Acoustic Impedance of Liquid He^3†

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We present a derivation of the acoustic impedance of liquid He^3 at very low temperatures, where the behavior of the liquid is of the collisionless type. Assuming the collective mode is dominant, we show that the energy transfer at the boundary can be calculated directly by equating the work done by the vibrating wall against the pressure of the liquid to the energy flowing from the wall in the collective mode. The comparison of our result with the exact result obtained by Bekarevich and Khalatnikov demonstrates the accuracy of our method. The application to experiment is discussed.

1. ANALYSIS OF BOUNDARY CONDITIONS

RECENTLY, Bekarevich and Khalatnikov[1] have studied in great detail the process of transfer of energy into liquid He^3 by a vibrating plane surface bounding it. We give here a simplified derivation which brings out some physical features of their results. They have solved the kinetic equation for the distribution of quasiparticles for both specular and diffuse reflection of the quasiparticles on the wall. Their results which provide the starting point of this paper are the following. They find from their calculation that the result depends only weakly on the character of the reflection, and, in the case of specular reflection, that the essential contribution arises from the coupling to the zero-sound collective mode.

Let us see how this can be directly interpreted. The distribution of the quasiparticles satisfies the equation

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial \mathbf{r}}\cdot\frac{\partial \epsilon}{\partial \mathbf{p}}-\frac{\partial n}{\partial \mathbf{p}}\cdot\frac{\partial \epsilon}{\partial \mathbf{r}}=I(n)\,, \tag{1}$$

valid for $z>0$, $z=0$ being the instantaneous position of the wall. n and $\epsilon=\epsilon_0+\delta\epsilon$ are, respectively, the distribution function and the energy of the quasiparticles and $I(n)$ is the collision integral. As usual, we linearize this equation by setting $n=n_0+\delta n$ and take a periodic behavior of the form $\exp[-i\omega t]$ where ω is the vibration frequency of the wall. Let θ be the polar angle of the momentum vector $\mathbf{p}$. Defining ν by $\delta n=\nu(\partial n_0/\partial\epsilon)$, and keeping the first two terms of the expansion in Legendre polynomials of the Landau correction $\delta\epsilon$ to the energy of a quasiparticle (for further details, see Refs. 1 and 2), we can rewrite the kinetic equation:

$$-i\omega\nu+v_0\cos\theta(\partial/\partial z)[\nu+F_0\nu_0+3F_1\nu_1\cos\theta]=I(\nu)\,, \tag{2}$$

where $v_0=(\partial\epsilon_0/\partial p)_{p_0}$ is the velocity at the Fermi surface and where we have used the notation

$$\nu_0=\int\nu\frac{d\Omega}{4\pi};\qquad \nu_1=\int\nu\cos\theta\frac{d\Omega}{4\pi}.$$

We are interested here in the limit $\omega\tau\gg1$ where τ^{-1} is the collision frequency of the excitations in the liquid. The collision integral $I(\nu)$ can then be neglected.

The boundary condition in the case of specular reflection reads as follows (see Ref. 1):

$$\Psi(\mu)+p_0u_z\mu=\Psi(-\mu)-p_0u_z\mu\,, \tag{3}$$

where $\mu=\cos\theta$, u_z is the maximum velocity of the wall, and

$$\Psi(\mu)=\nu(z=0,\mu)+F_0\nu_0(z=0)+3F_1\nu_1(z=0)\mu\,. \tag{4}$$

We shall now proceed to show that condition (3) can be satisfied to good accuracy by a plane wave flowing from the interface into the liquid. The corresponding zero sound distribution is obtained from Eq. (2) by neglecting the collision term and assuming for ν a plane-wave dependence $\exp[i(kz-\omega t)]$. ν then satisfies

$$(\cos\theta-s_0)\nu+\cos\theta(F_0\nu_0+3F_1\nu_1\cos\theta)=0\,, \tag{5}$$

where $s_0=\omega/kv_0$ is the zero sound velocity expressed in terms of v_0. Inserting Eq. (5) in Eq. (4), we find that

$$\Psi(\mu)-\Psi(-\mu)=2[(s_0F_0\nu_0+3s_0^2F_1\nu_1)/(s_0^2-\mu^2)]\mu\,. \tag{6}$$

