Quantum Approach to Amplification of Optical Phonons in Semiconductors

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It is shown by using a quantum-mechanical treatment that, in semiconductors with some ionic character, optical phonons can be amplified via their interaction with conduction electrons. The amplification of the optical phonons occurs when the drift velocity of the conduction electrons exceeds the phase velocity of the optical phonons. Both longitudinally and transversely polarized optical phonon modes are amplified via the electron-phonon interaction when the quantum treatment given is valid. The amplification of the optical phonon modes can be viewed as a phonon maser process, with the external drift field inverting the electron population. When the drift velocity of the electrons exceeds the optical-phonon phase velocity, the number of electrons that will emit optical phonons exceeds the number that will absorb optical phonons. The result is a net gain for those optical phonon modes whose phase velocity $s=\omega/q$ satisfies the condition $V_d>S$.

I. INTRODUCTION

THE observation of amplification of acoustic mode lattice vibrations through their interaction with conduction electrons[1–3] has lead to some speculation on whether the same mechanism could be used to amplify optical mode lattice vibrations.[4,5] Recent experiments on the current-voltage characteristics of GaAs and InP[6,7] at high drift fields have been interpreted on the basis of this mechanism. The existing theory developed to calculate the amplification of the optical phonons is based on a phenomenological treatment.[4,5] However, the phenomenological approach is valid only when the phonon wavelength is longer than the electron mean free path, i.e., $ql<1$, and the phonon frequency is smaller than the electron collision frequency, i.e., $\omega\tau<1$. Since the optical phonon frequencies in most materials are in the infrared, $\omega\approx10^{12}$–10^{13} sec^{-1}, and the electron relaxation time is about 10^{-12} sec, the condition for the phenomenological treatment to be valid will often be violated. Also, for short wavelength optical phonons, the conditions for the validity of the phenomenological treatment will always be violated.

In this paper, we give a quantum mechanical treatment of the interaction of the optical phonons with a gas of conduction electrons. The procedure used is that of the self-consistent field method described by Ehrenreich and Cohen[8] and applied by Takimoto[9] to the calculation of the absorption of acoustic phonons in the quantum limit. This treatment is valid when $\omega\tau>1$ and $ql>1$, and also when $\omega\tau<1$ and $ql>1$. In Sec. II, we calculate the conductivity tensor for an electron gas interacting with the optical phonons in the presence of a strong drift field. The dispersion relations for the optical phonon modes interacting with free electrons is derived in Sec. III, and the amplification coefficient is related to the components of the conductivity tensor. The amplification coefficient is calculated for both longitudinal and transverse polarized optical phonons in Sec. IV, and in Sec. V we give a discussion of the results obtained.

II. DERIVATION OF THE CONDUCTIVITY TENSOR

We treat the conduction electrons as a free electron gas of density N_0. The optical mode lattice vibration of frequency ω and wave number $\mathbf{q}$ manifests itself by means of a self-consistent field with scalar and vector potentials $\phi_1(\mathbf{r},t)$, $\mathbf{A}_1(\mathbf{r},t)\alpha\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$. The effect of the external drift fields is taken into account by using a drifted distribution function,[10] $f_0(\mathbf{k}-m\mathbf{V}d/\hbar)$, as the equilibrium electron distribution. Here, $\mathbf{V}_d$ is the drift velocity induced by an external electric field and $\mathbf{k}$ is the wave vector of the electron.

The electron-current density induced by the self-consistent field is obtained by taking the trace of the current-density operator and the single particle density matrix. The density matrix operator ρ must satisfy the equation of motion

$$i\hbar(\partial\rho/\partial t)=[\mathcal{H},\rho], \tag{2.1}$$

where $\mathcal{H}$ is the Hamiltonian of the system. We can separate the Hamiltonian into two parts,

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1, \tag{2.2}$$

where

$$\mathcal{H}_0=p^2/2m \tag{2.3}$$

is the free-electron Hamiltonian and,

$$\mathcal{H}_1=-\frac{e}{2c}(\mathbf{v}\cdot\mathbf{A}_1+\mathbf{A}_1\cdot\mathbf{v})+e\phi_1 \tag{2.4}$$

is the part of the Hamiltonian that contains the electron-phonon interaction to first order via the self-consistent fields.

