Tunneling Density of States for a Superconductor Carrying a Current*

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A current flowing in a superconducting film has a tendency to break pairs similar to the effect of paramagnetic impurities in a superconductor. The effect of the current on the tunneling density of states for a normal-superconducting tunneling junction has been computed by the Green's-function method of Abrikosov, Gorkov, and Maki, and found to be identical (for a short mean free path) to that of paramagnetic impurities. A simple relationship between the current parallel to the junction and the equivalent paramagnetic-impurity concentration is derived. Passing various amounts of current through the sample in the one experiment gives the same information as is obtainable in the other experiment only by preparing different samples with various impurity concentrations.

I. INTRODUCTION

IT has been known since the work of Rogers[1] and Bardeen[2] that a current in a superconductor, which requires a finite pairing momentum for the electron pairs, provides a mechanism for breaking the pairs. In this respect the effect of a current is comparable to that of paramagnetic impurities[3] and to that of a strong spin-exchange field,[4] where the pair breaking mechanism is provided by the interaction of the electron spins with the spins of the impurities or with an imposed field, respectively. The most striking effect of the inclusion of paramagnetic impurities in a superconductor is known to be the change in the tunneling density of states as measured by Reif and Woolf.[5] It is therefore of interest to calculate the tunneling density of states for a superconductor, if the latter is part of a superconductor–normal-metal tunneling junction and carries a current flowing parallel to the junction. In order to have a uniform current distribution in the superconductor one has to deal with a thin film and hence to assume a finite mean free path in the calculations. The problem is solved by using a Green's function formalism developed by Abrikosov and Gorkov[3] and Maki.[6] For the limiting cases of very short and infinitely long mean free path, simple expressions for the density of states can be derived. For short mean free path the density-of-state curves obtained in this way are identical with those one obtains for paramagnetic impurities in a superconductor. A simple relation will be derived which connects the supercurrent flowing parallel to the junction with the equivalent paramagnetic impurity concentration giving rise to the same tunneling density of states.

* Research supported in part by the U. S. Air Force Office of Scientific Research.

[1] K. T. Rogers, Ph.D. thesis, University of Illinois, 1960 (unpublished).
[2] J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).
[3] A. A. Abrikosov and L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].
[4] P. Fulde and R. A. Ferrell, Phys. Rev. **135**, A550 (1964).
[5] F. Reif and M. A. Woolf, Phys. Rev. Letters **9**, 315 (1962).
[6] K. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 10 and 333 (1963).

II. TUNNELING DENSITY OF STATES

Throughout the paper we are concerned with an experimental situation as depicted in Fig. 1. A current is flowing parallel to a tunneling junction in a superconducting film. Furthermore, we will restrict ourselves for simplicity to the zero-temperature case.

In order to determine the tunneling density of states $N_T(\omega)$ of the system we make use of a simple relationship noted by Kadanoff and Schrieffer[7] between $N_T(\omega)$ and the single-particle Green's function $G(\mathbf{p},\omega)$, and which can be written as

$$N_T(\omega)=-\frac{N(0)}{\pi}\int_{-\infty}^{+\infty}d\epsilon_p\int_{-1}^{+1}\frac{d\mu}{2}\,\mathrm{Im}G(\mathbf{p},\omega)\,. \qquad (1)$$

$N(0)$ denotes the density of states of free electrons at the Fermi surface and μ is the cosine of the angle between $\mathbf{p}$ and the direction of the current flow. In Eq. (1) it has been assumed that the single-particle Green's function is translational invariant, which is not quite true if scattering centers are present; but an average over all of their positions will make the Green's function translationally invariant and allow the use of the above formula.

