Nuclear Spectroscopy with Two-Nucleon Transfer Reactions

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The aspects of two-nucleon transfer reactions that depend on nuclear structure can be isolated in structure amplitudes G, which do not depend on the kinematics or scattering states of the reaction. The calculation of these amplitudes from microscopic nuclear models is illustrated in a number of examples. The structure amplitudes measure the parentage and the degree to which a nuclear state possesses the particular correlations predicated by the fact that the pair is transferred to or from a light nuclide which itself has definite and simple correlations among its nucleons. Several specific nuclear reactions are considered in some detail, and the structure amplitudes for many nuclear levels are given.

I. INTRODUCTION

IN an earlier paper the theory of direct two-nucleon-transfer reactions was developed in such a way as to give a central role to the structure of the nuclear states involved.[1,2] The purpose of the present paper is twofold. First, we discuss in more detail the form of the cross section in order to show how the nulcear structure can influence the intensity and multipolarity of the transitions. Second, we show in detail how to extract from nuclear wave functions, obtained from any particular microscopic model of the nucleus, the information that is relevant to the double-transfer reaction and thus to expose these functions to an experimental test.

The general features of two-nucleon-transfer reactions can be summarized as follows[1,3]: Nuclei and levels not easily studied by other means can be excited. The nuclei can be removed by two nucleons from stable targets. Levels having two nucleons excited can be formed which cannot appear (in lowest order) in single-nucleon transfer or inelastic reactions. The reaction is highly selective, favoring, in stripping reactions, those states having a large parentage based on the target in its ground state.

Just as for single-nucleon transfer, the angular distribution for two-nucleon-transfer reactions is characterized by the orbital angular momentum that is transferred. In the first case, the angular momentum is carried by a single nucleon, and the intensity of the reaction is proportional to the probability that the nucleon has that angular momentum in the nuclear state. But in the second case, the angular momentum is carried by the *pair* of nucleons, and many different configurations of the two nucleons can contribute to a given angular-momentum transfer. The resulting coherence can lead to very strong transition to levels for which it is constructive. It is based on correlations introduced by the angular-momentum coupling and the residual nucleon-nucleon interaction. The residual interaction is responsible for configuration mixing in the wave functions and consequently the two-nucleon-transfer reaction provides a mechanism for studying the nuclear wave functions in details not accessible to the single-transfer reaction.

General selection rules have been stated several times.[1,3,4] Certain additional rules, which hold under special circumstances, are discussed in the Appendix.

In the next sections we define the ingredients of the cross section and show how those that depend on the nuclear wave functions can be constructed. A number of different model wave functions will be considered. Our emphasis throughout is on the spectroscopy, but a brief discussion of the angular-momentum-transfer amplitude is included. Specific reactions are considered in later sections and comparison with experiment is made.

II. INGREDIENTS OF THE CROSS SECTION

It is well known that the *cross section* for single-nucleon-transfer reactions can be factorized into two parts: one contains the nuclear-structure information, and the other depends on the kinematics.[1,5,6] For two-nucleon-transfer reactions such a factorization is not possible in general. This is because of the coherence described in the Introduction. However, the stripping *amplitude* can still be factorized into a factor G that depends upon details of the nuclear structure, and a kinematic factor B. We concentrate as much of the structure information in G as is possible, thus leaving B to represent the probability amplitude for transferring

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[1] N. K. Glendenning, in *Annual Review of Nuclear Science*, edited by E. Segrè (Annual Reviews, Inc., Palo Alto, California, 1963), Vol. 13; a preliminary report of some of the material contained in the present paper was made at the Conference on Nuclear Spectroscopy with Direct Reactions, edited by F. E. Throw, Argonne National Laboratory Report ANL-6848, 1964 (unpublished), p. 188.

[2] Developments similar in varying degrees to that of Ref. 1 have been independently proposed by: J. Jänecke, Nucl. Phys. 48, 129 (1963); Ching Liang Lin and S. Yoshida, *Theory of Two Nucleon Stripping Reactions* (Institute for Nuclear Study, Tokyo, 1964, to be published); E. M. Henley and D. V. L. Yu, Phys. Rev. **133**, B1445 (1964); B. Bayman, in Proceedings of Conference on Nuclear Spectroscopy with Direct Reactions, edited by F. E. Throw, Argonne National Laboratory Report ANL-6848, 1964 (unpublished).

[3] N. K. Glendenning, Nucl. Phys. **29**, 109 (1962).

[4] H. C. Newns, Proc. Phys. Soc. (London) **A76**, 489 (1960).

[5] J. B. French, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960).

[6] M. H. MacFarlane and J. B. French, Rev. Mod. Phys. **32**, 567 (1960).

a structureless nuclide into the orbital state N, L in a structureless nucleus. This represents a complete separation of the nuclear-structure calculation from the spectroscopically uninteresting calculation of the transfer amplitude and the attendant distorted-wave method.

The differential cross section is found[1] to be an incoherent sum over L, S, J, and T of

$$\sum_M |\sum_N G_{NLSJT} B_{NL}{}^M(\mathbf{k}_1,\mathbf{k}_2)|^2, \qquad (2.1)$$

where L, S, J are the orbital, spin, and total angular momenta of the pair of transferred nucleons, and T is their isospin. The several radial states, characterized by N, contribute coherently to the cross section. The relative weights with which they contribute are determined by the structure factors G. These themselves are very sensitive to the correlations induced by the residual interaction that manifests itself in G by a sum over configuration amplitudes. The amplitude $B_{NL}{}^M$ is completely analogous to the similarly denoted amplitude in the theory of (d,p) reactions.[1,7,8] It contains the radial wave function $u_{NL}(R)$ for the center of mass of the pair, in place of the neutron radial function $u_{nl}(r)$. But whereas in single stripping, only one principal quantum number n is relevant, in two-nucleon stripping, almost always several radial functions are required to describe the center-of-mass motion of the transferred pair.[9]

It is easy to show that through the coherence, the details of the nuclear structure as manifested in the G's can have a marked effect on the cross section. According to Eq. (2.1), the cross section could be rewritten in terms of a transfer amplitude that contains projected wave functions

$$\tilde{u}_{LSJ}(R)=\sum_N G_{NLSJ} u_{NL}(R). \qquad (2.2)$$

(This plays the role of the so-called form factor in the distorted-wave calculation of B.) Suppose as an example that three radial states $N=1, 2, 3$ are required for a description of the center-of-mass motion of the transferred pair. The functions u_{NL} have signs $(-)^{N+1}$ at large radius. Therefore, if the nuclear wave function yields G_N's that have the same sign, then $\tilde{u}$ will be small in the nuclear surface, and large in the interior as illustrated in Fig. 1; whereas, if the G_N's had turned out to have alternating signs, $\tilde{u}$ would be concentrated at the nuclear surface. Because of the expected importance of the surface region, especially in reactions that have complex outgoing particles, such effects should show up in the cross section. It can influence the multipolarity of the transition when several L's are otherwise allowed.

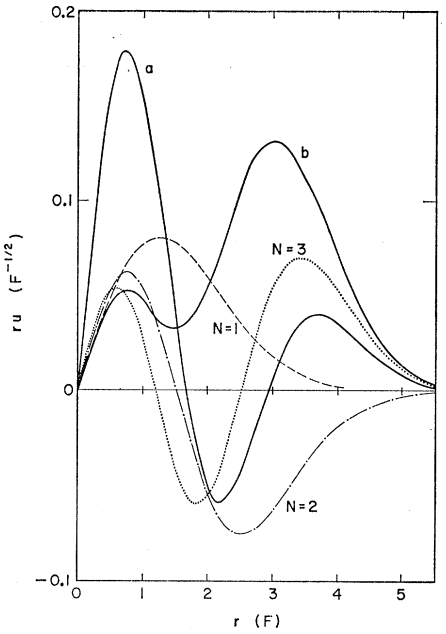

FIG. 1. Three s states of the center-of-mass motion of a pair of nucleons in N^{14} are shown. Curve (a) shows the projected wave function [Eq. (2.2)] corresponding to structure factors $G_N\equiv 1$, and curve (b) corresponds to $G_N=(-)^{N+1}$.

