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# Experimental Determination of the Fourth Sound Velocity in Helium II†

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Fourth sound is a pressure and density wave in He II, in which only the superfluid is in motion. Using packed rouge to lock the normal fluid, the existence of this wave mode has been experimentally confirmed by measuring the temperature dependence of a plane-wave resonance in a closed-closed cylindrical acoustic resonator. The measured phase velocity has a temperature dependence which agrees within 1% with that predicted theoretically. Pulse measurements using 40-kc/sec sine-wave bursts further corroborate the existence of this type of wave. The absolute value of the fourth sound velocity is affected by the coherent multiple scattering of the wave from the superleak material. The results of Twersky's and Saxon's scattering theories are compared with the data. It is found that the equation $n=(2-P)^{1/2}$, where $n$ is the index of refraction and $P$ is the porosity, fits the data within 3%. Values of the ratio of the superfluid to the He II density, derived from the fourth-sound data, agree within 2% with those obtained by other experimental techniques.

## INTRODUCTION

THE thermohydrodynamics of He II were first developed by Tisza[1] and Landau[2] independently. Neglecting irreversible effects, these authors predicted the existence of two wave modes in the unbounded medium. In He II $\gamma-1\ll1$ where $\gamma$ is the ratio of the specific heats; it is a valid approximation for many purposes to set $\gamma=1$. In this limit the density and entropy waves uncouple. The first mode—first sound—is a density (and pressure) wave with no entropy or temperature fluctuation; the normal fluid and superfluid move in phase. The second mode—second sound—is an entropy (and temperature) wave with no pressure or density fluctuations; the normal fluid and superfluid oscillate 180° out of phase with each other. These predictions have been supported by numerous experiments.[3–7]

Pellam[8] extended the analysis of wave propagation in He II to include the case in which the liquid filled a porous stationary matrix of solid material. The principal function of this matrix is to inhibit motion of the normal fluid through viscous drag. When this viscous drag is negligible, then, of course, one obtains the usual first and second sound with second-order damping. As the viscous drag increases (due to a decreased pore size) the velocities of the two modes change and attenuation of the waves increases. However, in the limit that the normal fluid becomes locked, an interesting situation occurs. The wave mode which was originally first sound suffers *no* attenuation. This mode has been called fourth sound by Atkins.[9] It has a phase velocity $C_4$ given by[10]

$$C_4=\{(\rho_s/\rho_0)C_1^2+(\rho_n/\rho_0)C_2^2[1-(2\beta_pC_1^2/\gamma s_0)]\}^{1/2}, \quad (1)$$

where $\rho_s$ is the superfluid density, $\rho_n$ is the normal fluid density, $\rho_0=\rho_s+\rho_n$, $C_1$ is the first-sound velocity, $C_2$ is the second-sound velocity, $\beta_p$ is the isobaric coefficient of expansion, $s_0$ is the ambient entropy per unit mass (see Appendix A). There is motion only of the superfluid and there are oscillations of density, pressure, temperature and entropy per unit mass. The wave mode which was originally second sound becomes, in the limit that the normal fluid is completely immobilized, a diffusion wave with a vanishingly small phase velocity.

The superfluid can be regarded as that component of He II in the ground state and in this sense fourth sound can be regarded as a wave motion in the ground state, the normal component (i.e., the excitations) maintaining its equilibrium density.

Figure 1 shows the temperature dependence of the velocity of first, second, and fourth sound. The values for fourth sound were calculated from Eq. (1). The values of $\rho_s/\rho_0$ were obtained from Andronikashvilli,[11] and the values of $C_1$ and $C_2$ are from Atkins.[12]

The grains of the superleak can be considered as incompressible, immobile scattering centers, and this will cause, as Pellam has pointed out,[8] a reduction in phase velocity. The grains are in all cases very small compared to the wavelength so that the long-wavelength limit of theoretical treatments is applicable. An important complication is the fact that the density of scatterers is sufficiently high that multiple scattering effects become important. Twersky[13,14] gives the following low-frequency results for identical scatterers, randomly dis-

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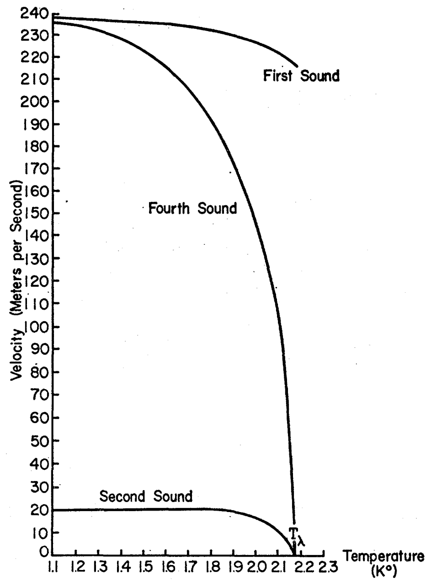



FIG. 1. Velocity of the wave modes in He II versus the absloute temperature. Fourth sound is the wave mode in which only the superfluid is in motion causing oscillations of density, pressure, temperature and entropy per unit mass. The curves for first and second sound represent the accepted experimental values. The fourth sound curve was calculated from Eq. (1).

tributed in space ($n$ is ratio of the phase velocity in the absence of the scatterers to that in their presence, i.e., the index of refraction; and $P$ is the fraction of volume occupied by fluid, i.e., the porosity):

Spheres:

$$n=\{(3-P)/2\}^{1/2}. \tag{2}$$

Needles (i.e., spheroids with eccentricity approximately equal to one) whose axes are perpendicular to the direction of sound propagation but arbitrarily oriented otherwise:

$$n=\{2-P\}^{1/2}. \tag{3}$$

These results do not hold for arbitrarily high density of scatterers.