The Green's functions for superconductors in the presence of impurities were calculated by Abrikosov and Gorkov[3] and later extended to nonzero pairing momenta by Maki.[6] If the electrons are paired around

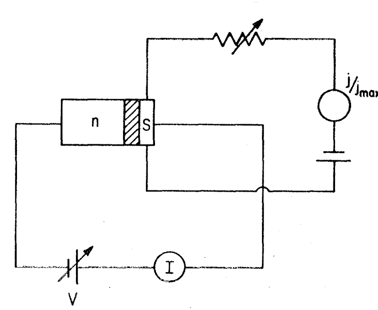

FIG. 1. Schematic representation of the proposed experiment for which the tunneling density of states is calculated. By varying $j/j_{\max}$ different density of states curves can be measured.

[7] J. R. Schrieffer, Rev. Mod. Phys. **36**, 200 (1964). (This paper gives a reference to unpublished work of L. P. Kadanoff.)

the pairing momentum $-\mathbf{q}$ in momentum space, it is advantageous to write the Green's function and Gorkov's F function in the form

$$G(\mathbf{x}-\mathbf{x}')=e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}G_q(\mathbf{x}-\mathbf{x}')\,,\qquad F^+(\mathbf{x},\mathbf{x}')=e^{-i\mathbf{q}\cdot(\mathbf{x}+\mathbf{x}')}F_q^{\;+}(\mathbf{x}-\mathbf{x}')\,. \tag{2}$$

The Fourier transforms $G(\mathbf{p},\omega)$ and $F^+(\mathbf{p},\omega)$ of $G_q(\mathbf{x}-\mathbf{x}')$ and $F_q^{\;+}(\mathbf{x}-\mathbf{x}')$ have a simple form and were calculated by Maki to be

$$G(\mathbf{p},\omega)=\frac{\tilde{\omega}+qv_F\mu+\xi_p}{(\tilde{\omega}+qv_F\mu)^2-\xi_p^{\,2}-\tilde{\Delta}^2},\qquad F^+(\mathbf{p},\omega)=\frac{-i\tilde{\Delta}}{(\tilde{\omega}+qv_F\mu)^2-\xi_p^{\,2}-\tilde{\Delta}^2}. \tag{3}$$

If one compares these expressions with those derived by Gorkov[8] one sees that Eq. (3) can be obtained from the latter by adding to ω a Galilean term $qv_F\mu$ and replacing ω and Δ by $\tilde{\omega}$ and $\tilde{\Delta}$. The latter quantities are functions of ω and Δ and take the self-energy caused by the impurity scattering into account. ξ_p is the single-particle energy ϵ_p minus the chemical potential.

If we restrict ourselves to s-wave scattering of the electrons by the impurities the relation between $\tilde{\omega}$, $\tilde{\Delta}$ and ω, Δ is given by

$$\omega-\tilde{\omega}=\frac{1}{2\pi\tau}\int_{-1}^{+1}\frac{d\mu}{2}\int_{-\infty}^{+\infty}d\epsilon_p G(\mathbf{p},\omega)\,,\qquad \Delta-\tilde{\Delta}=\frac{1}{2\pi\tau}\int_{-1}^{+1}\frac{d\mu}{2}\int_{-\infty}^{+\infty}d\epsilon_p F^+(\mathbf{p},\omega)\,. \tag{4}$$

Here τ is the mean free time between collisions. This allows us to write the tunneling density of states as

$$N_T(\omega)=N(0)\cdot 2\tau\cdot \mathrm{Im}\tilde{\omega}\,. \tag{5}$$

In order to determine $N_T(\omega)$ we have to establish a relation between $\tilde{\omega}$ and ω, Δ. To do this we calculate the integrals in Eq. (4) and obtain

$$\omega-\tilde{\omega}=\frac{-i}{4\tau qv_F}\{[(\tilde{\omega}+qv_F)^2-\tilde{\Delta}^2]^{1/2}-[(\tilde{\omega}-qv_F)^2-\tilde{\Delta}^2]^{1/2}\}\,,\qquad \Delta-\tilde{\Delta}=\frac{\tilde{\Delta}}{4\tau qv_F}\left\{\sin^{-1}\frac{\tilde{\omega}+qv_F}{\tilde{\Delta}}-\sin^{-1}\frac{\tilde{\omega}-qv_F}{\tilde{\Delta}}\right\}. \tag{6}$$