The structure factor G is a product of three overlap integrals:

$$G_{NLSJT}=g\sum_\gamma \beta_{\gamma LSJT}\Omega_n\langle n0,NL;L|n_1l_1,n_2l_2;L\rangle. \qquad (2.3)$$

The first overlap β is of the same form as appears in the theory of (d,p) reactions, and its square is proportional to the spectroscopic factor. It measures the parentage of the nucleus $(A+2)$ based on the nucleus (A) and having two nucleons in the state $\gamma(\equiv n_1l_1n_2l_2\cdots)$, L, S, J, T. In the next section we discuss at length this parentage factor, since it contains the information about the nuclear-coupling scheme.

The spatial part of the wave function for the two nucleons in the state γ, referred to above, can be transformed to the relative and center-of-mass coordinates r and R by

$$[\phi_{n_1l_1}(\mathbf{r}_1)\phi_{n_2l_2}(\mathbf{r}_2)]_L = \sum_{n\lambda N\Lambda}\langle n\lambda,N\Lambda;L|n_1l_1,n_2l_2;L\rangle[\phi_{n\lambda}(\mathbf{r})\phi_{N\Lambda}(\mathbf{R})]_L, \qquad (2.4)$$

where the square bracket denotes vector coupling. Here λ and Λ are the orbital angular momenta of the relative and center-of-mass motions, while n and N are their respective principal quantum numbers. The coordinates r and R are the most suitable for treating double-transfer reactions. The relative motion which, in the nucleus is described by $\phi_{n\lambda}(r)$, has to overlap with the motion of the pair in the light nuclide from which

[7] W. Tobocman, Phys. Rev. 94, 1655 (1954).

[8] R. Huby, M. Y. Rafai, and G. R. Satchler, Nucl. Phys. 9, 94 (1958).

[9] Consider two balls moving with different angular velocities in opposite directions around a circle. While their individual motion with respect to the center of the circle is very simple, the motion of their center of mass is evidently more complicated. In fact, it traces a many-petaled flower pattern that is periodic if the ratio of angular velocities is a rational fraction N/M and has $N+M$ petals.

TABLE I. The rms radius and size parameter for the light nuclides (unit of length is 10^{-13} cm).

	He^4	He^3	H^3
$\langle r^2\rangle^{1/2}$	1.61	1.97	1.68
η	0.233	0.206	0.242

they are transferred. This overlap is denoted by Ω_n. We shall assume that the relative motion in the light nuclides (of mass number $a=3, 4$) is pure s state. Consequently, the only part of the nuclear state which can contribute is that which corresponds to s-state motion ($\lambda=0$) in the *relative* coordinate.[10] This accounts for the appearance only of the $\lambda=0$ transformation brackets in Eq. (2.3). For harmonic-oscillator wave functions ϕ_{nl} in Eq. (2.4) the brackets can be obtained in closed form[11,12] and they have been tabulated.[13]

If the nuclear wave functions have definite symmetry under exchange of the two particles, then

$$\begin{aligned} g&=1, \quad \text{if} \quad n_1l_1j_1\equiv n_2l_2j_2 \\ &=\sqrt{2}, \quad \text{otherwise}. \end{aligned} \tag{2.5}$$

If the wave function does not have a definite symmetry (i.e., a neutron-proton configuration without definite isospin), then $g\equiv 1$.

The sum on γ in Eq. (2.3) is over the various configurations that may be present in the nuclear wave function of the pair of transferred nucleons. As we see from Eq. (2.3), the different configurations enter coherently. For some levels and their components, the coherence will be constructive so as to yield a large cross section, but for others it may be destructive. The sum γ, which is explicitly over $n_1l_1j_1n_2l_2j_2$, contains an implicit sum over n because of the connection[13]

$$2(n+N)+L=2(n_1+n_2)+l_1+l_2. \tag{2.6}$$

If we assume a Gaussian wave function for the light nuclide,

$$\phi_a\propto\exp(-\eta^2\textstyle\sum r_{ij}^2), \tag{2.7}$$

and harmonic-oscillator functions for the bound nucleons in the nucleus, we obtain

$$\Omega_n=\frac{[(2n-1)!]^{1/2}}{2^{n-1}(n-1)!}(xy)^{3/2}(1-x)^{n-1}, \quad n=1, 2, \cdots, \tag{2.8a}$$

which is a monotonically decreasing function of n. Here

$$x=2\nu/(2a\eta^2+\nu), \quad y=\eta(2a/\nu)^{1/2}, \tag{2.8b}$$

where $a(=3$ or $4)$ is the mass number of the light nuclide.

The oscillator parameter ν is defined so that the single-nucleon wave functions are proportional to $\exp(-\frac{1}{2}\nu r^2)$. This parameter is typically about $A^{-1/3}$ F^{-2}, which corresponds to an oscillator spacing $\hbar\omega\approx 41A^{-1/3}$ MeV. Of course if one is using shell-model wave functions obtained by diagonalization of the shell-model Hamiltonian, the same value of ν should be used in Eq. (2.8) as was used in the diagonalization.

The size parameter η of the light nuclide is connected to its mean-square radius by

$$\begin{aligned} \eta^2&=\frac{9}{64\langle r^2\rangle} \quad \text{for} \quad He^4 \\ &=\frac{1}{6\langle r^2\rangle} \quad \text{for} \quad He^3 \text{ or } H^3. \end{aligned} \tag{2.9}$$

The experimental rms radii[14,15] and the corresponding size parameter are listed in Table I.

It should be remarked that the structure factors depend on the properties of the light nuclide (in particular its size) through the overlap Ω_n. For nuclides heavier than He^4 the assumption concerning the dominance of the relative s state may be less valid. Indeed, one should use as a probe those nuclides whose properties are well enough known to allow an interpretation of the reaction in terms of the properties of the nucleus.

The full expression for the cross section is written in the Appendix.

III. CALCULATION OF THE PARENTAGE FACTOR

We shall refer to β_γ, which appears in the structure factor G [Eq. (2.3)], as the parentage factor connecting the nucleus $(A+2)$ to (A). To define our notation we denote the reaction by

$$(a)_{S_1}+(A)_{J_1T_1}\leftrightarrow(a-2)_{S_2}+(A+2)_{J_2T_2}. \tag{3.1}$$

Atomic-mass numbers are given in parentheses, and spins and isospins are indicated by subscripts. Then, for stripping reactions, β_γ measures the extent to which the nucleus $(A+2)$ in the state in which is it formed by the reaction, appears as the ground state of the nucleus (A), plus two nucleons in the state $\gamma(\equiv n_1l_1n_2l_2\cdots)$, L, S, J, T. For pickup reactions, β measures the degree to which the ground state of $(A+2)$ has as its parent the state of the nucleus (A) that is formed in the reaction, plus two nucleons with the above quantum numbers. More precisely:

$$\beta_{\gamma LSJT}(J_1,J_2)=\binom{A+2}{2}^{1/2}\int[\Psi_{J_1T_1}{}^*(A) \times\phi_{\gamma LSJT}{}^*(\mathbf{r}_1,\mathbf{r}_2)]_{J_2T_2}\Psi_{J_2T_2}(A,\mathbf{r}_1\mathbf{r}_2)dA\,d\mathbf{r}_1d\mathbf{r}_2, \tag{3.2a}$$

[10] The appearance of only the s state of the relative motion follows from our assumption that (a) the light nuclides of mass 4 or less are pure relative s and (b) the interaction responsible for the reaction acts between the centers of mass of the two parts of the light nuclide that are separated in the reaction. These assumptions also lead to the important connection of L with the parity change as discussed in Refs. 1 and 3.

[11] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

[12] M. Moshinsky, Nucl. Phys. **13**, 104 (1959).

[13] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Monografias del Instituto de Fisica, Mexico, 1960).

[14] R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956).

[15] H. Collard, R. Hofstadter, A. Johansson, R. Parks, M. Reynold, A. Walker, and M. R. Yearian, Phys. Rev. Letters **11**, 132 (1963).

where the square bracket denotes vector coupling.[16] The factor $\binom{A+2}{2}$ has to be understood as symbolic in the following sense: In case the isospin formalism is not used in constructing the wave functions, then

$$\binom{A+2}{2}\rightarrow\binom{N+\nu}{\nu}\binom{Z+\pi}{\pi}, \tag{3.2b}$$

where ν and π are the number of neutrons and protons transferred ($\nu+\pi=2$). *In any case*, if, as is usual, the overlap is computed with wave functions that refer only to a certain antisymmetrized subgroup of the total number of nucleons, then A (or N and Z) stands only for the number in the group to which the pair is added.