[1] A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961).
[2] A. M. Toxen and S. Tansal, Phys. Rev. Letters **10**, 481 (1963).
[3] J. H. McFee, J. Appl. Phys. **34**, 1548 (1963).
[4] J. B. Gunn, Phys. Letters **4**, 194 (1963).
[5] T. O. Woodruff, Phys. Rev. **132**, 679 (1963).
[6] J. B. Gunn, Sol. State Comm. **1**, 88 (1963) and private communication.
[7] H. Ikoma, I. Kuru, and K. Hataya, J. Phys. Soc. Japan **19**, 141 (1964).
[8] H. Ehrenreich and M. H. Cohen, Phys. Rev. **115**, 786 (1959).
[9] N. Takimoto, Progr. Theoret. Phys. (Kyoto) **25**, 327 (1961).
[10] B. V. Paranjape, Phys. Letters **5**, 32 (1963); K. Nakamura, Progr. Theoret. Phys. (Kyoto) **30**, 919 (1963).

The free-electron Hamiltonian has wave functions and energy eigenvalues

$$\psi(\mathbf{r})=\exp i\mathbf{k}\cdot\mathbf{r}, \quad E_k=(\hbar k)^2/2m. \tag{2.5}$$

The state of the electron is specified by the quantum numbers k_x, k_y, and k_z.

We can now solve (2.1) to first order in H_1 by expanding the operator ρ, $\rho=\rho_0+\rho_1$. Here ρ_0 is the value of ρ in the absence of the interaction, but in the presence of the drift fields. The equation of motion becomes

$$i\hbar(\partial\rho_1/\partial t)=[\mathcal{H}_0,\rho_1]+[\mathcal{H}_1,\rho_0], \tag{2.6}$$

to first order in the self-consistent fields. Taking the matrix elements of (2.6) in the representation (2.5), we find

$$\langle k'|\rho_1|k\rangle=\lim_{\delta\to 0+}\frac{(f_k-f_{k'})\langle k'|\mathcal{H}_1|k\rangle}{E_k-E_{k'}+\hbar\omega+i\hbar\delta}, \tag{2.7}$$

where f_k is the drifted distribution evaluated at wave number k.

The current density induced by the self-consistent fields is

$$\mathbf{J}_1(\mathbf{r},t)=\mathrm{Tr}[\mathbf{J}_{0p}\rho], \tag{2.8a}$$

and

$$\mathbf{J}_{0p}=-\tfrac{1}{2}e[(\mathbf{v}-e\mathbf{A}_1/mc),\delta(\mathbf{r}-\mathbf{x})]_+. \tag{2.8b}$$

In (2.8b), $[\ ]_+$ denotes the anticommutator. We can use (2.8a, b) together with (2.7) to obtain a relation between the induced current and the self-consistent electromagnetic field,

$$\mathbf{J}_1=\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}, \tag{2.9}$$

where

$$\boldsymbol{\sigma}=\frac{\omega_p^2}{4\pi i\omega}\left\{\mathbf{I}+\frac{m}{N}\sum_{kk'}\frac{(f_k-f_{k'})\langle k'|\mathbf{V}|k\rangle\langle k'|\mathbf{V}|k\rangle^*}{E_k-E_{k'}+\hbar\omega}\right\}. \tag{2.10}$$

The operator $\mathbf{V}$ in (2.10) is defined by

$$\mathbf{V}=\tfrac{1}{2}[\exp i\mathbf{q}\cdot\mathbf{r},\ \mathbf{v}]_+. \tag{2.11}$$

If we choose the direction of $\mathbf{q}$ to be the z axis of our coordinate system, then the matrix elements of $\mathbf{V}$ are

$$\langle k'|\mathbf{V}|k\rangle=\delta_{k',k+q}\frac{\hbar}{m}(\mathbf{k}+\tfrac{1}{2}\mathbf{q}). \tag{2.12}$$

The components of the conductivity tensor parallel and perpendicular to the direction of propagation are given by

$$\sigma_{zz}=-\frac{\omega_p^2\omega}{4\pi iq^2}\frac{m}{N_0\hbar}\sum_k\frac{(f_k-f_{k+q})}{(\hbar q/m)(k_z+\frac{1}{2}q)-\omega}, \tag{2.13a}$$

$$\sigma_\perp=\frac{\omega_p^2}{4\pi i\omega}\left\{1-\frac{m}{N_0\hbar}\sum_k\frac{(f_k-f_{k+q})(\hbar k_1/m)^2}{(\hbar q/m)(k_z+\frac{1}{2}q)-\omega}\right\}. \tag{2.13b}$$