Saxon[15] has investigated the scattering from a regular lattice of scatterers and has obtained results which are not seriously limited in this respect. For incompressible immobile spheres, in the long-wavelength limit his result is

$$n=\{1-g\}^{-1/2}, \tag{4}$$

where $g$ is a function of the relative geometry of the spheres and lattice, and depends on the direction of sound propagation, but is independent of the size of the spheres. For the cubic lattices, $g$ is also independent of the direction of sound propagation (see Appendix B).

It is worthy of note that Eqs. (2)–(4) show no dispersion.

[15] D. Saxon (private communications).

## EXPERIMENTAL TECHNIQUES

The experiment was performed as follows: solid dielectric transducers were placed at both ends of a standing wave tube, which was packed with superleak material. The composite package formed a closed-closed resonator. One transducer was electrically driven to produce pressure fluctuations in the liquid helium in the superleak, while the other was used as a receiver. The frequency of a plane-wave resonance was then determined as a function of temperature. The velocity of fourth sound was calculated from the observed frequency using the distance between the transducers and the harmonic number.

The experimental apparatus will be described under three headings: (A) Cryogenics, (B) Electronics, and (C) Acoustics.

(A) *Cryogenics.* The glass double Dewar system was of the standard type.[16] The temperature was lowered by pumping away the helium vapor with a vacuum pump which had sufficient pumping speed so that with the existing heat leaks it was possible to reach 1.1°K. The temperature was determined by measuring the vapor pressure with a Wallace and Tiernan FA 160 Gauge.

(B) *Electronics.* The electronics are shown in a block diagram in Fig. 2. The equipment consisted of a beat-frequency audio oscillator which was used to drive the source transducer. The input and output voltages were measured; the frequency was determined with a frequency counter. The output of the receiving transducer was fed into a cathode follower, amplified by 43 dB with an amplifier and a setup transformer, filtered and then displayed on one channel of a dual beam oscilloscope. The second channel of the oscilloscope was used to display the driving signal. The source transducer generated an acoustic signal whose frequency was twice that of the

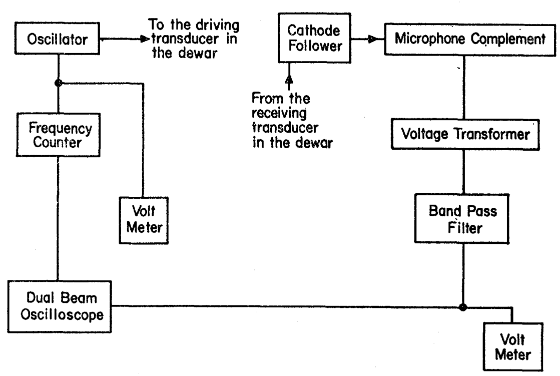
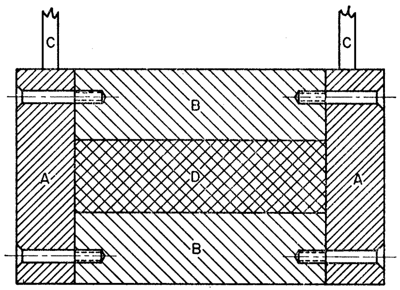



FIG. 2. Block diagram of the electronics. The driving transducer was always unpolarized and therefore the acoustic signal had twice the oscillator frequency. The input and received signals were compared on a dual beam oscilloscope and it was possible to distinguish the acoustic signal from the electromagnetic pickup by invoking the frequency doubling criterion.

[16] G. K. White, *Experimental Techniques in Low Temperature Physics* (Oxford University Press, Oxford, England, 1959), Chaps. 5, 4.




A. Transducers
B. Resonator Body
C. Support Tubes
D. Space For Superleak Material

FIG. 3. Schematic of the acoustic resonator. The normal fluid was immobilized in the tiny channels of the superleak, while the superfluid was free to move. Since observations were made only on the first and second harmonics of the resonator, the velocity of fourth sound $C_4$, was given by $C_4=2fl/q$ where $f$ is the frequency, $l$ is the distance between the transducers and $q$ is either one or two. $Q$'s up to 85 were obtainable with this setup.

driving voltage. By comparing the driving and received signals on the oscilloscope, it was possible to distinguish between acoustic signal and electromagnetic pickup since only the acoustic signal had double the drive frequency.

An Ohmite carbon resistor (27 Ω at room temperature) which was calibrated against the Wallace and Tiernan FA 160 manometer, served as an auxiliary thermometer. It was placed in close proximity to the resonator. A standard three-wire connection was used to reduce thermoelectric emf's and to couple the resistor to a Wheatstone bridge. A galvanometer, with microvolt sensitivity, was employed as a null detector for the bridge.

(C) *Acoustics.* The acoustic resonator consisted of a hollow copper cylinder with a transducer at each end, as shown in Fig. 3.

The body of the resonator had sufficient wall thickness to insure that the walls did not yield appreciably under the acoustic pressure swings, and hence did not affect the velocity of wave propagation.[17]

It is well known[18] that the resonant frequencies of a rigid cylindrical enclosure with plane rigid ends are given by

$$f=\tfrac{1}{2}C\{(q/l)^2+(\alpha_{mn}/a)^2\}^{1/2}, \qquad (5)$$

where $f$ is the resonant frequency, $C$ is the velocity of wave propagation, $l$ is the distance between the ends, $a$ is the radius of the cylinder, $q$ is zero or a positive integer, and $\alpha_{mn}$ is a root of $d[J_m(\pi\alpha)]/d\alpha=0$. $J_m(\pi\alpha)$ is a Bessel function of the first kind. The $\alpha_{mn}$ arise by requiring the radial velocity of the medium to be zero at the surface of the cylindrical enclosure. The plane wave modes (i.e., the harmonics), characterized by $q\neq 0$, $\alpha_{mn}=\alpha_{00}=0$, will be exclusively excited only when the driving frequency is smaller than that of the first nonsymmetric (i.e., $\alpha_{mn}=\alpha_{10}=0.5861$) and the first radial (i.e., $\alpha_{mn}=\alpha_{01}=1.2197$) modes. For the resonator used in this experiment ($l=3.32$ cm and $a=1.00$ cm) there are three harmonics below the first nonsymmetric mode and eight harmonics below the first radial mode. Since observations were made only on the first and second harmonics of the resonator, the velocity of wave propagation is given by

$$C=2fl/q, \qquad (6)$$

where $q$ is equal to either one or two.