For very long mean free path we can replace in both equations on the right-hand side $\tilde{\omega}$, $\tilde{\Delta}$ by ω, Δ. This gives the desired relation between $\tilde{\omega}$ and ω, Δ immediately. We will investigate $N_T(\omega)$ in this limit later, but we first want to concentrate on the more interesting case of a very short mean free path. In proceeding we adopt a notation used by Maki and define two new quantities χ, φ by

$$\tilde{\omega}/\tilde{\Delta}=\sin\varphi\cos\chi\,,\qquad qv_F/\tilde{\Delta}=\cos\varphi\sin\chi\,. \tag{7}$$

This simplifies the Eqs. (6) to

$$\omega=\tilde{\omega}(1-(1/2\tau qv_F)\tan\chi)\,,\qquad \Delta=\tilde{\Delta}(1-\chi/2\tau qv_F)\,. \tag{8}$$

Combining (7) and (8) we obtain

$$\frac{\omega}{\Delta}=\left[\sin^2\chi-\left(\frac{qv_F}{\Delta}\right)^2\left(1-\frac{\chi}{2\tau qv_F}\right)^2\right]^{1/2}\frac{2\tau qv_F\cot\chi-1}{2\tau qv_F-\chi}. \tag{9}$$

This equation together with Eq. (8) establishes the desired relation between $\tilde{\omega}$ and ω, Δ. $N_T(\omega)$ will be different from zero for all ω's which have complex χ solutions. In order to see where the onset of density of states will occur we first look for the real solutions of Eq. (9), assuming $\tau qv_F\ll 1$. Writing

$$\chi=2\tau qv_F(1-x)\,, \tag{10}$$

and expanding $\sin^2\chi$ and $\cot\chi$, we see that x has to be of the order τqv_F or smaller in order to obtain real solutions for x. Equation (9) therefore simplifies to

$$\omega/\Delta=[1-(x/2\tau\Delta)^2]^{1/2}[1-4(\tau qv_F)^2/3x]. \tag{11}$$

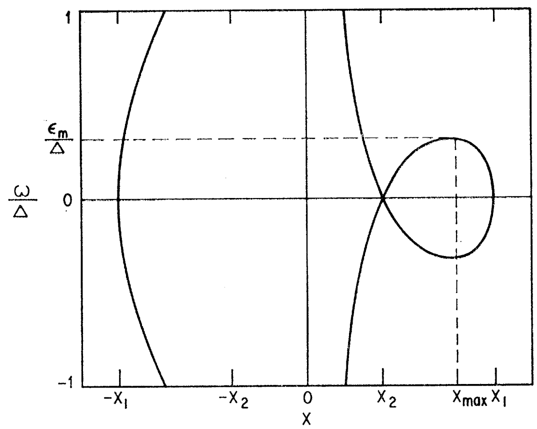

FIG. 2. Plot of the function $\omega/\Delta=[1-(x/2\tau\Delta)^2]^{1/2}[1-4(\tau qv_F)^2/3x]$. The function vanishes at $\pm x_1=\pm(\tau\Delta)$ and $x_2=\frac{4}{3}(\tau qv_F)^2$ and has a maximum ϵ_m/Δ at $x_{\max}$. For the special case $x_1=x_{\max}=x_2$ we find $qv_F/\Delta=(3/2\tau\Delta)^{1/2}$.