The real and imaginary parts of $\boldsymbol{\sigma}$ can be evaluated by using the relation[11]

$$\lim_{\delta\to 0+}\frac{1}{z-i\delta}=P\left(\frac{1}{z}\right)-i\pi\delta(z), \tag{2.14}$$

where $P(1/z)$ indicates that in any integration, the principal part of the integral is taken. The actual evaluation of σ_{zz} and $\sigma_\perp$ is done in the appendices. The results of the calculation for a degenerate electron gas give

$$\sigma_{zz}=\frac{3\omega_p^2\omega}{8\pi(qV_F)^2}\left\{\frac{\pi\omega\mu}{qV_F}-\frac{ik_F}{q}[F(a_+)+F(a_-)]\right\}, \tag{2.15a}$$

$$F(a)=a+\tfrac{1}{2}(1-a^2)\ln\left|\frac{1+a}{1-a}\right|.$$

$$\sigma_\perp=\frac{3\omega_p^2}{16\pi\omega}\left\{\frac{\pi\omega\mu}{qV_F}\left[1-\left(\frac{q}{2k_F}\right)^2-\left(\frac{\omega\mu}{qV_F}\right)^2\right]-\frac{ik_F}{4q}\left[G(a_+)+G(a_-)-\frac{16}{3}\frac{q}{k_F}\right]\right\}, \tag{2.15b}$$

$$G(a)=2a\left(-a^2+\frac{5}{3}\right)+(1-a^2)\ln\left|\frac{1+a}{1-a}\right|,$$

where $a_\pm=q/2k_F\pm\omega\mu/qV_F$, and $\mu=1-\mathbf{q}\cdot\mathbf{V}_d/\omega$. In (2.15), k_F is the Fermi momentum and V_F is the Fermi velocity. The real parts of σ_{zz} and $\sigma_\perp$ are different from zero only for wave numbers q lying in the range $\omega\mu/V_F\lesssim q\lesssim 2k_F$. Elsewhere, the real part of the conductivity tensor is zero.

III. DISPERSION RELATIONS FOR THE OPTICAL PHONONS

In a diatomic crystal with some ionic character, the optical lattice vibrations are accompanied by polarization waves. These polarization modes are described by the theory of Born and Huang[12] for a diatomic crystal with optical isotropy. In their theory, they introduce the vector $\mathbf{w}$, which is the relative displacement of the negative ions with respect to the positive ions multiplied by the square root of the reduced mass of the two ions per unit volume. The relevant equations are Maxwell's equations and the equations involving $\mathbf{w}$,

$$\mathbf{D}=\boldsymbol{\mathcal{E}}+4\pi\mathbf{P}, \tag{3.1}$$

$$\nabla\times\boldsymbol{\mathcal{E}}=-(1/c)(\partial\mathbf{B}/\partial t), \tag{3.2}$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}[\mathbf{J}+\partial\mathbf{P}/\partial t]+\frac{1}{c}\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t}, \tag{3.3}$$

$$\partial^2\mathbf{w}/\partial t^2=b_{11}\mathbf{w}+b_{12}\boldsymbol{\mathcal{E}}, \tag{3.4}$$

$$\mathbf{P}=b_{21}\mathbf{w}+b_{22}\boldsymbol{\mathcal{E}}. \tag{3.5}$$

[11] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 473.

[12] M. Born and K. Huang, *Dynamical Theory of Crystal Lattices* (Oxford University Press, New York, 1954).

Here, **D** is the electric displacement, **P** is the polarization, and **J** is the electron-current density induced by the electromagnetic fields. The coefficients b_{ij} are related by Born and Huang to the infrared dispersion frequency ω_0, the static dielectric constant ϵ_0 and the high-frequency dielectric constant ϵ_∞ in the following way:

$$b_{11}=-\omega_0^2, \tag{3.6a}$$

$$b_{12}=b_{21}=[(\epsilon_0-\epsilon_\infty)/4\pi]^{1/2}\omega_0, \tag{3.6b}$$

$$b_{22}=(\epsilon_\infty-1)/4\pi. \tag{3.6c}$$

The Eqs. (3.4)–(3.5) are valid for wavelengths much greater than a lattice spacing. Using (3.1)–(3.6) together with (2.9), we obtain the following dispersion relations for the longitudinal and transverse polarized waves, respectively:

Longitudinal waves

$$\omega^2=\omega_l^2+(4\pi/i\omega)(\sigma_{zz}/\epsilon_\infty)\omega_l^2\kappa/1-(4\pi\sigma_{zz}/i\omega\epsilon_\infty); \tag{3.7}$$

Transverse waves

$$\omega^2=\omega_0^2-\frac{\omega_l^2\kappa}{(cq/\omega)^2\epsilon_\infty^{-1}-1+(4\pi\sigma_1/(i\omega\epsilon_\infty)}. \tag{3.8}$$