The interior of the cylinder was always filled with some sort of porous medium whose purpose was to immoblize the normal component of the He II.

The transducers, which bolted directly to the body of the resonator, were mechanically identical and of the solid-dielectric-condenser type.[19] Figure 4 shows a cross section of one of the transducers. The copper housing A was machined from a single block of copper. Six equally spaced countersunk clearance holes were placed through the housing so that they would mate with the tapped holes in the body of the resonator. The nylon insert B insulated the back plate C from the copper housing. The back plate formed one side of the condenser. The insert and the back plate were "press-fit" in place and the front surface was machined so as to be flush with the front of the housing. The transducer was completed by laying a piece of prestretched, $\frac{1}{4}$-mil-thick Mylar,[20] which was gold-coated on one side, over the body of the resonator and bolting the transducer in place. The gold-coated side of the Mylar, which completed the condenser, was in contact with the body of the resonator, and therefore electrically grounded.

The tapped hole in the top of the transducer housing was the receptacle for the brass connect D, and also was

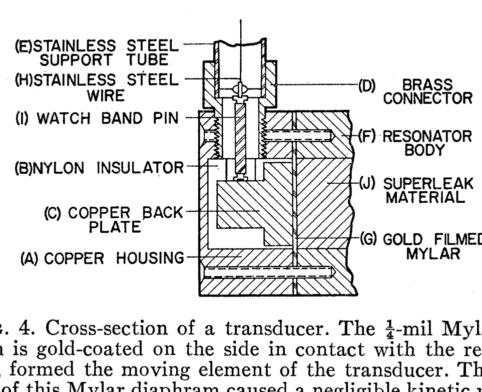



FIG. 4. Cross-section of a transducer. The $\frac{1}{4}$-mil Mylar film, which is gold-coated on the side in contact with the resonator body, formed the moving element of the transducer. The small mass of this Mylar diaphram caused a negligible kinetic reaction on the transducer body and consequently mechanical cross-talk was no problem.

[17] W. P. Mason *Electromechanical Transducers and Wave Filters* (D. Van Nostrand Company, Inc., New York, 1948), Sec. 4.22.

[18] P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., New York, 1948), 2nd ed., p. 397.

[19] W. Kuhl, G. R. Schodder, and F. K. Schroder, Acustica 4, 5, 519 (1954).

[20] Metalized Films; Hastings & Company, Inc., 2314 Market Street, Philadelphia, Pennsylvania.

the opening for the electrical connection to the back plate.

The brass connector was soldered to the outside of the stainless steel tube E. The tubes served as a support for the resonator, and also as the outer conductor of a coaxial line. The inner conductor, which was a stretched, 3 mil diameter, stainless steel wire, was soldered to feed throughs at the top and bottom of the tube. Electrical contact between the feed-through in the bottom of the tube and the back plate was made via the watch-band pin I.

The dead capacity of the tube-transducer assembly was small enough so that the acoustic signal was reduced by less than 2.5 dB.

The resistance to ground of the coaxial line used with the pickup transducer was of the order of $10^{10}\,\Omega$. This value of the leakage resistance maintained the polarizing voltage necessary for proper operation of this transducer.

An essential feature of the transducers is the very small mass of the diaphragm. This results in a very small kinetic reaction on the transducer body and mechanical cross talk is consequently no problem. Evidence for this is provided by the absence of spurious signals of significant level and the fact that when the temperature is allowed to drift above the lambda point the residual signals are of the order of 60 dB below those found through most of the temperature range below the lambda point.

With ordinary fluids, radiation of sound out of the resonator through motion in the annuli between the transducers and the tubular body of the resonator is prevented by (1) viscous drag, and (2) the mass reactance of the fluid in this annulus. Only the latter mechanism is operative in the He II at the acoustic levels used in this experiment. This mass reactance was made large by keeping the annulus thin (a good fit between the transducers A and the resonators) and long (thick resonator body). Evidence that these measures were effective is provided by the fact that the $Q$ of the resonant mode was 85 at 1.15°K and 25 at 1.74°K. In the absence of the superleak the tube had the predicted closed-closed tube resonances.

## PRESENTATION OF THE RESULTS

The viscous wavelength for the normal fluid is given by $\lambda_n = 2\pi\{\eta_n/(\rho_n\omega)\}^{1/2}$, where $\eta_n$ is the normal fluid viscosity, $\rho_n$ is the density of the normal fluid, and $f=\omega/2\pi$ is the frequency. $\lambda_n = 301 f^{-1/2}\,\mu$ at the lambda point, and is larger at lower temperatures. For frequencies less than 10 kc/sec the normal fluid viscous wavelength will be larger than 3.01 $\mu$, and therefore the pore sizes in the superleak should have a maximum value which does not exceed this.

Data obtained with a Polypore superleak (average pore size $0.45\pm0.02\,\mu$) have already been reported.[10] However, no multiple scattering correction was applied at that time. An empirical correction of 5.5% results in a 0.5% agreement between theory and experiment at lowest temperatures, but at 2.0°K the experimental points are 10% above the theoretical prediction. This discrepancy is probably due to the incomplete locking of the normal fluid in the larger pores of the superleak. The multiple scattering corrections are shown in Table I.

A new superleak was made using "green rouge" ($Cr_2O_3$) whose particles have a diameter of approximately 0.5 $\mu$ as determined from sedimentation techniques.[21] It was pounded into a hollow brass capsule. This capsule effectively filled the inside of the resonator. Its outside diameter was a mil smaller than the inside diameter of the resonator body. The ends of the capsule were terminated by thin brass plates perforated with the maximum possible number of 0.033-in. diameter holes. The walls and one end of the capsule were machined out of a solid piece of brass while the second end had its periphery threaded and screwed into the walls of the capsule.