The real solutions of this equation are sketched in Fig. 2. The right branch has a maximum ϵ_m/Δ for

$$x_{\max}=(4(\tau\Delta)(\tau qv_F)/\sqrt{3})^{2/3}. \tag{12}$$

If we define the quantities Z and $\bar{x}$ by

$$Z=[\tfrac{2}{3}(qv_F/\Delta)(\tau qv_F)]^{2/3}\,,\qquad x=x_{\max}\cdot\bar{x}\,, \tag{13}$$

[8] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 34, 735 (1958) [English transl.: Soviet Phys.—JETP 7, 505 (1958)].

then we can write Eq. (11) for x values of the order of $x_{\max}$ as

$$\omega/\Delta=[1-\bar{x}^2Z]^{1/2}(1-Z/\bar{x}). \tag{14}$$

Since this is an algebraic equation of fourth order in $\bar{x}$, we see from Fig. 2 that for $\omega/\Delta>\epsilon_m/\Delta$ we will obtain two complex and conjugate $\bar{x}$ solutions and hence two complex $\tilde{\omega}$ values. The onset of density of states is therefore at

$$\epsilon_m/\Delta=(1-Z)^{3/2}. \tag{15}$$

ϵ_m will be called the excitation gap. For $Z=1$ this gap vanishes. The formalism used here is not restricted to $Z<1$ values.

In order to calculate $N_T(\omega)$ for all ω, if a certain Z value determined by the experiment is given, we express $\tilde{\omega}$ in terms of $\bar{x}$ and Z. Expanding $\tan\chi$ in Eq. (8) and expressing χ in terms of $\bar{x}$ we obtain

$$\tilde{\omega}=\omega/\Delta[2\tau Z^{1/2}(\bar{x}-Z)]^{-1}. \tag{16}$$

The tunneling density of states therefore finally becomes

$$\frac{N_T(\omega)}{N(0)}=\frac{\omega}{\Delta}\frac{1}{Z^{1/2}}\operatorname{Im}\frac{1}{(\bar{x}-Z)}. \tag{17}$$

$\bar{x}$ is connected with ω/Δ by Eq. (14) which we also can write as

$$\bar{x}^4-2Z\bar{x}^3+\bar{x}^2\left[\frac{1}{Z}\left(\frac{\omega}{\Delta}\right)^2-\frac{1}{Z}+Z^2\right]+2\bar{x}-Z=0. \tag{18}$$

For an especially simple way to find the solutions of this equation we refer to Ref. 9. Results are plotted in Fig. 3. The different curves correspond to $Z=0.1$; 0.448; 1; 1.2.

Up to now we have dealt with the pairing momentum of the superconducting electrons. But it is known from

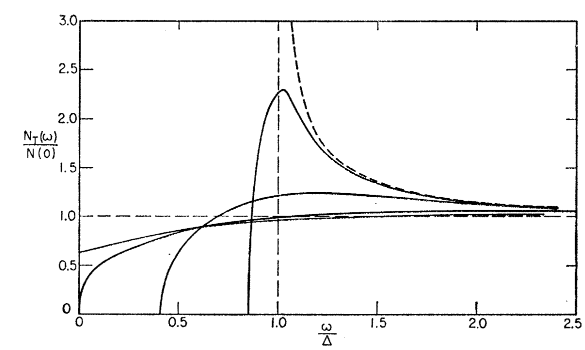

FIG. 3. Tunneling density of states in the limit of short mean free path for $\epsilon_m/\Delta=0.854(Z=0.1)$; $\epsilon_m/\Delta=0.410(Z=0.448)$; $\epsilon_m/\Delta=0(Z=1)$; $\epsilon_m/\Delta=0(Z=1.2)$. The dashed line represents the BCS density of states. $\epsilon_m/\Delta=0.410$ corresponds to $j/j_{\max}=1$. In order to measure smaller ϵ_m/Δ values a setup as proposed in Ref. 10 has to be used.

[9] L. Collatz, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1955), Vol. II, p. 325.