In (3.7)–(3.8), $\omega_l=(\epsilon_0/\epsilon_\infty)^{1/2}\omega_0$ is the longitudinal optical phonon frequency in the absence of the electron-phonon interaction, and $\kappa=1-\epsilon_\infty/\epsilon_0$ is the parameter which gives the strength of the electron-optical phonon interaction. Table I shows the value of κ for some typical III–V semiconductors. The absorption coefficient is obtained by taking the imaginary part of the frequency which one gets from the dispersion relations, $\omega=\omega_r-i\alpha$. Since κ is much less than unity for those ionic semiconductors which have very mobile electrons, we obtain the following for the absorption coefficient for small κ

$$\alpha_l=-\tfrac{1}{2}\omega_l\kappa\,\mathrm{Im}\frac{4\pi\sigma_{zz}/i\omega\epsilon_\infty}{1-(4\pi\sigma_{zz})/(i\omega\epsilon_\infty)}, \tag{3.9}$$

$$\alpha_t=\frac{\omega_l^2\kappa}{2\omega_0}\,\mathrm{Im}\frac{1}{(cq/\omega)^2\epsilon_\infty^{-1}-1+(4\pi\sigma_1)/(i\omega\epsilon_\infty)}. \tag{3.10}$$

The absorption coefficients for longitudinal and transverse optical phonons are evaluated in the next section.

TABLE I. Electron-optical phonon coupling constant, $\kappa=1-\epsilon_0/\epsilon_\infty$ in III–V compounds.[a]

InSb	0.123
InAs	0.191
InP	0.238
GaSb	0.079
GaAs	0.155
GaP	0.168
AlSb	0.149

[a] Values of ϵ_0 and ϵ_∞ taken from S. S. Mitra and R. Marshall (to be published).

IV. ABSORPTION COEFFICIENT

The absorption coefficient for longitudinal optical phonons can be evaluated by using (2.15a) in (3.9). The result is

$$\alpha_l=\frac{\pi\kappa\omega_l(q/q_d)^2(\omega\mu/qV_F)\epsilon_\infty}{\pi^2\left(\frac{\omega\mu}{qV_F}\right)^2+\left(\frac{k_F}{q}\right)^2\left[F(a_+)+F(a_-)+\left(\frac{2q}{k_F}\right)\left(\frac{q}{q_d}\right)^2\epsilon_\infty\right]^2}, \tag{4.1}$$

where $q_d=\sqrt{3}(\omega_p/V_F)$ is the Debye wave number. The absorption coefficient is positive, giving attenuation of the optical phonons when $\mu>0$. When $\mu<0$, the reverse is true and we get amplification of the optical phonons. Therefore, the condition for an optical phonon of frequency ω and wave number $\mathbf{q}$ to be amplified is that the component of electron drift velocity in the direction of propagation $\hat{q}\cdot\mathbf{V}_d$ exceed the phase velocity of the optical phonon $s=\omega/q$.

When $q/2k_F\ll1$ and $\omega\mu/qV_F\ll1$, (4.1) takes the simpler form

$$\alpha_l=\frac{(\pi/4)\kappa\omega_l(q/q_d)^2(\omega\mu/qV_F)\epsilon_\infty}{[1+\epsilon_\infty(q/q_d)^2]^2}. \tag{4.2}$$

In deriving the expression (4.2), we have used the following limiting form of $F(a)$ for small a:

$$F(a)\approx 2a, \quad a\ll1. \tag{4.3}$$

The amplification increases with wave number for q up to q_d and decreases with q thereafter. The wave-number dependence of α_l with q is shown in Fig. 1.

The absorption coefficient for transverse optical phonons is calculated by using (2.15b) in (3.10). We obtain the result

$$\alpha_t=\frac{\frac{3\pi}{8}\kappa\frac{\omega_l^2}{\omega_0}\left(\frac{\omega_p}{qc}\right)^2\left(\frac{\omega}{qc}\right)^2\epsilon_\infty\frac{\omega\mu}{qV_F}\left[1-\left(\frac{q}{2k_F}\right)^2-\left(\frac{\omega\mu}{qV_F}\right)^2\right]}{\left\{1+\frac{3}{16}\left(\frac{\omega_p}{qc}\right)^2\frac{k_F}{q}\left[\frac{16q}{3k_F}-G(a_+)-G(a_-)\right]-\left(\frac{\omega}{qc}\right)^2\epsilon_\infty\right\}^2+\frac{9\pi^2}{16}\left(\frac{\omega_p}{qc}\right)^4\left(\frac{\omega\mu}{qV_F}\right)^2\left[1-\left(\frac{q}{2k_F}\right)^2-\left(\frac{\omega\mu}{qV_F}\right)^2\right]^2}. \tag{4.4}$$