We immediately see that condition (3) can be satisfied by the zero-sound distribution inasmuch as the term μ^2 in the denominator can be neglected, that is, if $s_0^2\gg1$. This is generally very well verified as s_0^2 raises from $s_0^2=11$ at low pressure to $s_0^2=140$ at high pressure (see Table I), and thus zero sound appears to be the dominant mode involved in the transfer of energy from the vibrating wall.

TABLE I. Parameters of liquid He^3 at very low temperature.

$P_{(\mathrm{atm})}$	ρ/ρ_0[a]	m^*/m[b]	$v_0(m/s)$	$c_1(m/s)$[c]	F_1	F_0	s_1	s_0	$c_0(m/s)$
0.12	1	2.89	57.3	184	1.89	9.7	3.21	3.33	191
28	1.4	5.55	33.4	394	4.55	74	11.80	11.82	396

$\rho_0=0.0816$ g/cm^3

[a] R. H. Sherman and F. J. Edeskuty, Ann. Phys. (N. Y.) 9, 522 (1960).
[b] A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. 130, 495 (1963).
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[1] I. L. Bekarevich and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 39, 1699 (1960) [English transl.: Soviet Phys.—JETP 12, 1187 (1961)].
[2] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. 22, 329 (1959).

It is easy to see that the zero-sound distribution can fulfill the boundary condition for diffuse reflection but under the somewhat more stringent condition that $s_0 \gg 1$. This is still quite reasonable, and we shall now proceed a step further and take it as granted that, whatever the character of the reflection may be, the vibrating wall couples only to the collective mode.

II. DERIVATION OF THE ACOUSTIC IMPEDANCE

We shall thus ignore the detailed mechanism of energy transfer at the boundary and simply ensure the conservation of the flow of energy into the liquid. On the one hand, one knows the amount of energy flowing into the liquid per unit time, derived from the work done by the wall against the pressure of the liquid. On the other hand, there is, in the liquid, a definite flow of free energy associated with the quasiparticle distribution corresponding to the collective mode. Equating these two expressions provides us with a scale factor for the distribution function.

As we have seen in the preceding section, the accuracy of this treatment, when applied to zero sound, depends on the actual value of s_0. In the hydrodynamic regime it is, of course, exact, and as an illustration we shall apply this method simultaneously to first sound and zero sound.

In the hydrodynamic regime $\omega\tau \ll 1$, the collision term is chosen of the form

$$I(\nu) = -(\nu - \nu_0 - 3\nu_1 \cos\theta)/\tau \qquad (7)$$

in order to satisfy the basic conservation laws in collisions between quasiparticles. The first sound distribution function is then obtained from Eq. (2) assuming a plane-wave behavior:

$$(\cos\theta - \xi)\nu + \cos\theta(F_0\nu_0 + 3F_1\nu_1\cos\theta) = (1/\sigma)(\nu_0 + 3\nu_1\cos\theta), \qquad (8)$$

where

$$\xi = \frac{i\omega\tau - 1}{i\tau k v_0} \quad \text{and } \sigma = i\tau k v_0; \quad s_1 = \frac{\omega}{kv_0} = \frac{1+\xi\sigma}{\sigma}.$$

The density of momentum flux Π_{ik} is defined by the equation

$$\frac{\partial}{\partial t}\int p_i n d\tau + \frac{\partial \Pi_{ik}}{\partial x_k} = 0.$$

Π_{ik} is directly obtained using the linearized expression of the kinetic equation and taking into account the conservation of momentum in collisions between quasiparticles. We shall be interested in the component of the pressure directed along Oz. It is easy to show that for both distributions (5) and (8) it can be cast into the simple form

$$\Pi_{zz} = -3ns\nu_1, \qquad (9)$$

where n is the number of particles per unit volume. The free energy associated with the quasiparticle distribution is defined by

$$\delta F = \sum_p (\epsilon_p{}^0 - \mu)\delta n(p) + \tfrac{1}{2}\sum_{p,p'} f(p,p')\delta n(p)\delta n(p').$$