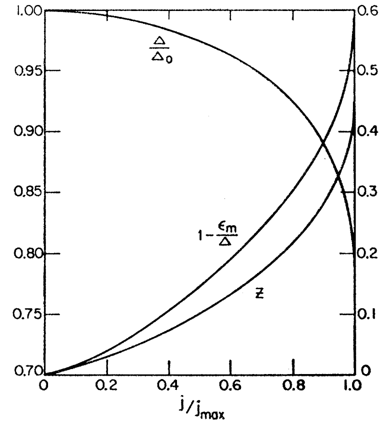

FIG. 4. The parameter Z, the reduced order parameter Δ/Δ_0 and the reduced difference $1-\epsilon_m/\Delta$ between order parameter and excitation gap as a function of the reduced current $j/j_{\max}$. Δ_0 denotes the order parameter in the absence of a current.

Refs. 1, 2, and 6 that the maximum current measured in an experiment as shown in Fig. 1 does not correspond to the largest pairing-momentum compatible with superconductivity. This implies that in such an experiment not all density of states curves shown in Fig. 3 will be realizable. Maki has shown that the Z value corresponding to the maximum current is $Z=0.448$ and that $(\epsilon_m/\Delta)_{j=j_{\max}}=0.410$. In order to measure also the density-of-states curves corresponding to larger pairing momenta, especially those in which the excitation gap vanishes, one has to use an experimental setup as proposed in Ref. 10.

Since the density of states in Fig. 3 is scaled with respect to the order parameter Δ, it is important to know how this order parameter depends on the current j. We have, therefore, plotted in Fig. 4 the dependence of Δ/Δ_0, Z and also $(1-\epsilon_m/\Delta)$ on the reduced current $j/j_{\max}$. In doing this the following relations derived by Maki for $Z<1$ were used:

$$\begin{aligned}\Delta/\Delta_0&=\exp(\tfrac{1}{4}\pi Z^{3/2})\\ j/j_{\max}&=0.947Z^{3/4}(\pi-\tfrac{4}{3}Z^{3/2})\exp(-\tfrac{3}{8}\pi Z^{3/2}).\end{aligned} \tag{19}$$

Δ_0 is the order parameter in the absence of a current. Thus if a value for $j/j_{\max}$ is given, Fig. 4 yields immediately the order parameter, the excitation gap and the Z value. These parameters in turn specify the density-of-states curve.

III. PARAMAGNETIC IMPURITIES

We want to show now that the tunneling density of states for a superconductor carrying a current is in the limit of a short mean free path the same as for a superconductor containing paramagnetic impurities. Using the theory of Abrikosov and Gorkov[3] we write for the Green's function of a superconductor with paramagnetic impurities

$$G(\mathbf{p},\omega)=(\tilde{\omega}+\xi_p)/(\tilde{\omega}^2-\xi_p{}^2-\tilde{\Delta}^2), \tag{20}$$

[10] P. Fulde and R. A. Ferrell, Phys. Rev. **131**, 2457 (1963).

where

$$\tilde{\omega}=\omega+(1/2\tau_1)u[1-u^2]^{-1/2},$$
$$\tilde{\omega}=\Delta+(1/2\tau_2)[1-u^2]^{-1/2}. \tag{21}$$

The quantity u is defined by

$$u=\tilde{\omega}/\tilde{\Delta}. \tag{22}$$

τ_1 is the inverse total scattering rate and τ_2 is the inverse difference between no spin-flip and spin-flip scattering rate. Using Eqs. (1) and (20) we obtain for the tunneling density of states

$$N_T(\omega)=N(0)\cdot \mathrm{Im}\tilde{\omega}/(\tilde{\Delta}^2-\tilde{\omega}^2)^{1/2}. \tag{23}$$

In terms of the quantity u this can be written as

$$N_T(\omega)=N(0)\cdot \mathrm{Im}u/(1-u^2)^{1/2}. \tag{24}$$

The relation of u with ω, Δ is given by combining Eqs. (21) and (22). We obtain

$$\frac{\omega}{\Delta}=u\left[1-\frac{1}{\tau_s\Delta}\frac{1}{(1-u^2)^{1/2}}\right]. \tag{25}$$