If the wave functions of (A) and $(A+2)$ are known, say from a shell-model calculation, then β can be computed. As a simple example, consider a nucleus (A) that has closed shells. Some states of the nucleus $(A+2)$ might therefore have the structure

$$\Psi_{J_2T_2}(A+2)=\Psi_0(A)\sum_{j_1j_2}C_{(j_1j_2)J_2T_2}\phi_{(j_1j_2)J_2T_2}(r_1,r_2), \tag{3.3}$$

where the C's are the mixture coefficients for the levels above the closed shells of (A). To calculate $\beta_{\gamma LSJT}$ we want to transform $\phi_{(j_1j_2)J}$ from the j-j scheme to the L-S scheme; this is achieved with the coefficients

$$\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix}=\{[L][S][j_1][j_2]\}^{1/2}\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix}, \tag{3.4}$$

where $[j]=2j+1$ and $\{\ \}$ is a 9-j coefficient.[17] Upon doing this and inserting the resulting expression for $\Psi_{J_2}(A+2)$ into Eq. (3.2), we can perform the integrations immediately, obtaining

$$\beta_{\gamma LSJT}(0,J_2)=C_{(j_1j_2)J_2T_2}\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix}\delta_{JJ_2}\delta_{TT_2}, \tag{3.5}$$

which is the parentage factor connecting the ground state of (A) and the state J_2T_2 of $(A+2)$.

It is very important to notice from Eqs. (2.1) and (2.3) that the configuration mixture coefficients C in the wave function contribute *coherently* to the structure factors. Thus, the two-nucleon stripping reaction is sensitive to the phases as well as the magnitudes of the mixture coefficients. The single-nucleon stripping reaction by contrast depends only on the absolute values of these coefficients. It should be evident however that, starting with experimental results, it is in general impossible to deduce the wave function. Even supposing that the experiment uniquely determined the G's, there is an infinity of ways in which the product of the three factors on the right side of Eq. (2.3) could be arranged to yield them. However, if we have a wave function obtained from a shell-model calculation, say, we can compute from it the structure factors, and thus test whether the wave function is compatible with the experimental results. In the next section this procedure is illustrated in detail for the N^{14} wave functions.

The parentage factors can be easily obtained when a pair of like nucleons is added or taken out of a given shell j. In particular, when n is even, the ground state is (assuming a pure configuration):

$$|(j^n);0\rangle=\sum_{v,j}((j^{n-2})vJ,(j^2)J\rangle\!\rangle(j^n)0)\,|(j^{n-2})vJ,(j^2)J;0\rangle, \tag{3.6}$$

where v is the seniority, and the bracket $\rangle\!\rangle$ is a coefficient of fractional parentage.[18] Again expanding the $(j^2)J$ configuration on an L-S basis, and inserting Eq. (3.6) into Eq. (3.2), we obtain immediately

$$\beta_{LSJ}[(j^{n-2})vJ\leftrightarrow(j^n)0]=\left[\frac{n(n-1)}{2}\right]^{1/2}((j^{n-2})vJ,(j^2)J\rangle\!\rangle(j^n)0)\begin{Bmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ L & S & J \end{Bmatrix}. \tag{3.7}$$

Similarly, the wave function for an excited state $|(j^n)v=2,J\rangle$ can be expanded and one finds

$$\beta_{LSJ}[(j^{n-2})v_1J_1\leftarrow(j^n)v_2J_2]=\left[\frac{n(n-1)}{2}\right]^{1/2}((j^{n-2})v_1J_1,(j^2)J\rangle\!\rangle(j^n)v_2J_2)\begin{Bmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ L & S & J \end{Bmatrix}. \tag{3.8}$$

Explicit formulas for coefficients of fractional parentage can be obtained for states of low seniority by methods

[16] In our earlier work (Ref. 1) the factor coming from antisymmetrization was left as a multiplying factor in front of the cross section. We now incorporate it into the definition of β in the same way that a similar factor is incorporated in the definition ofthe spectroscopic factor in single-nucleon stripping. That symbol denotes $\binom{m}{n}\equiv m!/[(m-n)!n!]$.

[17] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, New Jersey, 1957).

[18] G. Racah, Phys. Rev. **63**, 367 (1943).

discussed by Schwartz and de-Shalit.[19] One finds

$$((j^{n-2})vJ,(j^2)J\,\|\,(j^n)0)=\left\{\frac{2(n-2)}{n-1}\frac{2J+1}{(2j-1)(2j+1)}\right\}^{1/2}\quad\text{for}\quad v=2,\quad J\neq 0$$

$$=\left\{\frac{2j+3-n}{(n-1)(2j+1)}\right\}^{1/2}\quad\text{for}\quad v=0,\quad J=0, \tag{3.9}$$

$$((j^{n-2})v_1J_1,(j^2)J\,\|\,(j^n)v=2J_2)=\delta_{J_1J_2}\delta_{v_12}\left\{\frac{n-2}{n(n-1)}\frac{2j+1-n}{2j+1}\right\}^{1/2}\quad\text{for}\quad J=0$$

$$=\delta_{JJ_2}\delta_{v_10}\left\{\frac{2}{n(n-1)}\frac{(2j+1-n)(2j+3-n)}{(2j-1)(2j+1)}\right\}^{1/2}\quad\text{for}\quad J_1=0. \tag{3.10}$$

[See Eq. (36) of Ref. 19 for the case when $J\neq 0$, $J_1\neq 0$.]

Similarly to the above, when n is odd we obtain

$$\beta_{LSJ}[(j^n)v=1j\leftrightarrow(j^{n-2})v=1j]=\left[\frac{n(n-1)}{2}\right]^{1/2}((j^{n-2})v=1j,(j^2)J\,\|\,(j^n)v=1j)\begin{Bmatrix} l & \frac{1}{2} & j\\ l & \frac{1}{2} & j\\ L & S & J\end{Bmatrix}, \tag{3.11}$$

where

$$((j^{n-2})v=1j,(j^2)J\,\|\,(j^n)v=1j)=-\frac{2}{2j-1}\left\{\frac{(2J+1)(2j+2-n)}{n(2j+1)}\right\}^{1/2}\quad\text{for}\quad J\neq 0$$

$$=\left\{\frac{2j+2-n}{n(2j+1)}\right\}^{1/2}\quad\text{for}\quad J=0. \tag{3.12}$$

We now consider the situation in which the nucleons are transferred to or from different shells. Then

$$\beta_{\gamma LSJ}[(j_a{}^{n_a})J_a,(j_b{}^{n_b})J_b;J_2\leftrightarrow(j_a{}^{n_a-1})J_a',(j_b{}^{n_b-1})J_b';J_1]$$

$$=(n_an_b)^{1/2}((j_a{}^{n_a-1})J_a',j_a\,\|\,(j_a{}^{n_a})J_a)((j_b{}^{n_b-1})J_b',j_b\,\|\,(j_b{}^{n_b})J_b)\begin{Bmatrix} J_a' & j_a & J_a\\ J_b' & j_b & J_b\\ J_1 & J & J_2\end{Bmatrix}\begin{Bmatrix} l_a & \frac{1}{2} & j_a\\ l_b & \frac{1}{2} & j_b\\ L & S & J\end{Bmatrix}. \tag{3.13}$$

The coefficients of fractional parentage are exactly those familiar from (d,p) reactions,[6] and for states of lowest seniority can be written down [cf. Eq. (67) in Ref. 1].

