Anomalous Electron Scattering in Dilute Magnetic Alloys

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The Kondo theory of the resistance minimum has been extended to show the following: (1) The resistance anomaly is not affected by nonmagnetic scattering. (2) The magnetoresistance shows a logarithmic field dependence at strong fields and low temperatures. (3) The anomalous scattering affects the superconducting properties of the alloy in nearly the same way as the normal scattering. So the theory based on normal scattering is still qualitatively correct.

I. INTRODUCTION

RECENTLY, Kondo[1] gave a very clever explanation of the resistance minimum phenomenon in many dilute alloys of magnetic metals. It has long been suspected that the resistance anomaly must be somehow connected with the spin-dependent scattering of the conduction electrons by the magnetic moment of the impurities.[2] However, many efforts to explain the phenomenon in terms of the *s*-*d* exchange interaction have been unsuccessful because it was thought that the temperature dependence of the scattering cross section was due to the partial ordering of the spins.[3–5] Kondo showed, instead, that the *s*-*d* scattering cross section as calculated from the first two orders of the Born approximation contains a temperature-dependent term like $\ln T$. This result gives a very satisfactory fit to the experimental data. The anomalous term arises as a consequence of the sharpness of the Fermi surface, and the argument of the logarithmic function is simply the thermal broadening of the Fermi level. Therefore, the resistance anomaly is a one impurity spin effect rather than a many-spin collective effect.

Historically, a phenomenological model for the resistance anomaly was first proposed by Korringa[2] who postulated that the electron-impurity scattering undergoes a resonance when the electron has nearly the Fermi energy. It has been speculated whether collision broadening of the electron energy levels may smear out the resonance and thus cause the resistance anomaly to disappear. This effect has not been noticed experimentally.[6] In this paper we carry out a completely renormalized version of the Kondo theory to include the scattering by nonmagnetic impurities. The result shows that the resistance anomaly is unaffected by collision broadening. In fact, the anomalous scattering depends only on the sharp variation in electron population at the Fermi energy but not on the sharpness of the Fermi surface.

The magnetoresistance of these alloys has been carefully studied theoretically by Yosida[7] and Dekker.[4] We have calculated the next order term, and the result shows that at low enough temperatures and high enough fields the anomalous scattering gives rise to a $\ln H$ term. This effect should be noticeable by a careful analysis of the experimental data.

It is known that dissolved magnetic impurities reduce drastically the superconducting critical temperature of the solvent metal.[8] The theory of this effect based on the *s*-*d* scattering is qualitatively successful.[9] Recently, Merriam *et al.*[10] observed that the depression of the critical temperature of indium by manganese impurities depends sensitively on the mean free path of the electrons. When an inert impurity such as lead or tin is added to reduce the mean free path, the effect of manganese tends to disappear. The authors interpreted this result on the bases of Korringa model with collision broadening. We have analyzed this problem using the Kondo idea and shown that the effect of the anomalous scattering on the critical temperature is independent of the mean free path. Hence, it appears that the phenomenon observed by Merriam *et al.* is not connected with the scattering mechanism. Furthermore, the anomalous scattering affects the superconductivity in nearly the same way as the normal scattering. Hence, all the conclusions in the Abrikosov and Gor'kov paper remain qualitatively valid.

II. RESISTANCE ANOMALY

We studied the resistance anomaly of a simple model described by the following Hamiltonian

$$H=H_0+H'+H'', \tag{2.1}$$

where H_0 is the free-electron term, H' the *s*-*d* interaction term, and H'' the other interaction such as phonon and

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[9] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].
[10] M. F. Merriam, S. H. Liu, and D. P. Seraphim, Phys. Rev. **136**, A17 (1964).