TABLE I. The correction to the fourth sound propagation velocity due to the multiple scattering of the wave from the particles of the superleak. The theoretical results of Twersky (Refs. 13 and 14) and Saxon (Ref. 15) are included for comparison with the empirical correction. Equation (7) agrees with the empirical correction to within 3% over the entire range of porosities used.

| | | | | | | | Saxon | | |
|---|---|---|---|---|---|---|---|---|---|
| Data | Average particle size | Porosity | Empirical correction | Equation (7) | Twersky Spheres | Twersky Needles | Hexagonal close packed $z$ direction | Face centered cubic | Simple cubic |
| Ref. 10 | 0.45 $\mu$ | 85% | 5.5% | 7.5% | 3.75% | 7.5% | 0.50% | 1.5% | 5.5% |
| Green rouge, Fig. 5 | 0.5 $\mu$ | 54.6% | 23% | 20.5% | 11% | 20.5% | 3.2% | 2.6% | 11% |
| Green rouge, Fig. 6 | 0.5 $\mu$ | 49.6% | 25% | 22.2% | 11.8% | 22.2% | 3.8% | 2.5% | 13% |
| Pulse $Al_2O_3$ | 0.05 $\mu$ | 80.8% | 12% | 9.2% | 4.7% | 9.2% | | | |
| $Al_2O_3$ | 0.05 $\mu$ | 93.8% | 1.0% | 3.0% | 1.5% | 3.0% | | | |
| Green rouge | 0.5 $\mu$ | 80.0% | 11.5% | 9.5% | 4.9% | 9.5% | | | |
| Ferrero and Sacerdote (Ref. 27) | 3.7 mm | 39.0% | 26% | 27% | 14.0% | 27% | | | |
| | 1.95 mm and 2.55 mm | 38.5% | 28% | 27% | 14.0% | 27% | | | |

[21] P. R. Day, Soil Sci. Soc. Am. Proc. **20**, 167 (1956).

The capsule was weighed before and after packing and its packing density was determined to be 2.36 g/cm³; the porosity was 54.6%. The length of the actual superleak was 3.23 cm while the distance between the transducers was 3.41 cm. This left two 0.09-cm open spaces which were composed of the holes in the brass end plates and the unfilled space between the ends of the capsule and the fronts of the transducers.

The initial measurements showed that the normal fluid was not completely immobilized in the resonator because the clearance between the capsule and the resonator body was sufficiently large to allow the normal fluid component to move and therefore conduct first sound. To rectify this situation the capsule's surface was greased with vaseline and put into the resonator body. The open spaces near the transducers were filled

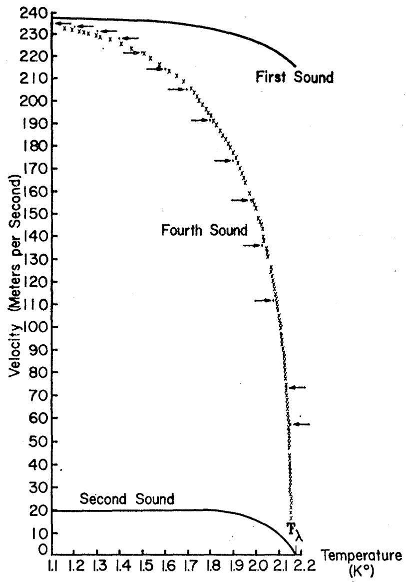



FIG. 5. The velocity of fourth sound. The $x$'s are the experimental points for the second harmonic which varied from 5402 cps at 1.16°K to 374 cps at 2.15°K. The arrows are precise points taken from Fig. 1. The superleak was made of 0.5 $\mu$ average diameter green rouge particles and had a porosity of 54.6%.

with wafers of Gelman AM-10 Polypore[22] filter material having 0.05-$\mu$ pores. Finally, in order to more effectively seal the ends of the resonator, the transducers were seated on 0.3-mm lead washers.

The resulting measurements are shown in Fig. 5. The $x$'s are the experimental points for the 2nd harmonic which varied from 5402 cps at 1.16°K to 374 cps at 2.15°K. The arrows are precise points taken from Fig. 1. It was necessary to apply an end correction and a multiple scattering correction to these data. The end correction was made by multiplying the length of the end plates of the brass capsule and the length of the Polypore wafers by the ratio of their porosity to that of the rouge.

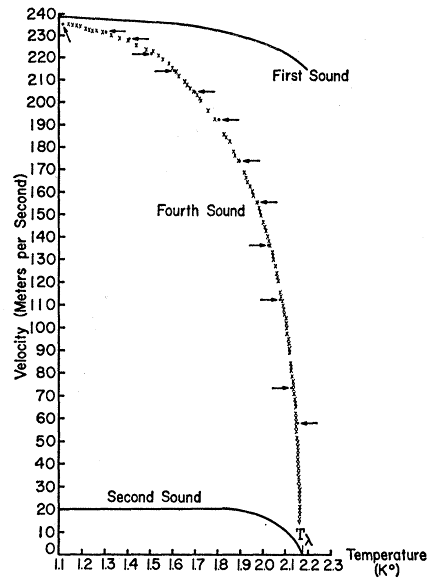



FIG. 6. The velocity of fourth sound. The $x$'s are the experimental points for the first harmonic which varied from 2748 at 1.14°K to 186 at 2.16°K. The arrows are precise points taken from Fig. 1. The superleak was made of 0.5 $\mu$ average diameter green rouge particles and had a porosity of 49.6%.

This yielded an effective length of 3.52 cm between the transducers. An empirical multiple scattering correction of 23.0% results in a fit to better than 1.0% at all temperatures. Table I summarizes the scattering corrections.