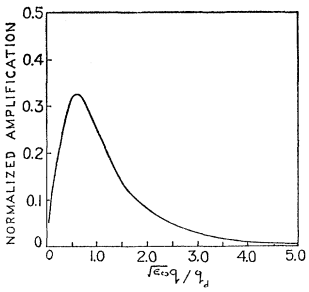

FIG. 1. The normalized amplification coefficient for longitudinal optical phonons, $(4\omega_p/\pi\kappa\omega_l\omega\mu\epsilon_\infty)\alpha_l$, is plotted as a function of the ratio of the optical phonon wavelength to the Debye wavelength.

In this case also, the optical phonons are absorbed when $\mu>0$, and amplified when $\mu<0$.

When $q/2k_F\ll 1$, $\omega/qc\ll 1$, and $\omega\mu/qV_F\ll 1$, we get the following for α_t:

$$\alpha_t=\frac{\frac{3\pi}{8}\kappa\frac{\omega_l^2}{\omega_0}\frac{\omega\mu}{qV_F}\left(\frac{\omega_p}{qc}\right)^2\left(\frac{\omega}{qc}\right)^2\epsilon_\infty}{\left[1+3\left(\frac{\omega_p}{qc}\right)^2\left(\frac{\omega\mu}{qV_F}\right)^2\right]^2+\frac{9\pi^2}{16}\left(\frac{\omega_p}{qc}\right)^4\left(\frac{\omega\mu}{qV_F}\right)^2}. \quad (4.5)$$

We have used the following limiting form of $G(a)$ in obtaining (4.5) from (4.4):

$$G(a)\approx 16a(1-a^2)/3, \quad a\ll 1. \quad (4.6)$$

The dependence of α_t on wave number falls into two regions according to whether $q\gtrless(\sqrt{|\mu|})q_0$, where $q_0=(q_d\omega/c)^{1/2}$. When $q<(\sqrt{|\mu|})q_0$, the amplification increases as q^3, while in the region $q>(\sqrt{|\mu|})q_0$, it decreases as q^{-5}. Figure 2 shows the dependence of α_t on wave number.

To get numerical estimates of α_l and α_t, we use the following values of the physical parameters, which are valid for GaAs: $\omega_l=5.4\times10^{13}$ sec^{-1}, $\omega_0=4.8\times10^{13}$ sec^{-1}, $\kappa=0.15$, $N_0=10^{16}$ cm^{-3}, and $\epsilon_\infty=11$. With these parameters, we get the values $\alpha_l=10^{10}$ sec^{-1} and $\alpha_t=10^6$ sec^{-1} when $q=10^5$ cm^{-1}, and $\alpha_l=10^{11}$ sec^{-1} and $\alpha_t=10$ sec^{-1} when $q=10^6$ cm^{-1}. From these estimates, it can be seen that the amplification coefficient for the optical phonons can be quite large. Also, the longitudinal optical phonons have larger amplification coefficients than the transverse optical phonons.

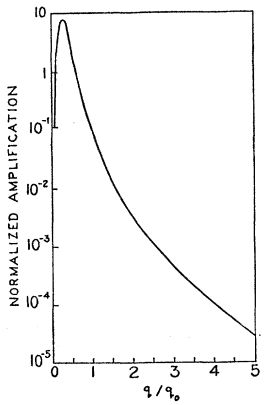

FIG. 2. The normalized amplification coefficient for transverse optical phonons, $(8/\pi)(c/V_F)\times(\omega_0/\omega_l^2\kappa\epsilon_\infty)\alpha_t$, is shown as a function of the ratio q/q_0. We have used the values $\omega=10^{13}$ sec^{-1}, $\omega_p=10^{13}$ sec^{-1}, $\mu=-0.1$, and $q_0=10^4$ cm^{-1}.

V. DISCUSSION

The treatment given in this paper of the interaction of optical-lattice vibrations with conduction electrons indicates that amplification can occur for all modes with q lying in the range $\omega/V_d<q<2k_F$. In practice, the lower limit on the wave number which can be amplified is set by considerations of power dissipation. Since the frequency is fairly independent of wave number for small q, it follows that V_d and therefore the electric fields, must be increased to amplify optical phonons of lower q. Power dissipation considerations would limit the drift velocities to not more than about 10^8 cm/sec. Therefore, the lower limit on the wave number of the optical phonons amplified is about $q=10^5$ cm^{-1}. The upper limit to q is set by considerations of momentum conservation. An electron can only absorb or emit a phonon whose maximum momentum is twice that of the electron. Since the maximum momentum of the electrons is k_F, phonons with $q>2k_F$ do not interact directly with electrons.