nonmagnetic impuritiy scatterings. Explicitly, in units with $\hbar=1$,

$$H_0=\sum_{\mathbf{k}s} \epsilon_k c_{\mathbf{k}s}{}^* c_{\mathbf{k}s}, \tag{2.2}$$

$$H'=-N^{-1}J\sum_j \sum_{\mathbf{k}\mathbf{k}'} [S_j{}^z(c_{\mathbf{k}'\uparrow}{}^*c_{\mathbf{k}\uparrow}-c_{\mathbf{k}'\downarrow}{}^*c_{\mathbf{k}\downarrow}) +S_j{}^+c_{\mathbf{k}'\downarrow}{}^*c_{\mathbf{k}\uparrow}+S_j{}^-c_{\mathbf{k}'\uparrow}{}^*c_{\mathbf{k}\downarrow}]e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}. \tag{2.3}$$

In these expressions $c_{\mathbf{k}s}{}^*$, $c_{\mathbf{k}s}$ are the creation and destruction operators for an electron in the momentum state $\mathbf{k}$ and spin state s; $\epsilon_k=k^2/2m$ is the energy of this state measured from the bottom of the band; $\mathbf{R}_j$ and $\mathbf{S}_j$ are the position and spin of the jth ion; $2J$ is the strength of the s-d interaction; and N is the total number of lattice sites in the sample. We shall use Greek letters κ, κ', etc. to denote both the momentum and the spin states of the electrons and write H' as

$$H'=\sum_{\kappa\kappa'} H'_{\kappa'\kappa}c_{\kappa'}{}^*c_\kappa. \tag{2.4}$$

It is clear that

$$H'_{\mathbf{k}'\uparrow\mathbf{k}\uparrow}=-N^{-1}J\sum_j S_j{}^z e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j},$$

etc. The concentration of magnetic impurities is assumed to be so low that their collective effect is always ignored.

We define the thermal Green function[11]

$$G_\kappa(\tau)=\langle Tc_\kappa(\tau)c_\kappa{}^*(0)\rangle, \tag{2.5}$$

where

$$c_\kappa(\tau)=e^{-\tau(\mu\mathfrak{N}-H)}c_\kappa e^{\tau(\mu\mathfrak{N}-H)}, \tag{2.6}$$

$$\mathfrak{N}=\sum_\kappa c_\kappa{}^*c_\kappa, \tag{2.7}$$

μ is the chemical potential or Fermi energy, and T is the ordering operator for τ such that

$$\begin{aligned} G_\kappa(\tau)&=\langle c_\kappa(\tau)c_\kappa{}^*(0)\rangle \qquad \tau>0\\ &=-\langle c_\kappa{}^*(0)c_\kappa(\tau)\rangle \quad \tau<0. \end{aligned}$$

The bracket denotes thermal average. This Green function can be expanded into a Fourier series

$$G_\kappa(\tau)=(1/\beta)\sum_n G_\kappa(\omega_n)e^{-i\omega_n\tau}, \tag{2.8}$$

where β is the inverse temperature in energy units, $\omega_n=(2n+1)\pi/\beta$, and n is an integer. In general $G_\kappa(\omega_n)$ has the form[12]

$$G_\kappa(\omega_n)=[\epsilon_k-\mu-\Sigma_\kappa(\omega_n)-i\omega_n]^{-1}, \tag{2.9}$$

where $\Sigma_\kappa(\omega_n)$ is the self energy due to interactions. In the present problem we may write

$$\Sigma_\kappa(\omega_n)=\Sigma_\kappa{}'(\omega_n)+\Sigma_\kappa{}''(\omega_n), \tag{2.10}$$

where $\Sigma_\kappa{}'(\omega_n)$ arises from H' and $\Sigma_\kappa{}''(\omega_n)$ from H''. The latter quantity is assumed to be known and the first quantity is to be calculated.

The diagrams of the first three orders of $\Sigma_\kappa{}'(\omega_n)$ are shown in Fig. 1. The lines denote electron Green functions, and the crosses represent the impurities. These diagrams are explicitly evaluated as follows:

$$\Sigma_\kappa{}'(\tau)=-\delta(\tau)\langle H_{\kappa\kappa}{}'\rangle+\sum_{\kappa'}\langle TH_{\kappa\kappa'}{}'(\tau)H_{\kappa'\kappa}{}'(0)\rangle G_{\kappa'}(\tau) -\sum_{\kappa'\kappa''}\int_0^\beta \langle TH_{\kappa\kappa''}{}'(\tau)H_{\kappa''\kappa'}{}'(\tau')H_{\kappa'\kappa}{}'(0)\rangle \times G_{\kappa''}(\tau-\tau')G_{\kappa'}(\tau')d\tau'. \tag{2.11}$$