The first term is an abbreviated way of writing

$$\frac{2}{(2\pi\hbar)^3}\int_{p_f}^{p_f+\delta n(p)} (\epsilon_p{}^0 - \mu)\mathbf{dp} = \frac{3n}{2m^* v_0{}^2}\int \nu^2 \frac{d\Omega}{4\pi}$$

and the second term has the value

$$(3n/2m^* v_0{}^2)(F_0\nu_0{}^2 + 3F_1\nu_1{}^2).$$

In the application of our results to the problem of the thermal boundary resistance (Sec. III), we shall use the value obtained by Khalatnikov[3] for the motion of the boundary. To be consistent with his quantization of elastic waves, we take the energy transferred into the liquid per unit time as given by $\Pi_{zz}u_z$ (where u_z is the maximum velocity of the wall), and the flow of energy carried by the sound wave as $c\delta F$, where $c = sv_0$. Equating these two expressions provides the required scale factor.

The formal expressions for the free energy will be different for the two cases considered here. In the case of first sound, for which one takes the limit $\sigma \to 0$, $\xi \to \infty$, $\sigma\xi \to -1$, the distribution can be replaced by its first two moments:

$$\nu = \nu_0 + 3\nu_1\cos\theta.$$

The free energy can then be written

$$\delta F = (3n/2m^* v_0{}^2)[(1+F_0)\nu_0{}^2 + 3(1+F_1)\nu_1{}^2]. \qquad (10)$$

Equating the two expressions for the flow of energy leads to the equation

$$-3ns_1\nu_1 u_z = (3n/2m^* v_0{}^2)s_1 v_0 \times[(1+F_0)\nu_0{}^2 + 3(1+F_1)\nu_1{}^2]. \qquad (11)$$

The two moments ν_0 and ν_1 are related through the law of conservation of mass which can be written for both distributions (5) and (8):

$$s\nu_0 = (1+F_1)\nu_1. \qquad (12)$$

Using Eq. (12) and the value of the velocity of sound

$$s_1{}^2 = \tfrac{1}{3}(1+F_0)(1+F_1), \qquad (13)$$

we immediately obtain the following value for the flow of energy:

$$Q = \Pi_{zz}u_z = ns_1 p_0 u_z{}^2/(1+F_1). \qquad (14)$$

Recalling the relations $p_0 = m^* v_0$ and $m^*/m = 1 + F_1$, one can cast Eq. (14) into the usual form

$$Q = \rho c_1 u_z{}^2. \qquad (15)$$

In the case of zero sound, the calculation is a little more tedious. The first term contributing to the free energy involves

$$\int \nu^2 \frac{d\Omega}{4\pi} = \frac{(F_0\nu_0)^2 + (3F_1\nu_1)^2 + 6F_0\nu_0F_1\nu_1 s_0}{s_0{}^2 - 1} - 2(F_0\nu_0{}^2 + 6F_1\nu_1{}^2).$$

[3] I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 22, 687 (1952).

Equating the two expressions for the energy flow leads to

$$-3ns_0v_1u_z=\frac{3n}{2m^*v_0^2}s_0v_0v_1^2\left\{\frac{([F_0(1+F_1)/s_0]+3F_1s_0)^2}{s_0^2-1}-\left[\frac{F_0(1+F_1)^2}{s_0^2}+9F_1(1+F_1)\right]\right\}, \quad (16)$$

where we have made use of Eq. (12). It is worth noticing that the coefficient in braces is directly proportional to the residue at the pole obtained by Bekarevich and Khalatnikov[1] in the case of the specular reflection [the first term of their Eq. (3.15)]. It is easy to show that the two results are identical, which demonstrates the accuracy of our present derivation.