The amplification of the optical lattice vibrations via the electron-phonon interaction can be viewed as a phonon maser process,[13,14] at least as far as the quantum treatment is concerned. As can be seen from (A1)–(A2) and the discussion immediately following in Appendix A, the externally induced drift field tends to invert the electron population. When V_d exceeds the phase velocity $s=\omega/q$ of an optical phonon mode, the electrons are able to emit more phonons of this mode than they absorb. The result is a net gain for those optical phonon modes satisfying the condition $V_d>s$. For the modes not satisfying this condition, the electrons absorb more phonons and the net result is a loss for these modes.

The quantum treatment gives a slightly different picture of the amplification of optical lattice modes than the phenomenological treatment. In the latter treatment, only the longitudinal modes are amplified, while our treatment predicts the amplification of both longitudinal and transverse modes. This is because in the phenomenological treatment, the interaction is between the lattice waves and space-charge waves induced by longitudinal bunching of electrons. Since the space-charge waves are longitudinal, the phonon modes which are amplified will also be longitudinal. An exception to the latter statement occurs when transverse lattice modes are coupled by some mechanism to the space charge waves. This is the case with acoustic lattice modes when there is piezoelectric[15] or deformation potential coupling[16] to the electrons. It would also be the case for optical phonons if there exists an optical-deformation potential. In the phonon-maser picture,

[13] A. B. Pippard, Phil. Mag. 8, 161 (1963).
[14] H. N. Spector, Phil. Mag. 9, 1057 (1964).
[15] D. White, J. Appl. Phys. 33, 2547 (1962).
[16] H. N. Spector, Phys. Rev. 127, 1084 (1962).

however, the emission of both longitudinal and transverse optical phonons can occur if the electrons, whose population is being inverted, couple to these phonons. The coupling mechanism for both longitudinal and transverse polarization is the electromagnetic fields induced by the optical-lattice vibrations in crystals with ionic character. As a result, both longitudinal and transverse optical phonons can be amplified as long as their frequencies and wave numbers are such that the quantum treatment is applicable, i.e., $ql>1$.

The amplification of the longitudinal optical phonons is greater than that of the transverse optical phonons. This is because the electromagnetic fields generated by the transverse optical modes of interest are much weaker than those generated by the longitudinal modes. Those transverse optical modes which generate strong electromagnetic fields have wave numbers too small to be amplified directly by this mechanism at reasonable drift fields. The longitudinal mode of wave number $q\approx q_d$ has the maximum interaction with the electrons. For $q\ll q_d$, the electromagnetic fields generated by the optical lattice vibrations are screened by those generated by the electronic current. When $q\gg q_d$, the screening breaks down and the amplification coefficient decreases rapidly with wave number. The transverse mode which has the maximum interaction with the electrons is that with $q\approx q_0$. For wave numbers less than q_0, the transverse polarization fields are screened by the fields arising from the electronic currents, while for wave numbers greater than q_0, the screening breaks down. The interaction is a maximum at these wave numbers, i.e., $q\approx q_d$ for longitudinal modes and $q\approx q_0$ for transverse modes, because the electrons are interacting collectively with the optical phonons. For larger wave numbers, the interaction is between individual electrons and phonons and is weaker.

The electrons have been treated as forming a degenerate Fermi gas in the preceding calculations. However, considering the electrons to form a Maxwell-Boltzmann gas of particles would not alter our main conclusions. The arguments following (A1)–(A2) remain valid as the form of the distribution function is not specified. All that is required is that, in the absence of the drift field, the distribution be such that states of lower energy have a higher probability of being occupied than states of high energy. This condition is satisfied both for Fermi and for Maxwell-Boltzmann particles. The main change is that there is no longer an upper limit on the momentum of the phonon that can be emitted or absorbed. However, the amplification coefficient falls off exponentially for phonon wave numbers $q>2k_T$, where $k_T=(2mk_BT/\hbar^2)^{1/2}$, k_B is Boltzmann's constant and T is the absolute temperature. This effectively sets an upper limit to the wave number of the phonons which interact with the electrons. The only other changes, besides some numerical factors of order unity, that the use of Maxwell-Boltzmann statistics makes in our results for the absorption coefficient, (4.2) and (4.5) is the replacement of the Fermi velocity v_F by the mean thermal velocity $v_T=(2k_BT/m)^{1/2}$. The expressions for the components of the conductivity tensor using Maxwell-Boltzmann statistics are derived in Appendix B.