The parentage factor for configuration mixed-wave functions based upon the above configurations can easily be found from those given for the pure configurations. Thus, for example, if

$$|J_2\rangle=\sum_{j_aj_bJ_aJ_b}C_{j_aj_bJ_aJ_b}{}^{(2)}|(j_a{}^{n_a})J_a,(j_b{}^{n_b})J_b;J_2\rangle,$$

$$|J_1\rangle=\sum_{j_aj_bJ_a'J_b'}C_{j_aj_bJ_a'J_b'}{}^{(1)}|(j_a{}^{n_a-1})J_a',(j_b{}^{n_a-1})J_b';J_1\rangle, \tag{3.14a}$$

then

$$\beta_{\gamma LSJ}[J_2\leftrightarrow J_1]=\sum_{J_aJ_bJ_a'J_b'}C^{(1)}C^{(2)}\beta_{\gamma LSJ}[J_aJ_b;J_2\leftrightarrow J_a'J_b';J_1]. \tag{3.14b}$$

For several other configurations that might rise in the conventional shell model, we have given the corresponding parentage factors elsewhere.[1]

In regions of the periodic table removed by more than several nucleons from closed shells, the conventional shell model becomes very cumbersome. In such situations, the Bardeen-Cooper-Schrieffer method has been applied to the nuclear-structure problem.[20,21] With some sacrifices, one can obtain a solution to the many-body problem. Using this nuclear model, Yoshida[22] has considered the two-nucleon stripping reaction and ob-

[19] C. Schwartz and A. de-Shalit, Phys. Rev. **94**, 1257 (1954).

[20] S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **31**, No. 11 (1959).

[21] L. S. Kisslinger and R. A. Sorenson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **32**, No. 9 (1960).

[22] S. Yoshida, Nucl. Phys. **33**, 693 (1962).

tains "spectroscopic factors," which he calls $B(J,j_1j_2)$ for various types of nuclei. The other details of the reaction he develops in an interesting but unnecessarily approximate fashion. We can, however, use directly his expression for the "spectroscopic factors" in terms of which our parentage factor is given as

$$\beta_{\gamma LSJ}=B(J,j_1j_2)\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix}. \quad (3.15)$$

He finds, for example, that if both nuclei (A) and $(A+2)$ are in zero quasiparticle states (i.e., ground states of even nuclei)

$$B(0jj)=(j+\tfrac{1}{2})^{1/2}U_j(A)V_j(A+2). \quad (3.16)$$

If, however, $(A+2)$ is in a zero quasiparticle state and (A) is in a two-quasiparticle state with configuration $(j_1j_2)J$, then

$$B(Jj_1j_2)=-(2J+1)^{1/2}V_{j_1}(A+2)V_{j_2}(A+2), \quad (3.17)$$

which is appropriate for pickup from an even nucleus. If the nucleus $(A+2)$ is in a two-quasiparticle state, while (A) is in the ground state, then

$$B(Jj_1j_2)=U_{j_1}(A)U_{j_2}(A). \quad (3.18)$$

Other situations are also treated, notably collective vibrational states.

IV. ANGULAR-MOMENTUM TRANSFER AMPLITUDE

The second factor in Eq. (2.1), $B_{NL}{}^M$, is the amplitude for transfer of a pair of nucleons between the light nuclide and the nucleus, when their center-of-mass motion in the nucleus is characterized by the quantum numbers N, L, M. It contains no detailed reference to the nuclear structure, since this information has been concentrated in the structure factors G. The B's are expected, as in single-nucleon-transfer reactions, to depend in a characteristic way on the angular momentum L that is transferred. In addition, they depend on the number of nodes $(N-1)$ in the radial function $u_{NL}(R)$ for the center of mass of the pair of transferred nucleons in the nucleus. In general (as already remarked), several different radial states N are required to define the center-of-mass motion, and these enter *coherently* with weights and phases that depend upon the details of the nuclear wave functions as expressed in G.

The actual calculation of B requires the use of distorted waves to describe the motion of the incident and outgoing nuclides, a method well known from other work.[7,23–26] There are, however, several uncertainties that arise when the nuclides are strongly absorbed in the nucleus. In the first place, there is a whole set of optical potentials that give essentially the same elastic scattering and differ from each other in the characteristic that one additional half-wavelength of each pertinent partial wave is pulled into the potential for successively deeper potentials.[27] In the second place, it is entirely possible that inside the nucleus no optical potential can give an adequate description of the wave function. If this be so, then fortunately, for the same reason that it is so, the interior should play a very minor role in the direct transfer process, while compound nucleus contributions to which the interior would contribute will usually contribute little intensity to any given channel above a few MeV bombarding energy.[24] This should be especially so if the outgoing particle is composite. In such a situation it would be appropriate to introduce a cutoff, or otherwise damp the contributions to B coming from the nuclear interior.

An attempt to find a prescription for calculating B will be the subject of a subsequent publication.

V. METHOD OF ANALYSIS

Following the preceding discussion we shall assume that the interior of the nucleus makes a negligible contribution to the angular-momentum transfer amplitude $B_{NL}{}^M$. In this case the dependence on N becomes trivial. The wave functions $u_{NL}(R)$ at large radius have the sign $(-)^{N-1}$ (in the convention used by most authors), and outside the nucleus they obey the field-free Schrödinger equation with negative energy corresponding to the separation energy of the two nucleons from the nucleus. We therefore write

$$B_{NL}{}^M \rightarrow (-)^{N-1}W_{NL}(\nu,\kappa)\tilde{B}_L{}^M(\kappa,R_N), \quad (5.1)$$

where $\tilde{B}_L$ has the same structure as B_{NL} except that the wave function $u_{NL}(R)$ is replaced by the spherical Hankel function $-i^Lh_L{}^{(1)}(i\kappa R)$, where $\kappa^2=4M^*\epsilon_B/\hbar^2$, ϵ_B is the separation energy of the pair from the nucleus, and M^* is the reduced nucleon mass. (More accurately, $2M^*$ is the mass of the transferred pair that possibly includes some binding energy.) The integration in $\tilde{B}$ extends from R_N to infinity. (In the plane-wave approximation, $\tilde{B}$ is proportional to the Butler Wronskian.) The quantity $W_{NL}(\nu,\kappa)$, which depends on the separation energy and the size parameter of the nucleus (see Sec. II), is a positive normalizing constant found by matching the interior (harmonic-oscillator) function to the Hankel function, and renormalizing the combination to the original normalization. It is tabulated in Table II.

From $\tilde{B}_L{}^M$ a reduced cross section can be calculated, which depends on the distortion of the incident and outgoing nuclides by their interaction with the target, the Q of the reaction, the separation energy of the transferred pair from the nucleus, and of course the scattering

[23] C. A. Levinson and M. K. Banerjee, Ann. Phys. (N. Y.) **2**, 471 (1957); **2**, 499 (1957); **3**, 67 (1958).
[24] N. K. Glendenning, Phys. Rev. **114**, 1297 (1959).
[25] E. Rost and N. Austern, Phys. Rev. **120**, 1375 (1960).
[26] R. H. Bassel, D. H. Drisko, and G. R. Satchler, Oak Ridge National Laboratory Report ORNL-3240, 1962 (unpublished).
[27] R. M. Drisko, G. R. Satchler, and R. H. Bassel, Phys. Letters **5**, 347 (1963).

TABLE II. Constants W_{NL} for matching Hankel-function tail to harmonic-oscillator function. The oscillator constant is $\nu=0.32$ F^{-2}. The transferred pair has the wave number $\kappa=0.287(12.492-E_x)^{1/2}$ where E_x is the excitation energy in N^{14}. Intermediate values of κ can be interpolated easily when log$W$ is plotted against κ.

		$W_{NL}(\nu,\kappa)$		
κ	L	$N=1$	$N=2$	$N=3$
1	0	4.49	13.0	29.0
	1	5.38	15.0	32.7
	2	4.83	13.7	30.3
	3	3.42	10.1	22.9
	4	2.30	7.07	16.5
0.8	0	2.30	5.40	10.3
	1	2.31	5.44	10.3
	2	1.71	4.23	8.29
	3	1.01	2.66	5.44
	4	0.538	1.50	3.21
0.6	0	1.13	2.14	3.45
	1	0.918	1.82	3.02
	2	0.530	1.15	2.01
	3	0.245	0.579	1.07
	4	0.0975	0.250	0.493
0.5	0	0.765	1.29	1.92
	1	0.545	0.997	1.54
	2	0.270	0.550	0.908
	3	0.106	0.238	0.422
	4	0.0352	0.0868	0.165

angle, but which is completely independent of the nuclear structure. We denote the reduced cross section by

$$\tilde{\sigma}_L(\kappa,R_N,\theta)\propto\sum_M|\tilde{B}_L{}^M|^2. \tag{5.2}$$

The actual cross section is proportional to

$$\frac{d\sigma}{d\Omega}\propto\sum_L\xi_L\tilde{\sigma}_L(\kappa,R_N,\theta), \tag{5.3a}$$

where

$$\xi_L=\sum_{SJT}C_{ST}{}^2\Big|\sum_N(-)^N W_{NL}G_{NLSJT}\Big|^2 \tag{5.3b}$$

now contains all the structure information on which the cross section depends, under the assumption that the nuclear interior makes a negligible contribution to the reaction. We admit that, while this assumption is plausible, it is not known to be true. It is not an easy point to investigate, because of the difficulties inherent in treating reactions as a many-body problem. We emphasize, however, that the analysis of nuclear wave functions to yield the structure factors, G is independent of this question.