The quantity $\langle TH'_{\kappa\kappa'}(\tau)H'_{\kappa'\kappa}(0)\rangle$ and the third-order product are Green functions for the impurity spins. For $\kappa=\mathbf{k}\uparrow$ and $\kappa'=\mathbf{k}'\downarrow$ the second-order product is

$$(N^{-1}J)^2\sum_j\langle TS_j{}^-(\tau)S_j{}^+(0)\rangle =N^{-1}cJ^2\langle TS^-(\tau)S^+(0)\rangle, \tag{2.12}$$

where c is the concentration of magnetic impurities and

$$\begin{aligned}\langle TS^-(\tau)S^+(0)\rangle&=\langle S^-S^+\rangle \quad \tau>0\\ &=\langle S^+S^-\rangle \quad \tau<0.\end{aligned}$$

The average is taken over the random orientation of $\mathbf{S}$. The correlation between different spins is ignored. Since the different components of $\mathbf{S}$ do not commute, the τ-ordering is sometimes not trivial. This is best illustrated by evaluating a third-order product. For $\kappa=\mathbf{k}\uparrow$, $\kappa'=\mathbf{k}'\downarrow$, $\kappa''=\mathbf{k}''\downarrow$, we get a term

$$\langle TH_{\kappa\kappa''}(\tau)H'_{\kappa''\kappa'}(\tau')H_{\kappa'\kappa}(0)\rangle =N^{-2}cJ^3\langle TS^-(\tau)S^z(\tau')S^+(0)\rangle.$$

Then, for $\tau>\tau'>0$, the spin product equals

$$\langle S^-S^zS^+\rangle=\langle S^-S^+\rangle+\langle S^zS^-S^+\rangle=\tfrac{1}{3}S(S+1),$$

while for $\tau'>\tau>0$ the product equals

$$\langle S^zS^-S^+\rangle=-\tfrac{1}{3}S(S+1).$$

Thus, the dynamical property of the spin operator manifests itself in the third-order terms. This dynamical effect is the cause of the $\ln T$ divergence in resistivity.

We may now evaluate all the spin T-products and obtain an explicit expression for the self energy. It can be easily shown that the self energy is the same for both spin states of the electron, and we may drop the spin subscript henceforth. Thus,

$$\Sigma_{\mathbf{k}}{}'(\tau)=N^{-1}J^2cS(S+1)\sum_{\mathbf{k}'} G_{\mathbf{k}'}(\tau)-N^{-2}J^3cS(S+1) \times\sum_{\mathbf{k}'\mathbf{k}''}\int_0^\tau G_{\mathbf{k}''}(\tau-\tau')G_{\mathbf{k}'}(\tau')d\tau'+N^{-2}J^3cS(S+1) \times\sum_{\mathbf{k}'\mathbf{k}''}\int_\tau^\beta G_{\mathbf{k}''}(\tau-\tau')G_{\mathbf{k}'}(\tau')d\tau'. \tag{2.13}$$

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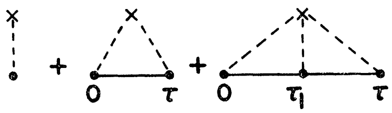
[11] See, for example, L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

[12] This equation is obtained from solving the Dyson equation

$$G_\kappa(\omega_n)=G_\kappa{}^{(0)}(\omega_n)+G_\kappa{}^{(0)}(\omega_n)\Sigma_\kappa(\omega_n)G_\kappa(\omega_n),$$

where $G_\kappa{}^{(0)}(\omega_n)$ is the zero-order Green function. The validity of this equation for magnetic impurity scattering in the first Born approximation was proved by Abrikosov and Gor'kov (A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960), [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)]). Although the proof seems to be valid only for classical spins, it can be generalized to the case of spin operators provided that the correlation between different spins is ignored. This essentially assumes that the various spins scatter the electrons independently. Under this assumption the proof can be extended with no difficulty to the next order Born approximation.