Here τ_s is proportional to the inverse spin-flip scattering rate and is defined by

$$1/\tau_s=1/2\tau_1-1/2\tau_2. \tag{26}$$

In order to show the equivalence as stated above we have only to write Eqs. (17) and (18) in such a way that they look similar to Eqs. (24) and (25). This can be done by setting

$$Z^{1/2}\bar{x}=(1-u^2)^{1/2}. \tag{27}$$

With this substitution Eqs. (14) and (18) read

$$\omega/\Delta=u[1-Z^{3/2}(1-u^2)^{-1/2}], \tag{28}$$

while Eq. (17) can now be written as

$$N_T(\omega)=N(0)\ \mathrm{Im}u/(1-u^2)^{1/2}. \tag{29}$$

This shows that a superconducting film carrying a current $j/j_{\max}$ will have the same tunneling density of states as a superconductor containing such a concentration of paramagnetic impurities that

$$1/\tau_s\Delta=Z^{3/2}. \tag{30}$$

The dependence of the order parameter Δ on τ_s is derived in Ref. 3.

This equivalence shows that a tunneling density of states as defined by Eqs. (17) and (18) is more general than originally thought. It suggests that other depairing mechanisms also will lead to such a density of states in the limit of a short mean free path. Among these the case of a superconducting film in a magnetic field is of special interest and calculations of the tunneling density of states are under way. If we compare the two equivalent experiments analyzed in this paper, it would seem that the experiment on a current carrying film would possess some practical advantage relative to the experiment on impurity-containing films. In the first case, the relevant parameter can be varied without changing the sample while in the second case the equivalent measurements can be carried out only by comparing different samples containing different concentrations of the paramagnetic impurity.

IV. INFINITE MEAN FREE PATH

As mentioned in the discussion of Eq. (6) the relation between $\tilde{\omega}$ and ω, Δ is easily found in the limit of long mean free path. We obtain in that case from Eq. (6)

$$\tilde{\omega}=\omega+(i/4\tau qv_F)\{[(\omega+qv_F)^2-\Delta^2]^{1/2}-[(\omega-qv_F)^2-\Delta^2]^{1/2}\}, \tag{31}$$

where we have not yet passed to the limit $\tau\to\infty$. With the help of Eq. (5) the tunneling density of states becomes now

$$N_T(\omega)=(N(0)/2qv_F)\ \mathrm{Re}\{[(\omega+qv_F)^2-\Delta^2]^{1/2}-[(\omega-qv_F)^2-\Delta^2]^{1/2}\},$$

independent of τ in the limit $\tau\to\infty$. It is convenient to add a small positive imaginary part to ω in order to find the right branch of the square roots. Using the above formula we have plotted in Fig. 5 the density of states for different values of the parameter qv_F/Δ ($qv_F/\Delta=0.146$; 0.646; 1.20). The values were chosen such that the onset of density of states occurs at the same point as for $Z=0.1$; 0.5; 1.2 in the short mean-free-path limit. A comparison with Fig. 3 illustrates the effect of the mean free path. Although there are quantitative changes in the density of states for the two limiting cases, the essential features of the latter are independent of mean free path.

It might be mentioned that the tunneling density of states in the limit of an infinite mean free path as represented by Eq. (32) can also be obtained without the use of Green's functions. One can start directly from the excitation spectrum. The latter is identical to that of the BCS theory plus an additional Galilean term which takes the nonzero pairing momentum into

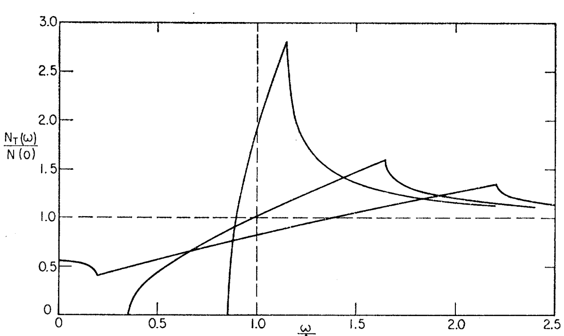

FIG. 5. Tunneling density of states in the limit of an infinite mean free path for different values of the parameter qv_F/Δ ($qv_F/\Delta=0.146$; 0.646; 1.20).