In order to eliminate the necessity of an end correction, the resonator was completely filled with green rouge. A porosity of 49.6% obtained. The results for this superleak are shown in Fig. 6. This mode was the fundamental and varied from 2748 cps at 1.14°K to 186 cps at 2.16°K. The distance between the transducers was 3.43 cm. It was once again necessary to use a one-parameter fit in order to empirically correct for multiple scattering. The empirical scattering correction for this porosity was 25%. Once again Table I summarizes the multiple scattering corrections.

In Eq. (1) the dominant term in the curly brackets is the first term, $(\rho_s/\rho_0)C_1^2$. The remaining terms account for 3% of $C_4$ at 2.16°K, and decrease progressively as the temperature decreases being 1.7% at 2.0°K. It becomes zero at 1.1°K because of the reversal in sign of $\beta_p$ at 1.17°K. It is thus clear that to the extent that $C_4$ and $C_1$ can be accurately determined such data offer a procedure alternative to the pendulum measurements of Andronikashvilli,[11] or second sound,[23] for obtaining accurate values of $\rho_s/\rho_0$. The experiments reported here have a high degree of internal consistency and accordingly values of $\rho_s/\rho_0$ were calculated from these data and are presented in Figs. 7 and 8. The arrows mark the

[22] Gelman Polypore Filter Material, Gelman Instrument Company, 106 North Main Street, Chelsea, Michigan (1963).

[23] D. DeKlerk, R. P. Hudson, and J. R. Pellam, Phys. Rev. 93, 28 (1954).

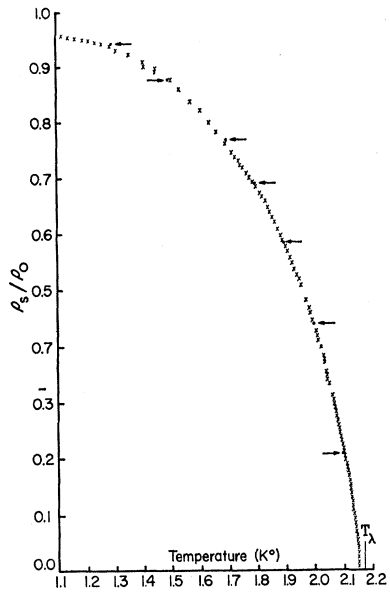



FIG. 7. The ratio of the superfluid to the total liquid density $\rho_s/\rho_0$, versus the absolute temperature. The $x$'s are values calculated from the fourth sound data in Fig. 5. The arrows mark the values derived from Andronikashvilli's pendulum measurements, and are seen to be in substantial agreement with the present data.

data of Andronikashvilli[11] and are seen to be in substantial agreement with the present data.

## PULSE MEASUREMENTS

The fourth sound propagation velocity has also been measured using 40 kc/sec sine wave bursts. Two Clevite PZT-4 ceramic transducers, in the form of cylinders 1-in. diameter $\frac{1}{2}$-in. thick, were mounted vertically and coaxially with the opposing plane surfaces 4.20 cm apart, as described previously.[24] When a transducer was excited the lowest frequency in its natural decay was 76 kc/sec and the time constant was sufficiently long so as to interfere with the measurements. Accordingly 40-kc/sec signals were used with an appropriate filter in the receiver transducer circuit to eliminate the higher frequencies.

Superleak material (0.05 $\mu$ Linde $Al_2O_3$ 0.05B polishing compound)[25] was pressed, using 1000 pounds per square inch, into a hollow brass cylinder whose inside diameter was 10 mil larger than the outside diameter of the PZT-4 ceramic. The resultant porosity was 80.0%. This superleak rested on the flat face of the bottom transducer; the top transducer pressed lightly on the other side of the superleak holding it in place. A large hollow brass cylinder which enclosed the entire assembly, was filled with green rouge ($Cr_2O_4$). The rouge filled the remaining open spaces around the transducers, and was sufficiently loosely packed so that the normal fluid was only partially locked; hence, any acoustic radiation outside the tightly packed superleak cartridge would be attenuated because of viscous damping.

The gating for the sine-wave bursts was synchronized with the 40-kc/sec signal to avoid jitter in the observed signal. Figure 9 is a photograph of an oscilloscope trace showing six well-defined and evenly spaced pulses of fourth sound at $T=1.2$°K. Near the seventh and eighth pulses spurious signals begin to occur but fairly accurate measurements can be made on the earlier arrivals.

Measurements made in this way are in substantial agreement with the results obtained using the resonator but are not reported here because they were not made with comparable accuracy. The necessary scattering correction was found to be 12% and is listed in Table I.

The use of (1) fine particles (0.05 $\mu$ $Al_2O_3$) and (2) packing under high pressure (1000 lbs per sq in.) were found necessary in order to completely lock the normal fluid at all temperatures when working at 40 kc/sec. Earlier measurements made under circumstances in which these two criteria were not adequately met showed significant departures from the fourth sound velocity at higher temperatures. However, the measurements have value in that at lower temperatures one can obtain multiple scattering corrections and these are shown in Table I for $Al_2O_3$ with a porosity of 93.8% and green rouge with a porosity of 80%.

## FURTHER DISCUSSION OF SCATTERING

An interesting byproduct of the investigation of fourth sound has been the data obtained on multiple

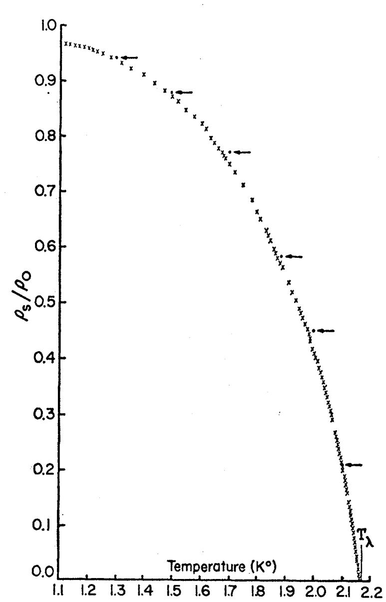



FIG. 8. The ratio of the superfluid to the total liquid density $\rho_s/\rho_0$, versus the absolute temperature. The $x$'s are values calculated from the fourth sound data in Fig. 6. The arrows mark the values derived from Andronikashvilli's pendulum measurements, and are seen to be in substantial agreement with the present data.