The use of degenerate statistics is valid for highly doped semiconductors at low temperatures. As an example, for GaAs, which has been doped to 10^{16} cm^{-3}, the use of degenerate statistics is valid at temperatures below 27°K. However, for semiconductors at temperatures near room temperature, one would have to use Maxwell-Boltzmann statistics. At room temperature, the thermal velocity of the electrons in GaAs is $v_T=3\times10^7$ cm/sec. Since this is about the same as the Fermi velocity of GaAs doped to 10^{16} cm^{-3} at temperatures below the degeneracy temperature, the values of the absorption coefficient would be about the same as the numerical estimates given at the end of Sec. IV.

In addition to the effect of the electron-phonon interaction on the amplification coefficient, (3.7)–(3.8) give the effect of the interaction on the optical-phonon dispersion relations. For $q\ll q_d$, the longitudinal optical-phonon frequency is changed from ω_l to ω_0. This occurs because the electrons screen out the electromagnetic fields so that the ions see only the short range forces which are lumped together in the coefficient b_{11}.

APPENDIX A

In this appendix, we evaluate the σ_{zz} and $\sigma_\perp$ components of the conductivity tensor. The sign of the absorption coefficient can be seen to depend on the sign of the real part of the conductivity tensor. Therefore, we will calculate the real and imaginary parts of σ_{zz} and $\sigma_\perp$ separately.

We first change from a summation to an integration over k. Performing the integration over k_z, and using (2.14), we have

$$\mathrm{Re}\sigma_{zz}=\frac{3\omega_p{}^2\omega}{8(qV_F)^3}\frac{\hbar}{m}\int dk_\perp k_\perp[f_0(\epsilon_k)-f_0(\epsilon_k+\hbar\omega\mu)], \quad \text{(A1)}$$

$$\mathrm{Re}\sigma_\perp=\frac{3\omega_p{}^2}{16\omega qV_F{}^3}\left(\frac{\hbar}{m}\right)^3\int dk_\perp k_\perp{}^3[f_0(\epsilon_k)-f_0(\epsilon_k+\hbar\omega\mu)], \quad \text{(A2)}$$

where $\epsilon_K=(\hbar K)^2/2m+(\hbar k_\perp)^2/2m$, and $K=-\frac{1}{2}q+m\omega\mu/\hbar q$. In obtaining (A1) and (A2), our variable of integration was changed from $\mathbf{k}$ to $\mathbf{k}-m\mathbf{V}_d/\hbar$. From (A1) and (A2), the real part of the conductivity tensor is positive when $\mu>0$, and becomes negative when $\mu<0$. The first term in the brackets inside the integral gives the number of electrons of that energy which can absorb a phonon of frequency ω and wave number $\mathbf{q}$ while the second term gives the number that can emit this phonon. When

$\mu<0$, the number of electrons that can emit phonons exceeds the number that can absorb phonons.

For a degenerate Fermi gas, we get from (A1) and (A2)

$$\mathrm{Re}\sigma_{zz}=\frac{3\omega_p^2\omega k_F V_F}{16(qV_F)^3}\begin{cases}0 & a_+, \quad a_->1\\ 1-a_-^2 & a_+>1,\ a_-<1\\ -(1-a_+^2) & a_+<1,\ a_->1\\ a_+^2-a_-^2 & a_+, \quad a_-<1,\end{cases} \tag{A3}$$

$$\mathrm{Re}\sigma_\perp=\frac{3\omega_p^2}{64\omega}\frac{k_F}{q}\begin{cases}0 & a_+, \quad a_->1\\ (1-a_-^2)^2 & a_+>1,\ a_-<1\\ -(1-a_+^2)^2 & a_+<1,\ a_->1\\ (1-a_-^2)^2-(1-a_+^2)^2 & a_+, \quad a_-<1.\end{cases} \tag{A4}$$