account. Because of the asymmetry of the excitation energy in momentum space one has to take an average of BCS-type density-of-states curves with the origin on the ω axis shifted by the amount of the Galilean term. In this way one obtains the same results as with the method used above. For further details we refer to Ref. 4 where the second method has been applied.

Note added in proof. A simple calculation using a Green's function formalism as developed by K. Maki, Progr. Theoret. Phys. (Kyoto) **31**, 731 (1964), shows that a superconducting film in a uniform magnetic field parallel to its surface gives again the same type of tunneling density of states in the limit of a short mean free path as do paramagnetic impurities or a uniform current. There is again a simple relationship between the paramagnetic impurity concentration and the size of the magnetic field.

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Off-Diagonal Long-Range Order and Persistent Currents in a Hollow Cylinder*

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Some properties of the reduced density matrix are considered in their general relation to the current produced by a system of charged particles in a thin-walled long cylinder. The common connection of persistent currents and off-diagonal long-range order to the velocity distribution function will be particularly stressed. Essential features of this function, characteristic for the superconducting state, are discussed and exemplified by applying an analogous procedure to the case of boson and fermion systems.

I. INTRODUCTION

BY introducing the concept of off-diagonal long-range order (ODLRO), Yang[1] has furnished an important key to the characteristic features which underlie the phenomena of superconductivity and flux quantization.[2,3] Based upon his considerations, the present paper shall serve to further illustrate the usefulness of this concept and to show its direct bearing upon the existence of persistent currents under particularly simple conditions.

For this purpose we consider the axially symmetrical equilibrium state of N identical particles with mass m and charge e in a circular cylinder of length L, large compared to the outer radius R and the wall thickness d. This geometry was used by Bloch and Rorschach (B.R.)[4] in an earlier investigation of the ideal charged boson gas at zero temperature according to Schafroth's model, and their method to account for magnetic effects will be likewise applied here. For simplicity, the discussion will be restricted to very thin cylinder walls,[5] i.e., to the case where d is assumed to be small not only compared to R but also to the penetration depth $\lambda \ll R$. While the modifications necessary for thicker walls are of considerable interest, they would not add essentially new elements to the considerations presented below and, therefore, will be mentioned only briefly in the last section.

On the other hand, the following treatment refers to a system of arbitrarily interacting particles at any temperature and obeying either Bose or Fermi statistics. It leads to quite general qualitative conclusions, and Schrafroth's model, as well as a corresponding example for the fermion system, will be used primarily for the purpose of illustration.

II. REDUCED DENSITY MATRIX, VELOCITY DISTRIBUTION, AND CURRENT

Starting from Yang's general defintion[6] of the reduced density matrices ρ_1 and ρ_2, each of the indices $i, j, \cdots$ stands in a complete coordinate representation for a set of three position variables and, generally, of an additional spin variable ζ. In view of the chosen cylindrical geometry, two of the former, denoted by r and z, shall measure the distance from the cylinder axis and a distance in the direction parallel to the axis, respectively. Instead of the angle θ around the axis, we use the length $x=r\theta$ as the third position variable and denote the corresponding tangential direction as the x direction. Since r varies within the cylinder only by the

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[1] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[2] B. Deaver and W. M. Fairbank, Phys. Rev. Letters **7**, 43 (1961).
[3] R. Doll and M. Näbauer, Phys. Rev. Letters **7**, 51 (1961).
[4] F. Bloch and H. E. Rorschach, Phys. Rev. **128**, 1697 (1962).
[5] Reference 4, Sec. IVA.
[6] Reference 1, Eqs. (1) and (2).