The factor $C_{ST}{}^2$ in Eq. (5.3b) is simply an isospin factor which is written down in the Appendix.

The W_{NL} in Table II are independent of the cutoff radius R_N that is used in computing the modified transfer amplitudes $\tilde{B}_L{}^M$. This independence is in line with our design to keep the structure calculation independent of the distorted-wave calculation. The price paid for this is that the table may be (mistakenly) interpreted as implying that the cross section decreases with increasing L, because the W_{NL}'s do. The point is that the $\tilde{B}_L{}^M$'s contain the Hankel function $h_L(i\kappa R_N)$ which increases with L and compensates the decrease in W_{NL}. On the other hand, the W_{NL} for given L increase with N. This is significant and is not compensated by $\tilde{B}_L{}^M$, which is (by design) independent of N.

VI. ANALYSIS OF THE C$^{12}(\alpha,d)$N^{14} REACTION

Here some aspects of the two-nucleon stripping reactions discussed in the foregoing are illustrated in greater detail by considering the C$^{12}(\alpha,d)$N^{14} reaction to various excited states. This reaction has been chosen because of the availability of experimental results[28,29] as well as shell-model calculations.[30,31] Our object is to test the appropriateness of the wave functions by extracting the spectroscopic information, relevant to the two-nucleon-transfer reaction, and to compare the results with the experimental cross section to the various levels.

Since C^{12} and the two light nuclides have isospin $T=0$, only states in N^{14} of the same isospin can be excited. In his shell-model calculation, True[29] assumed that many of the states in N^{14} could be described as an inert C^{12} core plus a neutron and proton in the shells beyond (i.e., $p_{1/2}$, $d_{5/2}$, $s_{1/2}$, $d_{3/2}$). The parentage factor for states of this structure is given by Eq. (3.5).[32]

True's wave functions were obtained with an oscillator parameter $\nu=0.27$ F^{-2} for the s and d orbits, and $\nu=0.32$ F^{-2} for the p orbit. We use the latter value to avoid the unnecessary complication of using two such parameters. From Eq. (2.7), we then find for Ω_n the values shown in Table III.

The $\langle|\rangle$ in Eq. (2.3) can be obtained from the tables

TABLE III. Spectroscopic data for $(d_{5/2}{}^2)_J$, $T=0$ ($S=1$) states.

J	L	β_{LSJ}	N	n	Ω_n	$\langle\|\rangle$[a]	G_{NLSJ}
1	0	0.529	3	1	0.983	0.408	0.212
			2	2	0.182	−0.745	−0.072
			1	3	0.031	0.408	0.007
	2	−0.4	2	1	0.983	0.289	−0.114
			1	2	0.182	−0.441	0.032
3	2	0.785	2	1	0.983	0.289	0.223
			1	2	0.182	−0.441	−0.063
	4	−0.151	1	1	0.983	0.612	−0.091
5	4	1.0	1	1	0.983	0.612	0.602

[a] This is the bracket appearing in Eq. (2.3).

[28] B. G. Harvey and J. Cerny, Phys. Rev. **120**, 2162 (1960); B. G. Harvey, J. Cerny, R. H. Pehl, and E. Rivet, Nucl. Phys. **39**, 160 (1962).

[29] R. H. Pehl, E. Rivet, J. Cerny, and B. G. Harvey, refer to Phys. Rev. **137**, B114 (1965); following article.

[30] W. W. True, Phys. Rev. **130**, 1530 (1963).

[31] W. W. True (private communication, 1964). We are indebted to Dr. True for making these calculations available.

[32] A convenient table of LS-jj transformation coefficients is given by G. Racah, Physica **16**, 655 (1950). He uses the coupling convention $(sl)j$. The coefficients in the other convention, $(ls)j$, differ by the phase $(-)^\sigma$, $\sigma=l_1+l_2+j_1+j_2+1+L+S+J$. True uses the latter convention.

of Brody and Moshinsky.[13] (Our notation is slightly different: in particular our n is related to theirs by $n=n'+1$.)

The three factors obtained in such a way for the $T=0$ states of the configuration $d_{5/2}^2$ are gathered together in Table III, along with the resulting structure factors G. From earlier discussions of the selection rules,[1] we know that only the triplet ($S=1$) part of the wave function contributes in (α,d) reactions and that for configurations j^2, only states with $J=$odd have $T=0$.

TABLE IV. Structure amplitudes for pure configurations of isospin $T=0$ ($S=1$) in N^{14} ($\nu=0.32$ F^{-2}).

			G_{NLSJ}			
j_1j_2	J	L	$N=1$	$N=2$	$N=3$	$N=4$
$(p_{1/2})^2$	1	0	0.0248	−0.1337		
		2	0.5981			
$p_{1/2}2s$	0	1	0.0304	0.6343		
	1	1	0.0248	0.5179		
$p_{1/2}d_{5/2}$	2	1	−0.0942	0.3930		
		3	0.1501			
	3	3	0.5673			
$(2s)^2$	1	0	0.0141	0.0304	0.4486	
$(d_{5/2})^2$	1	0	0.0067	−0.0719	0.2123	
		2	0.0322	−0.1135		
	3	2	−0.0632	0.2229		
		4	−0.0910			
	5	4	0.6018			
$2sd_{5/2}$	2	2	0.0136	−0.3356		
	3	2	−0.0215	0.5307		
$p_{1/2}d_{3/2}$	1	1	−0.0555	0.2316		
	2	1	0.0192	−0.0802		
		3	0.7353			
$d_{3/2}d_{5/2}$	1	0	0.0101	−0.1087	0.3209	
		2	−0.0426	0.1501		
	2	2	−0.0804	0.2836		
	3	2	−0.0486	0.1715		
		4	0.2363			
	4	4	0.6017			
$2sd_{3/2}$	1	2	0.0215	−0.5307		
	2	2	−0.0166	0.4110		
$(d_{3/2})^2$	1	0	−0.0036	0.0385	−0.1135	
		2	−0.0602	0.2123		
	3	2	0.0105	−0.0371		
		4	0.5459			
$p_{1/2}f_{7/2}$	3	2	−0.0796	0.2807		
		4	0.0860			
	4	4	0.4485			
$2sf_{7/2}$	3	3	0.0298	−0.2785		
	4	3	−0.0456	0.4255		
$d_{5/2}f_{7/2}$	1	1	−0.0080	0.0666	−0.1661	
	2	1	0.0083	−0.0692	0.1726	
		3	0.0302	−0.0940		
	3	3	0.0189	−0.0587		
	4	3	−0.0808	0.2514		
		5	−0.0959			
	5	5	−0.0587			
	6	5	0.7768			
$d_{3/2}f_{7/2}$	2	1	0.0111	−0.0923	0.2301	
		3	−0.0227	0.0705		
	3	3	−0.0654	0.2034		
	4	3	−0.0487	0.1516		
		5	0.1590			
	5	5	0.5382			
$(f_{7/2})^2$	1	0	−0.0006	0.0090	−0.0534	0.1087
		2	−0.0040	0.0285	−0.0628	
	3	2	0.0062	−0.0438	0.0964	
		4	0.0179	−0.0503		
	5	4	−0.0575	0.1619		
	5	6	−0.0580			
	7	6	0.5494			

TABLE V. Structure amplitudes for the configuration mixed states of N^{14} having isospin $T=0$ ($S=1$).