For the imaginary parts of σ_{zz} and $\sigma_\perp$, we obtain for a degenerate Fermi gas

$$\mathrm{Im}\sigma_{zz}=\frac{3\omega_p^2\omega}{8(qV_F)^3}\frac{\hbar}{m}\int_0^{k_F}dk_\perp k_\perp\left[\ln\left|\frac{K_0+\frac{1}{2}q-(m\omega\mu/\hbar q)}{K_0-\frac{1}{2}q-(m\omega\mu/\hbar q)}\right|+\ln\left|\frac{K_0+\frac{1}{2}q+(m\omega\mu/\hbar q)}{K_0-\frac{1}{2}q-(m\omega\mu/\hbar q)}\right|\right], \tag{A5}$$

$$\mathrm{Im}\sigma_\perp=\frac{\omega_p^2}{4\omega}\left\{1-3/(4qk_F^3)\int_0^{k_F}dk_\perp k_\perp^3\times\left[\ln\left|\frac{K_0+\frac{1}{2}q-(m\omega\mu/\hbar q)}{K_0-\frac{1}{2}q+(m\omega\mu/\hbar q)}\right|+\ln\left|\frac{K_0+\frac{1}{2}q+(m\omega\mu/\hbar q)}{K_0-\frac{1}{2}q-(m\omega\mu/\hbar q)}\right|\right]\right\}, \tag{A6}$$

where $K_0=[k_F^2-k_\perp^2]^{1/2}$. The integrals in (A5)–(A6) can easily be evaluated by changing the variable of integration from $k_\perp$ to K_0. The results (2.15) are then obtained.

APPENDIX B

In this appendix, we evaluate the σ_{zz} and $\sigma_\perp$ components of the conductivity tensor for Maxwell-Boltzmann statistics. As in Appendix A, we calculate the real and imaginary parts of σ_{zz} and $\sigma_\perp$ separately.

To evaluate the real part of the conductivity tensors, we use the distribution function for Maxwell-Boltzmann statistics

$$f_0(\mathbf{k})=\left(\frac{\hbar^2}{2\pi mk_BT}\right)^{3/2}N_0\exp-\frac{(\hbar k)^2}{2mk_BT} \tag{B1}$$

in (A1) and (A2). Performing the integration over k_z, we obtain

$$\mathrm{Re}\sigma_{zz}=\frac{\pi\omega_p^2\omega}{4q^3}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{k_BT}{\hbar}\exp-\frac{(\hbar K)^2}{2mk_BT}\times\left(1-\exp-\frac{\hbar\omega\mu}{k_BT}\right), \tag{B2}$$

$$\mathrm{Re}\sigma_\perp=\frac{\omega_p^2}{4\omega q}\left(\frac{m}{2\pi k_BT}\right)^{1/2}\frac{k_BT}{\hbar}\exp-\frac{(\hbar K)^2}{2mk_BT}\times\left(1-\exp-\frac{\hbar\omega\mu}{k_BT}\right). \tag{B3}$$

From (B2) and (B3), we see that the general conclusions concerning the sign of the real part of the conductivity as a function of the sign of μ hold for classical statistics as well as for degenerate statistics.

For the imaginary parts of σ_{zz} and $\sigma_\perp$, we obtain for a classical electron gas

$$\mathrm{Im}\sigma_{zz}=\frac{\omega_p^2\omega}{2q^3}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{k_BT}{\hbar}\left\{F\left(\frac{m\omega\mu}{\hbar k_Tq}-\frac{q}{2k_T}\right)-F\left(\frac{m\omega\mu}{\hbar k_Tq}+\frac{q}{2k_T}\right)\right\}, \tag{B4}$$

$$\mathrm{Im}\sigma_\perp=\frac{\omega_p^2}{4\pi\omega}\left\{1-\frac{2\pi^2}{q}\left(\frac{m}{2\pi k_BT}\right)^{1/2}\frac{k_BT}{\hbar}\left[F\left(\frac{m\omega\mu}{\hbar k_Tq}-\frac{q}{2k_T}\right)-F\left(\frac{m\omega\mu}{\hbar k_Tq}+\frac{q}{2k_T}\right)\right]\right\}, \tag{B5}$$

where[17]

$$F(x)=2(\sqrt{\pi})e^{-x^2}\int_x^\infty dte^{t^2}. \tag{B6}$$

When the conditions $q/2k_T\ll1$, $\omega\mu/qv_T\ll1$, and $\hbar\omega\mu\ll k_BT$ hold, then using (B2)–(B5) for the conductivity tensor gives the same result as (4.2) and (4.5) except for a numerical factor of order unity with the thermal velocity v_T replacing the Fermi velocity v_F. When the first two conditions above are not satisfied, then the absorption coefficient goes exponentially to zero as $\exp-(K/k_T)^2$. Therefore, the basic form of the absorption coefficient is the same for both degenerate and classical electron gases.

Note added in proof. After this manuscript had been submitted for publication, we discovered a paper by I. Yokota [Phys. Rev. Letters **10**, 27 (1964)] in which he obtained results similar to ours for the longitudinal optical phonons.

[17] H. N. Spector, Phys. Rev. **125**, 1880 (1962).