N}$ is

$$\begin{aligned}\frac{f_1}{[(p_{10}+M)(p_{20}+M)]^{1/2}}&=\frac{1}{2W}\frac{A^{\pi N}}{4\pi}+\frac{W-M}{2W}\frac{B^{\pi N}}{4\pi}\,,\\ \frac{f_2}{[(p_{10}-M)(p_{20}-M)]^{1/2}}&=\frac{-1}{2W}\frac{A^{\pi N}}{4\pi}+\frac{W+M}{2W}\frac{B^{\pi N}}{4\pi}\,.\end{aligned}\tag{A8}$$

The partial wave expansion of f_1 and f_2 is given by

$$\begin{aligned} f_1&=\sum_{l=0}^{\infty}f_{l+}P_{l+1}'(y)-\sum_{l=2}^{\infty}f_{l-}P_{l-1}'(y)\,,\\ f_2&=\sum_{l=1}^{\infty}(f_{l-}-f_{l+})P_l'(y)\,,\\ f_{l+}&=\frac{1}{2}\int_{-1}^{1}dy[f_2P_{l+1}(y)+f_1P_l(y)]\,,\\ f_{l-}&=\frac{1}{2}\int_{-1}^{1}dy[f_2P_{l-1}(y)+f_1P_l(y)]\,,\end{aligned}\tag{A9}$$

where $f_{l\pm}$ is the amplitude for the partial wave with orbital angular momentum l and total angular momentum $J=l\pm\frac{1}{2}$. The symmetric isospin amplitude $f_{l\pm}^{(+)}$ is given in terms of the isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$ amplitudes by

$$f_{l\pm}^{(+)}=\tfrac{2}{3}f_{l\pm}^{(3/2)}+\tfrac{1}{3}f_{l\pm}^{(1/2)}\,.\tag{A10}$$

Finally, we need the inverse of Eq. (A8) for the amplitude $A^{\pi N}$,

$$\frac{A^{\pi N}}{4\pi}=\frac{(W+M)f_1}{[(p_{10}+M)(p_{20}+M)]^{1/2}}-\frac{(W-M)f_2}{[(p_{10}-M)(p_{20}-M)]^{1/2}}\,.\tag{A11}$$

A. Equations for Threshold Subtraction and Static Limit

Let us first consider the case when $k^2=-M_\pi^2$ and derive the equations used in the threshold subtraction and the static limit treatments of the dispersion relations. Below the two-pion threshold,

$$f_{l\pm}^{(I)}=\exp[i\delta_{l\pm}^{(I)}]\sin\delta_{l\pm}^{(I)}/|\mathbf{q}|\,,\tag{A12}$$

where $\delta_{l\pm}^{(I)}$ is the phase shift. The scattering length $a_{l\pm}^{(I)}$ is defined by

$$a_{l\pm}^{(I)}=\lim_{|\mathbf{q}|\to0}\frac{f_{l\pm}^{(I)}}{|\mathbf{q}|^{2l}}\,.\tag{A13}$$

Using the facts that

$$\cos\varphi=[(2M\nu_B+M_\pi{}^2)/|\mathbf{q}|^2]+1\,, \tag{A14}$$

and that the leading term of $P_l'(y)$ for large y is

$$P_l'(y)\sim[l(2l)!/2^l(l!)^2]y^{l-1}\,, \tag{A15}$$

we find that, at threshold,

$$\begin{aligned}
[f_1{}^{(+)}]_T=&\sum_{l=0}^{\infty}[\tfrac{2}{3}a_{l+}{}^{(3/2)}+\tfrac{1}{3}a_{l+}{}^{(1/2)}]\\
&\times\frac{(l+1)[2(l+1)]!}{2^{l+1}[(l+1)!]^2}[2M\nu_B+M_\pi{}^2]^l\,,\\
\left[\frac{f_2{}^{(+)}}{|\mathbf{q}|^2}\right]_T=&\sum_{l=1}^{\infty}\{\tfrac{2}{3}[a_{l-}{}^{(3/2)}-a_{l+}{}^{(3/2)}]\\
&+\tfrac{1}{3}[a_{l-}{}^{(1/2)}-a_{l+}{}^{(1/2)}]\}\\
&\times\frac{l(2l)!}{2^l(l!)^2}[2M\nu_B+M_\pi{}^2]^{l-1}\,,\\
\frac{[A^{\pi N(+)}]_T}{4\pi}=&\left(1+\frac{1}{2M}\right)[f_1{}^{(+)}]_T-2M\left[\frac{f_2{}^{(+)}}{|\mathbf{q}|^2}\right]_T.
\end{aligned} \tag{A16}$$

When $\nu_B=0$, this is just the result stated in Eq. (41) of the text.