					G_{NLSJ}		
$J\pi$	E[a] MeV	Dominant configuration[b]	L	$N=1$	$N=2$	$N=3$	$N=4$
0−	3.1	$p_{1/2}2s$	1	0.030	0.634		
1+	0	$(p_{1/2})^2$	0	−0.027	0.150	−0.118	0.007
			2	−0.586	0.114	−0.004	
	5.5	$(2s)^2$	0	0.013	−0.018	0.578	−0.011
			2	−0.113	0.008	0.006	
	9.3	$(d_{5/2})^2$	0	−0.002	−0.098	0.065	−0.011
			2	−0.079	−0.073	0.006	
	12	$2sd_{3/2}$	0	−0.006	0.052	−0.046	−0.003
			2	−0.046	−0.569	0.002	
	14	$d_{3/2}d_{5/2}$	0	−0.004	0.047	−0.084	0.004
			2	0.012	0.095	−0.002	
1−	4.5	$p_{1/2}2s$	1	0.016	0.549	−0.017	
	12	$p_{1/2}d_{3/2}$	1	0.059	−0.157	0.032	
2+	8.8	$2sd_{5/2}$	2	−0.044	0.546		
	14	$2sd_{3/2}$	2	0.058	−0.232		
	16	$d_{3/2}d_{5/2}$	2	−0.041	−0.081		
2−	3.6	$p_{1/2}d_{5/2}$	1	−0.092	0.400	−0.077	
			3	0.215	−0.001		
	7.3	$p_{1/2}d_{3/2}$	1	−0.027	0.119	−0.032	
			3	−0.715	0.014		
3+	6.0	$2sd_{5/2}$	2	−0.081	0.656	−0.009	
			4	0.019	0.005		
	11	$(d_{5/2})^2$	2	0.063	0.027	0.006	
			4	0.048	−0.003		
	14	$p_{1/2}f_{7/2}$	2	0.050	−0.095	0.001	
			4	−0.320	−0.000		
	16	$d_{3/2}d_{5/2}$	2	−0.005	0.018	−0.000	
			4	−0.335	0.000		
3−	5.1	$p_{1/2}d_{5/2}$	3	0.572	−0.048		
4+	11	$d_{3/2}d_{5/2}$	4	0.750			
	17	$p_{1/2}f_{7/2}$	4	−0.040			
4−	15	$2sf_{7/2}$	3	−0.092	0.511		
			5	−0.024			
5+	8.5	$(d_{5/2})^2$	4	−0.604	0.016		
			6	−0.006			
	27	$(f_{7/2})^2$	4	−0.002	−0.161		
			6	0.058			
6−	14	$d_{5/2}f_{7/2}$	5	0.777			

[a] Energies are calculated ones. With several exceptions only states calculated to lie below 16 MeV are shown.

[b] In some cases the functions are very strongly mixed so that there is no configuration that is dominant.

Therefore, for such configurations we have calculated G_{NLSJ} only for $S=1$, $J=$odd. For other two-nucleon configurations, jj', the $T=0$ states can have both odd and even J. For any $T=0$ level, since only $S=1$ is allowed for (α,d) reactions, the multipolarity of the transition is limited to one value, $L=J$, if the spin and parity is $J,(-)^J$ while it can have two values, $L=J\pm1$, if the spin and parity is $J,(-)^{J+1}$. (Note that J is the total angular momentum carried by the transferred pair, and is necessarily the spin of one of the nuclei in the reaction only if the other has spin zero. This is the situation for the reaction discussed here.)

The above selection rules are reflected in the entries in Table IV, where the structure factors for other configurations relevant to N^{14} are given.

The structure factors for a configuration mixed state can be found by weighting the factors for the pure configurations by their *amplitudes* in the mixed state. True has computed energy levels and wave functions of N^{14}

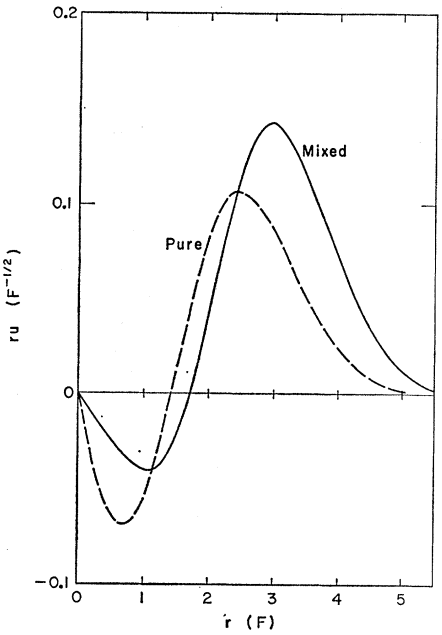

FIG. 2. Projection of the N^{14} ground-state wave functions for the center-of-mass of the last neutron and proton appropriate to the $C^{12}(\alpha,d)N^{14}$ reaction. The $L=0$ part is shown for the pure configuration $(p_{1/2})^2$, and for the configuration mixed-wave function of Ref. 30.

on the basis of the conventional shell model.[30] In a more recent unpublished calculation, he has included the $f_{7/2}$ level, which is not of much importance for the low-lying levels, but enters as an important component of some of the higher ones.[31] The structure factors corresponding to these wave functions are presented in Table V. The energies quoted are the calculated ones, and are somewhat different from the original published calculation. The correspondence between some of these states and experimental levels can be found in True's paper and in the following paper.[29] The energy eigenvalues for the higher lying levels could be in error by several MeV. The calculation in the region above, say, 9 MeV should in fact be regarded as qualitative.

A comparison of the structure factors for the configuration mixed states in Table V with the structure factors for their dominant configuration which can be found in Table IV reveals that important differences can be introduced even by small admixtures. This is because, as already emphasized, the detailed structure of the wave function induced by the nucleon-nucleon interaction enters coherently in determining the transition rate for transfer of the two nucleons. Thus, if we refer to True's paper for the mixture amplitudes of the ground state of N^{14}, we see that he finds it has an amplitude of 0.96666 for $(p_{1/2})^2$. However, the sum of the absolute values of the other amplitudes, which have only a 7% probability, is 0.48; it is this number, compared to the dominant amplitude, that is important for coherent effects, not the probability. Figure 2 shows the $L=0$ part of the configuration mixed-wave function for the ground state of N^{14} obtained by weighting the functions of Fig. 1 by the structure factors listed in Table IV For comparison, the $L=0$ part of the dominant configuration is shown also. In this case, the effect of the small components has been to concentrate the wave function further out near the edge of the nucleus. In contrast, the $L=2$ part is pulled in by the additional configurations. These are striking effects considering they are caused by 7% admixtures in the wave function, and they manifest themselves in the crucial surface region.

For more strongly mixed states, the coherence can be of even much more importance. Consider for example the two 4+ states. The wave function of the lower one is

$$0.64(p_{1/2}f_{7/2})+0.77(d_{3/2}d_{5/2})$$

and that of the upper one is the orthogonal function. The first has a structure factor for an $L=4$ transition of 0.75, while the second has 0.04. The cross section is proportional to the squares of these numbers.

An examination of the structure factors will suggest which levels will be strongly or weakly made, and what the dominant multipolarity of the transition is for those cases where it is mixed. Several points have to be kept in mind when reading the tables for this purpose: (a) for given L, the component with the larger N is favored because the corresponding wave function is peaked closer to the nuclear surface; (b) for given L, alternating signs for the N components corresponds to constructive interference in the surface region, and leads to stronger transitions; (c) the higher L's are often kinematically favored by the energy of the experiment and the Q value of the reaction. For the reaction considered here, and 40–50-MeV alpha particles, $|\mathbf{k}_\alpha-\mathbf{k}_d|R\cong 4$, where R is the nuclear radius, so that $L=3$, 4, 5 are favored, all other things being equal. Thus, an $L=4$, $N=1$ transition would be favored over an $L=2$, $N=1$, but possibly not over an $L=2$, $N=2$ or $L=0$, $N=3$.

Concerning comparisons with experiment, it is very important to keep in mind the purely statistical factor $(2J+1)$ which is contained in the experimental intensities, and which very much favors the high-spin states. Thus, the 5+ level at ~9 MeV dominates the spectrum.[29] However, if this factor is removed, the intensity is only about 1.5 stronger than the ground-state intensity. From the nuclear structure point of view, this is the relevant comparison.