[24] R. D. Finch, R. Kagiwada, M. Barmatz, and I. Rudnick, Phys. Rev. **134**, A1425 (1964). See Fig. 2.

[25] Linde, Production Polishing Semiconductors, Linde Company, Division of Union Carbide Corporation, Crystal Products Department, 4120 Kennedy Avenue, E. Chicago, Indiana (1961).

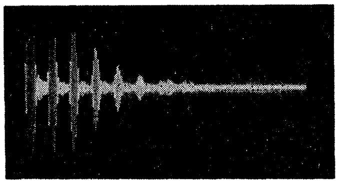


FIG. 9. Photograph of an oscilloscope trace showing 40 kc/sec fourth sound pulses at approximately 1.2°K. These signals were propagated through a superleak of 0.05 $\mu$ average diameter $Al_2O_3$ which had a porosity of 80.0%. Near the seventh and eighth pulse spurious signals appear, but measurements made on the earlier arrivals yield values of the fourth sound velocity in agreement with the resonator data.

scattering. Since viscous and heat conductive effects in ordinary fluids change the velocity of wave propagation through capillary porous media[26] the effect of multiple scattering on the velocity is not always apparent. Measurements of the fourth-sound velocity come very close to the situation in which viscous and thermal effects can be neglected.

The empirical scattering corrections listed in Table I can be fit to 3%, by the expression

$$n=\{2-P\}^{1/2}, \tag{7}$$

where $n$ is the index of refraction and $P$ is the porosity. Although this formula is identical with Twersky's[14] result for needles [cf. Eq. (3)] this agreement appears to be fortuitous since the particles of the superleak are not in general needles, and if they were, it would be difficult to explain why they should all have their major axis perpendicular to the direction of the original unscattered wave.[27]

## CONCLUSION

The existence of fourth sound and the temperature dependence of the velocity of propagation of fourth sound are firmly established by the work described in the preceding sections.

The problem of obtaining accurate absolute values of the velocity was complicated by the multiple scattering of the wave from the particles of the superleak. The results of Twersky's[13,14] and Saxon's[15] scattering theories are compared with the data and summarized in Table I. Equation (7) is an analytic expression which yields the empirical scattering corrections to within 3% for all the cases discussed.

The values of $\rho_s/\rho_0$ derived from the corrected values of the experimentally determined fourth-sound velocity agree within 2% with those of other investigators.

[26] L. E. Kinsler and A. R. Frey, *Fundamentals of Acoustics* (John Wiley & Sons, Inc., New York, 1950), Chap. 9.

[27] M. A. Ferrero and G. G. Sacerdote, Acustica **1**, 137 (1951). Equation (7) also accounts for the lowering of the velocity of propagation of sound waves in air, when the air is trapped in the open spaces in a container filled with a random arrangement of small lead balls. The experimental velocity agrees to within 3%, with Eq. (7), if an order of magnitude correction for viscous and thermal effects is also applied.

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## APPENDIX A: DERIVATION OF THE VELOCITY OF FOURTH SOUND

The linearized equations governing the reversible thermohydrodynamics of He II in unbounded space are written below in the Eulerian representation[28]

$$\partial \mathbf{v}_s/\partial t = -(\mathrm{grad}p)/\rho + s\,\mathrm{grad}T, \tag{A1}$$

$$\partial \mathbf{v}_n/\partial t = -(\mathrm{grad}p)/\rho - (\rho_s s\,\mathrm{grad}T)/\rho_n, \tag{A2}$$

$$\partial \rho/\partial t = -\rho_s\,\mathrm{div}\mathbf{v}_s - \rho_n \mathrm{div}\mathbf{v}_n, \tag{A3}$$

$$\partial(\rho s)/\partial t = -\rho s\,\mathrm{div}\mathbf{v}_n, \tag{A4}$$

$$\rho = \rho_s + \rho_n \tag{A5}$$

where $\rho$ is the density of He II, $\mathbf{v}_n$ is the normal fluid velocity vector, $\mathbf{v}_s$ is the superfluid velocity vector, $s$ is the entropy per unit mass of the He II, $\rho_s$ is the superfluid density, $\rho_n$ is the normal fluid density, $p$ is the total pressure in the He II, and $T$ is the temperature of the He II. The first equation is the force equation for the superfluid, the second is the force equation for the normal fluid, the third equation expresses conservation of mass and the fourth equation, conservation of entropy.

In order to write the equations for He II in a porous medium only Eq. (A2) has to be altered by adding $-R\mathbf{v}_n$ to the right-hand side of the equation, where $R$ is the flow resistance for the normal fluid[29] (multiple scattering effects are neglected). Hence the porous medium acts only on the normal fluid, by impeding its flow.

Now, introduce the coordinates

$$\dot{\mathbf{r}}_1 = (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)/\rho \tag{A6}$$

and

$$\dot{\mathbf{r}}_2 = \rho_s(\mathbf{v}_n - \mathbf{v}_s)/\rho, \tag{A7}$$

where $\mathbf{v}_n$ is the normal fluid velocity vector and $\mathbf{v}_s$ is the superfluid velocity vector. $\dot{\mathbf{r}}_1$ refers to "center-of-mass" velocity of the liquid and $\dot{\mathbf{r}}_2$ to a "relative motion" with no mass flow. These are the "normal coordinates" Tisza[30] used in his description of first and second sound.

[28] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. 2, p. 83.

[29] G. L. Pollack, California Institute of Technology thesis, p. 27, 1962 (unpublished).

[30] L. Tisza, Phys. Rev. **72**, 838 (1947).