The static limit of $A^{\pi N(+)}$ is easily derived. According to Eqs. (A9–A11), when all partial wave amplitudes except $f_{3,3}\equiv f_{1+}{}^{(3/2)}$ are neglected, $A^{\pi N(+)}$ is given by

$$\frac{A^{\pi N(+)}}{4\pi}=\left[3\frac{W+M}{p_{20}+M}\,\frac{2M\nu_B+q_0{}^2}{|\mathbf{q}|^2}+\frac{(W-M)(p_{20}+M)}{|\mathbf{q}|^2}\right]\frac{2}{3}f_{3,3}\,. \tag{A17}$$

In the static limit, when $\nu_B=0$, this is

$$A^{\pi N(+)}\approx\frac{16}{3}\frac{M\pi\omega}{|\mathbf{q}|^2}f_{3,3}\,.$$

Since in the static limit (when $\nu_B=0$) $\nu\approx\omega$ and $\nu_0\approx1$, we have

$$\frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'}\,\mathrm{Im}A^{\pi N(+)}(\nu',0,-1)\approx\frac{32}{3}M\pi\cdot\frac{1}{\pi}\int_1^{\infty}d\omega\frac{\mathrm{Im}f_{3,3}}{|\mathbf{q}|^2}\,. \tag{A18}$$

B. Equations for Extrapolation off Mass Shell

Now let us consider $k^2\neq-M_\pi{}^2$ and derive the equations used for going off mass shell in k^2. According to our model, we wish to calculate

$$\mathrm{Im}\Delta(\nu,k^2)\equiv[\mathrm{Im}A_{3,3}{}^{\pi N(+)}(\nu,0,k^2)/K^{NN\pi}(k^2)]-\mathrm{Im}A_{3,3}{}^{\pi N(+)}(\nu,0,-1) \tag{A19}$$

at $k^2=0$. From Eqs. (A9–A11) and Eq. (63) of the text, $\mathrm{Im}\Delta(\nu,k^2)$ is given by

$$\mathrm{Im}\Delta(\nu,k^2)=\frac{4\pi}{|\mathbf{q}|^2}\frac{2}{3}\,\mathrm{Im}f_{3,3}\left[\frac{3(W+M)q_0{}^2}{p_{20}+M}+\omega(p_{20}+M)\right](L-1)\,, \tag{A20}$$

with

$$L=\frac{1}{K^{NN\pi}(k^2)}\frac{f_{3,3}{}^B(\nu,k^2)}{f_{3,3}{}^B(\nu,\,k^2=-M_\pi{}^2)}\frac{|\mathbf{q}|}{|\mathbf{k}|}\times\frac{\dfrac{3(W+M)q_0k_0}{[(p_{10}+M)(p_{20}+M)]^{1/2}}+\omega[(p_{10}+M)(p_{20}+M)]^{1/2}}{\dfrac{3(W+M)q_0{}^2}{p_{20}+M}+\omega(p_{20}+M)}\,. \tag{A21}$$

The Born approximations are computed by substituting the isospin $\frac{3}{2}$ part of the Born approximation

$$\begin{aligned}
B^{\pi NB(3/2)}&=-g_r{}^2K^{NN\pi}(k^2)/|\mathbf{q}||\mathbf{k}|\,(y+a)\,,\\
a&=(2p_{20}k_0+k^2)/2|\mathbf{q}||\mathbf{k}|\,,
\end{aligned} \tag{A22}$$

into Eq. (A8) to calculate $f_1{}^{B(3/2)}$ and $f_2{}^{B(3/2)}$. The $J=\frac{3}{2}$ projection is then done by using Eq. (A9). The result is

$$\frac{1}{K^{NN\pi}(k^2)}\frac{f_{3,3}{}^B(\nu,k^2)}{f_{3,3}{}^B(\nu,\,k^2=-M_\pi{}^2)}=\frac{|\mathbf{q}|}{|\mathbf{k}|}\frac{N}{N'}\,, \tag{A23}$$

$$\begin{aligned}
N=&\,\omega[(p_{10}+M)(p_{20}+M)]^{1/2}A(a)\\
&+(W+M)[(p_{10}-M)(p_{20}-M)]^{1/2}C(a)\,,\\
N'=&\,\omega(p_{20}+M)A(a')+(W+M)(p_{20}-M)C(a')\,,
\end{aligned}$$

where

$$\begin{aligned}
a'&=(2p_{20}q_0-M_\pi{}^2)/2|\mathbf{q}|^2\,,\\
A(a)&=1-\frac{a}{2}\ln[(a+1)/(a-1)]\,,\\
C(a)&=-\frac{1}{2}\left[3a+\left(\frac{1-3a^2}{2}\right)\ln\left(\frac{a+1}{a-1}\right)\right].
\end{aligned} \tag{A24}$$