Expression (16) can be considerably simplified by making use of the equation determining s_0:

$$W(s_0)=\frac{s_0}{2}\ln\frac{s_0+1}{s_0-1}-1=\left(F_0+\frac{3F_1}{1+F_1}s_0^2\right)^{-1} \quad (17)$$

which is obtained from Eq. (5) using Eq. (12). Upon eliminating F_0 from Eq. (16) by using Eq. (17) and expanding $W(s_0)$ in powers of s_0^{-1}, one finally obtains[4]

$$Q=\alpha\rho c_0u_z^2, \quad (18)$$

where

$$\alpha=1-\frac{12}{7\times25}\frac{1+F_1}{s_0^4}+0(1/s_0^6).$$

At low pressure, the correction to $\alpha=1$ is $\sim2\times10^{-3}$, at high pressure $\sim3\times10^{-5}$, and is therefore negligible. We thus see that the acoustic impedance of He³ in the collisionless regime is $Z=\rho c_0$.

The most immediate application of this result is, of course, to the experiment performed by Keen, Matthews, and Wilks.[5] There is a slight discrepancy between their direct determination of $Z/\rho\sim200$ *m/s* and the value of c_0 computed from F_0 and F_1, $c_0=191$ *m/s*, which we believe comes from the lack of accuracy in the determination of F_0 and F_1.

Note added in proof. Dr. G. A. Brooker suggests that the calculated value of C_0 should be corrected by taking account of the coefficient F_2, here neglected. (Abstract C.N. 10 of the IXth International Conference on Low Temperature Physics, Columbus 1964, and private communication.)

III. APPLICATION TO THE DETERMINATION OF THE THERMAL BOUNDARY RESISTANCE

We now turn to the application of the preceding results to the determination of the thermal boundary resistance between liquid He³ and a solid body. Khalatnikov[3] has developed a theory for this effect, taking into account the emission and absorption of thermal phonons at the boundary. His result expresses the thermal boundary resistance R in terms of the acoustic impedance Z:

$$R=\frac{\rho_s}{Z}\frac{15}{16\pi^5}\frac{(2\pi\hbar c_T)^3}{k(kT)^3}\frac{1}{F}, \quad (19)$$

where F is a function of the elastic constants of the solid and ρ_s and c_T are, respectively, the density and the velocity of the transverse vibrations of the solid. This formula, as it is written, can be applied to the whole scale of temperature.

Using our previous results, we thus see that the thermal boundary resistance should *decrease* slightly as one goes from the hydrodynamic to the collisionless behavior of He³. This result is in contradiction with that of Bekarevich and Khalatnikov,[1] but the discrepancy appears to be mainly a question of a factor 2. They use the definition $Q=\frac{1}{2}\Pi_{zz}u_z$ which, we think, is not consistent with Khalatnikov's definitions.[3]

TABLE II. Thermal boundary resistance between Cu and liquid He³ below 0.1°K.

	RT^3 (10^{-5} cgs units) Exp.[a]	Theor.
$P=0.12$ atm	2	11
$P=28$ atm	1.25	3.8

[a] See Ref. 6.

We shall only consider the collisionless region below 0.1°K, where a T^{-3} behavior of the thermal boundary resistance has been observed.[6] Using the latest available data listed on Table I, we compute the thermal boundary resistance from Khalatnikov's expression. As in Ref. 6, we choose $F=1.6$. For the solid, we use the Debye temperature of Cu, $\Theta_D=340$°K. The results thus obtained are listed on Table II together with the experimental values from Ref. 6. The agreement is still rather poor, but one has the feeling that it could be improved by taking into account the interaction of the conduction electrons with the vibrating boundary.[7,8]

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[4] The zero sound contribution a' to the coefficient a of Bekarevich and Khalatnikov in their specular reflection calculation is related to α by the following equation: $a'=s_0\alpha/3(1+F_1)$.

[5] B. E. Keen, P. W. Matthews, and J. Wilks, Phys. Letters **5**, 5 (1963).

[6] A. C. Anderson, J. I. Connolly, and J. C. Wheatley, Phys. Rev. **135** A910 (1964).

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