Upon substitution of Eqs. (A6) and (A7) into Eqs. (1), (2), (3), and (4) [remembering to add $-R\mathbf{v}_n$ to Eq. (A2)] the linearized thermohydrodynamic equations become

$$\ddot{\mathbf{r}}_1-(\rho_n\ddot{\mathbf{r}}_2/\rho_s)=-\mathrm{grad}p/\rho+s\,\mathrm{grad}T\,, \tag{A8}$$

$$\ddot{\mathbf{r}}_1+\ddot{\mathbf{r}}_2=(-\mathrm{grad}p/\rho)-(\rho_s s\,\mathrm{grad}T/\rho_n)-R(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2)\,, \tag{A9}$$

$$\partial\rho/\partial t=-\rho\,\mathrm{div}\dot{\mathbf{r}}_1\,, \tag{A10}$$

$$\partial(\rho s)/\partial t=-\rho s\,\mathrm{div}(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2)\,. \tag{A11}$$

Eliminating $T$ from Eqs. (A8) and (A9), expanding $p$ in terms of $\rho$ and $s$, and using Eqs. (A10) and (A11) to express $\rho$ and $s$ as functions of $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ yields

$$\dot{\mathbf{q}}_1=C_1^2\{\nabla^2\dot{\mathbf{r}}_1+(T_0s_0\beta_p\nabla^2\dot{\mathbf{r}}_2)/C_p\}-\{\rho_nR(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2)/\rho_0\}\,, \tag{A12}$$

where $C_1$ is the first-sound velocity, $T_0$ is the ambient temperature, $s_0$ is the ambient entropy per unit mass, $\beta_p$ is the isobaric expansion coefficient, $C_p$ is the specific heat at constant pressure, $\rho_0$ is the ambient density, and $\dot{\mathbf{q}}_1=\partial\ddot{\mathbf{r}}_1/\partial t$.

Eliminating $p$ from Eqs. (A8) and (A9), expanding $T$ in terms of $\rho$ and $s$, and using Eqs. (A10) and (A11) to eliminate $\rho$ and $s$ gives

$$\dot{\mathbf{q}}_2=C_2^2[\nabla^2\dot{\mathbf{r}}_2+(C_1^2\beta_p\nabla^2\dot{\mathbf{r}}_1)/\gamma s_0]-[\rho_sR(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2)/\rho_0]\,, \tag{A13}$$

where $C_2$ is the second-sound velocity, $\gamma$ is the ratio of specific heats, and $\dot{\mathbf{q}}_2=\partial\ddot{\mathbf{r}}_2/\partial t$.

Seeking a plane-wave solution, put

$$\dot{\mathbf{r}}_1=\dot{r}_{10}\hat{e}_x\exp[j(\omega t-kx)]\,, \tag{A14}$$

and

$$\dot{\mathbf{r}}_2=\dot{r}_{20}\hat{e}_x\exp[j(\omega t-kx)]\,, \tag{A15}$$

where $\hat{e}_x$ is a unit vector in the $x$ direction, $\dot{r}_{10}$ and $\dot{r}_{20}$ are the amplitudes of $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$, $\omega$ is the angular frequency and $k$ is the wave number. Substituting Eqs. (A14) and (A15) into Eqs. (A12) and (A13) yields

$$\dot{r}_{20}=-\dot{r}_{10}\frac{(C_1^2C_2^2\beta_pk^2/\gamma s_0)+(R\rho_sj\omega/\rho_0)}{(R\rho_sj\omega/\rho_0)+C_2^2k^2-\omega^2}\,, \tag{A16}$$

$$\dot{r}_{20}=-\dot{r}_{10}\frac{(R\rho_nj\omega/\rho_0)+C_1^2k^2-\omega^2}{(C_1^2T_0s_0\beta_pk^2/C_p)+(R\rho_nj\omega/\rho_0)}\,. \tag{A17}$$

Equations (A16) and (A17) form a pair of linear homogeneous equations in $\dot{r}_{10}$ and $\dot{r}_{20}$. These equations have nontrivial solutions only if the determinant of the coefficients of $\dot{r}_{10}$ and $\dot{r}_{20}$ is identically zero. This condition gives the result written below:

$$C^4-\frac{C^2}{1-(jR/\omega)}\left[(C_1+C_2^2)-jR\frac{\rho_sC_1^2+\rho_nC_2^2}{\rho_0\omega}+2j\rho_nC_1^2C_2^2\frac{\beta_pR}{\rho_0\omega\gamma s_0}\right]+\frac{C_1^2C_2^2}{1-(jR/\omega)}\left(1-\frac{\beta_p^2T_0C_1^2}{\gamma C_p}\right)=0\,, \tag{A18}$$

where $C^2=\omega^2/k^2$.

The two roots of Eq. (A18) for $C^2$ are, in general, difficult to evaluate. However, two limiting cases are of interest. When $(R/\omega)\to 0$

$$C^2=\tfrac{1}{2}(C_1^2+C_2^2)\pm\tfrac{1}{2}[(C_1^2+C_2^2)^2-4C_1^2C_2^2(1-\epsilon)]^{1/2}\,, \tag{A19}$$

where $\epsilon=C_1^2\beta_pT_0/\gamma C_p$. When $\epsilon\ll 1$ this result leads to the following modes:

$$C_+^2=C_1^2+\epsilon[C_1^2C_2^2/(C_1^2-C_2^2)] \tag{A20}$$

and

$$C_-^2=C_2^2-\epsilon[C_1^2C_2^2/(C_1^2-C_2^2)]\,, \tag{A21}$$

where $C_+^2$ is the root corresponding to the $+$ sign in Eq. (A19) and $C_-^2$ is the root corresponding to the $-$ sign in Eq. (A19). The results were first derived by Tisza.[1] $\epsilon$ represents a coupling term which is equal to $1.5\times 10^{-2}$ a tenth millidegree below the lambda point and is smaller at lower temperatures. Hence $\epsilon$ can be neglected, and the modes reduce to first and second sound.