Referring now to Table V, we see that among the 1+ states, for example, the one at 0 MeV will be excited by both $L=0$ and $L=2$ transitions, the one at 5.5 MeV will go predominantly by $L=0$, the one at 9.3 MeV will be weakly excited, the one at 12 MeV will go predominantly by $L=2$ and the one at 14 MeV will be weakly excited.

For the most part the spectrum of $T=0$ levels in N^{14} below 9 MeV is understood. Prior to the present cal-

culation and a preliminary report of it,[1] the region above ~9 MeV was unexplored by this reaction. From the tables we see that in the region from about 9 to 16 MeV, the (α,d) reaction should excite the following levels, listed roughly in order of expected descending intensity: $6-(L=5)$, $4-(L=3)$, $4+(L=4)$, two $3+(L=4)$, $2+(L=2)$, and $1+(L=2)$. A number of transitions are observed in this energy region.[29] We shall not speculate as to their assignments, however, since the calculated energies at this high excitation cannot be trusted to within several MeV.

VII. THE $C^{12}(He^3,p)N^{14}$ REACTION

In this reaction, the transferred pair can carry both $T=0$ and 1; and since the target has isospin 0, both $T=0$ and 1 levels in N^{14} can be reached. The calculation of the structure factors for the $T=0$ levels is identical to that in the example of Sec. VI, except that the overlap integrals Ω_n will be somewhat different owing to the different sizes of He^4 and He^3 (see Table I). Therefore, the numerical values of the structure factors will be somewhat different for the He^3 initiated reaction considered here.

We concentrate attention on the structure factors for the $T=1$ levels. As discussed in the Appendix, the total angular momentum of the transferred pair when they carry $T=1$ is subject to the selection rule

$$J+\Delta\pi=\text{even}.$$

Since the target has zero spin, J in this case is the spin of the final nucleus. Therefore, the $T=1$ levels with the spin and parity 0^-, 1^+, $2^-\cdots$ cannot be excited. The structure factors for the remaining levels of N^{14} are given in Table VI, and correspond to the second of True's calculations.[31]

As in the preceding example, we can form a rough idea of which states will be most strongly populated.

TABLE VI. Structure amplitudes for the $T=1$ ($S=0$, $L=J$) levels of N^{14} excited in the (He^3,p) reaction ($\nu=0.32$ F^{-2}).

$J\pi$	E MeV	Dominant configuration	G_{NLSJ} $N=1$	$N=2$	$N=3$	$N=4$
0+	2.7	$(p_{1/2})^2$	−0.049	−0.348	0.189	−0.013
	7.9	$(2s)^2$	−0.024	−0.115	−0.525	0.012
	10	$(d_{5/2})^2$	−0.0163	−0.187	−0.129	0.0164
1−	7.0	$p_{1/2}2s$	−0.006	0.401	−0.021	
	12	$p_{1/2}d_{3/2}$	−0.058	−0.247	0.053	
2+	9.6	$2sd_{5/2}$	0.035	0.548	−0.006	
	12	$(d_{5/2})^2$	−0.038	−0.035	0.004	
	16	$2sd_{3/2}$	−0.007	−0.226	0.003	
3−	7.4	$p_{1/2}d_{5/2}$	0.488	−0.030		
4+	12	$(d_{5/2})^2$	−0.563	0.005		
	15	$d_{3/2}d_{5/2}+p_{1/2}f_{7/2}$	0.454	−0.005		

Of the 0+ states, the second should be the strongest; of the 1− states, the first; and of the 2+ states, the first. The 3− and both 4+ states should be strongly populated.

VIII. THE $Pb^{208}(p,t)Pb^{206}$ REACTION

As a final example of the construction of the structure factors, we consider the above reaction. In this case Pb^{208} is doubly magic so its wave function can be assumed to have completely closed shells

$$|\mathrm{Pb}^{208}\rangle=|p_{1/2}{}^2f_{5/2}{}^6p_{3/2}{}^4\cdots;0\rangle, \tag{8.1}$$

where only neutron configurations are listed, since they alone are involved in the reaction.

The wave functions for the levels of Pb^{206} have been obtained by True and Ford,[33] and are of the form

$$|\mathrm{Pb}^{206};J\rangle=\sum_i a_i|(j_i{}^{-2})J\rangle+\sum_{ik} b_{ik}|(j_i{}^{-1}j_k{}^{-1})J\rangle, \tag{8.2}$$

where closed shells have not been mentioned. The parentage factor for components of the first type in Eq. (8.2) is

$$\beta_{j_i{}^2LSJ}=\left\{\binom{N_i}{2}\right\}^{1/2}((j_i{}^{N_i-2})J,(j_i{}^2)J\,]](j_i{}^{N_i})0)\begin{bmatrix} l_i & \frac{1}{2} & j_i \\ l_i & \frac{1}{2} & j_i \\ L & S & J \end{bmatrix}=(2J+1)^{1/2}\begin{bmatrix} l_i & \frac{1}{2} & j_i \\ l_i & \frac{1}{2} & j_i \\ L & S & J \end{bmatrix}, \tag{8.3}$$

where $N=2j+1$ and the coefficient of fractional parentage is given by Eq. (3.9).

For terms of the second type we obtain

$$\beta_{j_ij_kLSJ}=\left\{\binom{N_i}{1}\binom{N_k}{1}\right\}^{1/2}\begin{bmatrix} j_i & j_i & 0 \\ j_k & j_k & 0 \\ J & J & 0 \end{bmatrix}\begin{bmatrix} l_i & \frac{1}{2} & j_i \\ l_k & \frac{1}{2} & j_k \\ L & S & J \end{bmatrix}$$

$$\times((j_i{}^{N_i-1})j_i,j_i\,]](j_i{}^{N_i})0)((j_k{}^{N_k-1})j_k,j_k\,]](j_k{}^{N_k})0)=(2J+1)^{1/2}\begin{bmatrix} l_i & \frac{1}{2} & j_i \\ l_k & \frac{1}{2} & j_k \\ L & S & J \end{bmatrix}. \tag{8.4}$$

[33] W. W. True and K. W. Ford, Phys. Rev. **109**, 1675 (1958).

The fractional parentage coefficients here are unity.

The structure factors for the configuration mixed states are, according to Eq. (2.3),

$$G_{NLSJ}=\sum_i a_i\beta_{j_i{}^2LSJ}\Omega_n\langle n0,NL;\,L|\,n_il_i,n_il_i;\,L\rangle$$
$$+\sqrt{2}\sum_{ik} b_{ik}\beta_{j_ij_kLSJ}\Omega_n\langle n0,NL;\,L|\,n_il_i,n_kl_k;\,L\rangle. \quad (8.5)$$

IX. SUMMARY

In the reactions we have considered, a pair of nucleons is transferred between a nucleus and a light nuclide. The pair is presented or taken away from the nucleus in a specifically correlated condition predicated by the properties of the light nuclides. Nuclear states will have greatly varying proportions of the appropriate correlation, thus accounting in part for the wide range of intensities observed for levels in a given nucleus. In addition, strongly excited states must have a parentage based on the lighter nucleus. The wave functions of a nucleus obtained from a microscopic model must reproduce the observed intensities which depend on rather intimate details. These reactions therefore provide a severe check of the wave functions.

A measure of the appropriate correlation and parentage is provided by the structure amplitude G which appears as a factor multiplying the transfer amplitudes B_{NL}. The latter quantity, which depends upon the scattering states and the kinematics, is divorced from our main discussion.

In any microscopic nuclear model, the correlations are reflected in the wave functions by mixtures of several of the basic states of the model. Once these wave functions have been provided, the structure amplitudes can be computed as a linear combination of the structure amplitudes of the basic states.

The calculation of G has been illustrated in a number of possible situations in Sec. III, and particular reactions were considered in the final sections. To make a conclusive check on whether the wave functions correctly reproduce the observed intensities, one would have to carry out the calculation of the transfer amplitudes B, perhaps along lines suggested in Secs. IV and V. However, the structure factors alone are sufficient to suggest which states will be strongly populated and with what multipolarity.

From the point of view of this paper, the most important experiments to do are those using nuclei from regions of the periodic table where detailed nuclear structure calculations are possible.