When $(R/\omega)\to\infty$

$$C^4-C^2C_4^2=0\,, \tag{A22}$$

where $C_4$ is the velocity of fourth sound,

$$C_4=\left[\frac{\rho_sC_1^2}{\rho_0}+\frac{\rho C_2^2}{\rho_0}\left(1-\frac{2\beta_pC_1^2}{\gamma s_0}\right)\right]^{1/2}. \tag{A23}$$

Hence

$$C_+^2=C_4^2 \tag{A24}$$

and

$$C_-^2=0\,. \tag{A25}$$

This solution also leads to two modes. The "+" mode is a wave motion which propagates with a velocity given by Eq. (A23). The "−" mode does not propagate. (This mode might be nicknamed "No Sound.")

It should be noted that the "+" mode, which is $C_1^2$ when $R/\omega\to 0$, goes over to $C_4^2$ when $R/\omega\to\infty$, and that the "−" mode, which is $C_2^2$ when $R/\omega\to 0$, does not propagate when $R/\omega\to\infty$.

Equation (A23) can be written as

$$C_4=\left(\frac{\rho_s C_1^2}{\rho_0}+\frac{\rho_n C_2^2}{\rho_0}\right)^{1/2}, \tag{A26}$$

since the difference in the value of Eqs. (A23) and (A26) is 1.5% at 2.16°K and becomes smaller as the temperature is lowered. Equation (A26) is the Pellam's formula for fourth sound.

### APPENDIX B: SAXON'S FORMULA FOR THE COHERENT MULTIPLE-SCATTERING CORRECTION TO THE PROPAGATION VELOCITY

The effect of coherent multiple scattering from a periodic arrangement of fixed, rigid spheres has been analyzed to a reasonable approximation by Saxon[15] using a Green's function technique. When the general formulation, which is valid for any allowable porosity, is simplified to the case of long wavelengths, the ratio of the propagation velocity in the fluid medium in the periodic lattice $C$, to the propagation velocity in free space $C_0$, is

$$C_0/C=(1-g)^{-1/2}. \tag{B1}$$

The symbol $g$ is given by

$$g=\left[\frac{3(1-P)}{2\pi N_0}\right]^2 \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{(\hat{e}_k\cdot \mathbf{x}_s)^2 j_1^2(2\pi|\mathbf{x}_s|)}{\mathbf{x}_s^4}, \tag{B2}$$

where $P$ is the porosity, $N_0$ is the number of spheres per primitive cell, $\hat{e}_k$ is a unit vector in the direction of propagation, and $j_1(2\pi|\mathbf{x}_s|)$ is the spherical Bessel function of the first kind and first order. The vector $\mathbf{x}_s$ is expressed as

$$\mathbf{x}_s=(s_1\boldsymbol{\alpha}_1 a+s_2\boldsymbol{\alpha}_2 a+s_3\boldsymbol{\alpha}_3 a)/K, \tag{B3}$$

where $s_1$, $s_2$, and $s_3$ are positive or negative integers not simultaneously zero, $a$ is the sphere radius, and $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\boldsymbol{\alpha}_3$ are the primitive translation vectors of the reciprocal lattice for the lattice in question. $2a$ has been chosen as the side of the conventional unit cube.[31] The $\boldsymbol{\alpha}$'s are expressed in terms of the sphere radius $a$ and the periodicity of the lattice. $K$ is a parameter which adjusts the volume of a primitive cell for the correct porosity and is given by

$$K=a\left[\frac{4\pi N_0(\boldsymbol{\alpha}_1\times\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_3)}{3(1-P)}\right]^{1/3}. \tag{B4}$$

Hence, $g$ depends only on the direction of propagation and on the relative geometry of the spheres and lattice.

$g$ has the form of a symmetric second-rank tensor expressed relative to the principle axes. It can be shown[32] that the principle components of such a tensor are equal in cubic lattices. Thus, the quantity $g$ is independent of the direction of sound propagation in such lattices.

Equation (B1) is identical to Eq. (5) of the Introduction.

It is of interest that Rayleigh[33] solved the problem of the effect on the propagation velocity of multiple scattering of a wave from a simple cubic lattice composed of rigid, fixed spheres. Rayleigh's result is

$$\frac{C_0^2}{C^2}=\frac{3-P}{2}\left[\frac{1-(0.394(1-P)^{10/3}/(3-P))}{1-(0.394(1-P)^{10/3}/2P)}\right], \tag{B5}$$

where $C_0$, $C$, and $P$ have been defined above. This equation is valid when powers of $(1-P)$ higher than 10/3 may be neglected. The quantity in the square bracket has a maximum value of 1.025. If powers of $(1-P)^{10/3}$ are neglected this equation is identical to Twersky's formula for rigid fixed spheres [cf.: Eq. (2) of the Introduction].

Although Rayleigh's result [i.e., Eq. (B5)] does not formally resemble Saxon's formula [cf.: Eq. (B1)] when evaluated for the simple cubic lattice, numerical analyses show they agree to better than 2% for all allowed porosities.

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[31] C. Kittel, *Introduction to Solid State Physcis* (John Wiley & Sons, Inc., New York, 1956), Chap. 12; beware of errors in the formulas for the primitive translation vectors in the crystal and in the reciprocal lattice.

[32] J. F. Nye, *Physical Properties of Crystals* (Oxford University Press, Oxford, 1952), Chap. 1.

[33] Lord Rayleigh, Phil. Mag. **34**, 481 (1892); *Collected Works* (Cambridge University Press, Cambridge, England), Vol. 4, p. 19.

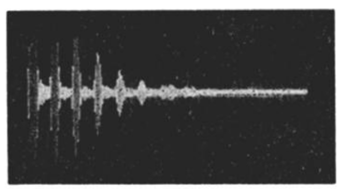

FIG. 9. Photograph of an oscilloscope trace showing 40 kc/sec fourth sound pulses at approximately 1.2°K. These signals were propagated through a superleak of 0.05 $\mu$ average diameter $Al_2O_3$ which had a porosity of 80.0%. Near the seventh and eighth pulse spurious signals appear, but measurements made on the earlier arrivals yield values of the fourth sound velocity in agreement with the resonator data.