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APPENDIX

We write down in more detail the form of the differential cross section for two-nucleon transfer in the direct-reaction mode. The notation for the reaction is defined in Eq. (3.1). The cross sections for the stripping and pickup reactions are given by

$$\frac{d\sigma}{d\Omega}=\frac{k_2}{k_1}\frac{2J_2+1}{2J_1+1}\left(\frac{d\sigma}{d\Omega}\right)_0, \quad \text{(stripping)}, \quad (A1)$$

$$\frac{d\sigma}{d\Omega}=\frac{k_1}{k_2}\frac{2S_1+1}{2S_2+1}\left(\frac{d\sigma}{d\Omega}\right)_0, \quad \text{(pickup)}, \quad (A2)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_0=\frac{m_1{}^*m_2{}^*}{(2\pi\hbar^2)^2}\Omega_d{}^2$$
$$\times\sum_{LSJT} C_{ST}{}^2\sum_M\Big|\sum_N G_{NLSJT}B_{NL}{}^M\Big|^2, \quad (A3)$$

$$C_{ST}=\langle T_1T_{Z_1},TT_Z\,|\,T_2T_{Z_2}\rangle b_{ST}. \quad (A4a)$$

Here $m_1{}^*$ and $m_2{}^*$ are the reduced masses of the light nuclides and the bracket is a Clebsch-Gordan coefficient for the isospins, where the transferred pair carries TT_Z. The quantity $b_{ST}{}^2$ is an overlap factor involving the spin-isospin functions of the light nuclides and is given by

$$b_{ST}{}^2=\begin{cases}\delta_{S0}\quad\delta_{T1}, & (t,p)\text{ or }(\mathrm{He}^3,n),\\ \tfrac{1}{2}(\delta_{S0}\quad\delta_{T1}+\delta_{S1}\quad\delta_{T0}), & (t,n)\text{ or }(\mathrm{He}^3,p),\\ \delta_{S1}\quad\delta_{T0}, & (\alpha,d),\end{cases} \quad (A4b)$$

where we have assumed that the spatial wave functions are totally symmetric. In case the isospin formalism is not used, factors involving T should be dropped (and the counting factor in β should be rewritten, as explained in Sec. III, in terms of neutron and proton numbers[34]).

The transfer amplitude $B_{NL}{}^M$ is defined for stripping by

$$B_{NL}{}^M(\mathbf{k}_1,\mathbf{k}_2)=i^{-L}(2L+1)^{-1/2}$$
$$\times\int[\psi_2{}^{(-)}(\mathbf{k}_2,\mathbf{R}_2)u_{NL}(2\nu R^2)Y_L{}^M(\hat{R})]^*$$
$$\times V(\rho)\psi_1{}^{(+)}(\mathbf{k}_1,\mathbf{R}_1)\varphi(\rho)d\mathbf{R}_1d\mathbf{R}_2 \quad (A5)$$

and for pickup by

$$(-)^{L+M}B_{NL}{}^{-M}(-\mathbf{k}_1,-\mathbf{k}_2). \quad (A6)$$

Here $\psi^{(\pm)}$ refer to the scattering solutions; R, R_1, and R_2 refer to the center-of-mass coordinates of the transferred pair and the two light nuclides of Eq. (3.1); and $\rho=|\mathbf{R}-\mathbf{R}_2|$. The wave function $\varphi(\rho)$ refers to that part of the internal wave function for the light nuclide (a)

[34] The result for b^2 looks different than that given in Ref. 1. Here we have absorbed the factor $\binom{n}{\nu}\binom{p}{\pi}/(2S+1)$ into its definition.

which depends on ρ. This separation is possible when we use a Gaussian wave function for the nuclide (a):

$$\varphi_a = N\exp(-\eta^2\textstyle\sum r_{ij}^2)$$
$$= \varphi_{10}(4\eta^2r^2)\varphi_{10}(4\eta^2r_2^2)\varphi_{10}(8\eta^2\rho^2), \quad (a=4)$$
$$= \varphi_{10}(3\eta^2r^2)\varphi_{10}(4\eta^2\rho^2), \quad (a=3). \quad \text{(A7)}$$

Here r is the relative coordinate between the transferred pair of nucleons, and r_2 is the relative coordinate between the pair of nucleons in the nuclide $(a-2)$ in the case $a=4$. For $a=3$ this coordinate is absent. The functions φ_{nl} are harmonic-oscillator functions:

$$\varphi_{nl}{}^m(\nu r^2) = u_{nl}(\nu r^2)Y_l{}^m(\hat{r})$$
$$= \left[\frac{2\nu^{3/2}(n-1)!}{\Gamma(n+l+\tfrac{1}{2})}\right]^{1/2}(\nu^{1/2}r)^l$$
$$\times L_{n-1}{}^{l+1/2}(\nu r^2)e^{-\frac{1}{2}\nu r^2}Y_l{}^m(\hat{r}) \quad \text{(A8a)}$$

$$L_{n-1}{}^{l+1/2}(x) = \sum_{k=0}^{n-1}\binom{n+l-\frac{1}{2}}{n-k-1}\frac{(-x)^k}{k!}, \quad (n\geqq 1). \quad \text{(A8b)}$$

The factor Ω_d in Eq. (A3) is the overlap between the deuteron in (α,d) reactions, and the relevant part of the α-wave function, whereas if $a=3$ it is unity:

$$\Omega_d = \int u_d{}^*(r_2)u_{10}(4\eta^2r_2^2)r_2^2dr_2, \quad (a=4)$$
$$= 1, \quad (a=3). \quad \text{(A9)}$$

It acts only as an over-all normalizing factor. The overlap integral on the coordinate r in Eq. (A7) is called Ω_n,

$$\Omega_n = \int u_{n0}(\tfrac{1}{2}\nu r^2)u_{10}(a\eta^2r^2)r^2dr, \quad \text{(A10)}$$

and is discussed in Sec. II.

In our formulation, the wave functions u_{NL} describing the center-of-mass motion of the transferred pair are harmonic-oscillator functions. This choice was made because of their convenient analytic properties. These functions are good representations of the single-particle wave functions in any potential well of the type usually assumed for the shell-model central potential, except in the surface region, where they decay too rapidly. This fault can be easily remedied by replacing the oscillator by the appropriate Hankel (or Coulomb) function beyond the point in the surface region where their logarithmic derivatives match.

TABLE VII. Selection rules for two-nucleon-transfer reactions.

Reaction	S[a]	T[a]	ΔT[b]	j^2	J[a] j_1j_2
(α,d)	1	0	0	odd	···
(t,p) or (He^3,n)	0	1	1 or 0 if $T_1\neq 0$	even	$J+\Delta\pi$ even
(t,n) or (He^3,p)	1	0	0	odd	···
	0	1	1 or 0 if $T_1\neq 0$	even	$J+\Delta\pi$ even

[a] Belongs to transferred pair.
[b] Isospin change of nucleus.

Alternatively, one could from the beginning use single-particle wave functions corresponding to, say, a Woods-Saxon potential. The convenience of the oscillator functions could still be exploited by expanding the former in terms of the oscillator functions. In this case, in Eq. (2.3) the replacement

$$\Omega_n\langle n0,NL;L|n_1l_1n_2l_2;L\rangle \rightarrow \sum_{\bar{n}\bar{n}_1\bar{n}_2} a_{\bar{n}_1l_1}a_{\bar{n}_2l_2}$$
$$\times\Omega_{\bar{n}}\langle\bar{n}0,NL;L|\bar{n}_1l_1\bar{n}_2l_2;L\rangle \quad \text{(A11)}$$

should be made. Here $a_{\bar{n}l}$ are the expansion coefficients.

General selection rules for two-nucleon-transfer reactions have been given elsewhere.[1,3,4] In special cases additional rules hold.

If both particles are transferred to (or from) the same state to form $(j^2)J$ then the additional rule

$$J+S=\text{even} \quad \text{(A12)}$$

governs the total spin and angular momentum. Because of the selection rules on S as dictated by the particular reaction, this restricts the squared configurations to only certain spins J.

For any configuration, if $S=0$, then J must obey the parity rule: $J+\Delta\pi=$ even.

These are summarized in Table VII. We emphasize that J, S, and T belong to the transferred pair and are connected to the nuclear properties by

$$\mathbf{J}_1=\mathbf{J}_2+\mathbf{J}, \quad \mathbf{T}_1=\mathbf{T}_2+\mathbf{T}. \quad \text{(A13)}$$