FIG. 1. The diagrams for the magnetic scattering contribution to the self energy of conduction electrons.



To evaluate the τ' integrals, it is convenient to introduce the spectral representation of the Green functions

$$G_{\mathbf{k}}(\tau)=\int\frac{d\omega}{2\pi}e^{-\omega\tau}(1-f(\omega))A(\mathbf{k},\omega)\qquad \tau>0$$

$$=-\int\frac{d\omega}{2\pi}e^{-\omega\tau}f(\omega)A(\mathbf{k},\omega)\qquad \tau<0, \quad (2.14)$$

where

$$f(\omega)=[e^{\beta\omega}+1]^{-1}$$

and

$$A(\mathbf{k},\omega)=\Gamma_{\mathbf{k}}(\omega)/\{[\omega+\mu-\epsilon_k-\Delta_{\mathbf{k}}(\omega)]^2+[\tfrac{1}{2}\Gamma_{\mathbf{k}}(\omega)]^2\}. \quad (2.15)$$

In the expression for $A(\mathbf{k},\omega)$ the energy shift $\Delta_{\mathbf{k}}(\omega)$ and level width $\Gamma_{\mathbf{k}}(\omega)$ are related to the Fourier components of the total self energy by

$$\Delta_{\mathbf{k}}(\omega)=\tfrac{1}{2}[\Sigma_{\mathbf{k}}((1/i)\{\omega+i\delta\})+\Sigma_{\mathbf{k}}((1/i)\{\omega-i\delta\})]$$

$$\Gamma_{\mathbf{k}}(\omega)=(1/i)[\Sigma_{\mathbf{k}}((1/i)\{\omega+i\delta\})-\Sigma_{\mathbf{k}}((1/i)\{\omega-i\delta\})] \quad (2.16)$$

$$\Sigma_{\mathbf{k}}(\omega_n)=\int_0^\beta\Sigma_{\mathbf{k}}(\tau)e^{i\omega_n\tau}d\tau.$$

In general Δ and Γ are weakly dependent on $\mathbf{k}$, so we may regard them as functions of ω only and drop the subscript $\mathbf{k}$. Putting Eq. (2.14) into Eq. (2.13) we can easily integrate over τ'. We then take the Fourier component of the result and obtain

$$\Sigma_{\mathbf{k}}'(\omega_n)=-N^{-1}cJ^2S(S+1)\Bigg\{\sum_{\mathbf{k}'}\int\frac{d\omega'}{2\pi}\frac{A(k',\omega')}{i\omega_n-\omega'}+N^{-1}J\sum_{\mathbf{k}'\mathbf{k}''}\int\frac{d\omega'}{2\pi}\int\frac{d\omega''}{2\pi}A(\mathbf{k}',\omega')A(\mathbf{k}'',\omega'')\frac{1}{\omega''-\omega'}\times\left[\frac{1-2f(\omega')}{i\omega_n-\omega''}-\frac{1-2f(\omega'')}{i\omega_n-\omega'}\right]\Bigg\}. \quad (2.17)$$

The sums over $\mathbf{k}'$ and $\mathbf{k}''$ are worked out as follows. Since $A(\mathbf{k}',\omega')$ peaks strongly at $\epsilon_{k'}\cong\mu+\omega'$, and Δ and Γ are independent of $\mathbf{k}'$, we may write

$$\sum_{\mathbf{k}'}A(\mathbf{k}',\omega')\cong N(\omega')\int_0^\infty A(\mathbf{k}',\omega')d\epsilon_{k'}$$

$$=2\pi N(\omega'), \quad (2.18)$$

where

$$N(\omega')=m\Omega(2m(\mu+\omega'))^{1/2}/2\pi^2 \quad (2.19)$$

is the density of state at the energy ω' from the Fermi level, Ω is the volume of the sample. Notice that Eq. (2.18) is only valid if the level width is small compared with the conduction band width, so that the limits of integration may be replaced by infinity. It will turn out that the level width tends to diverge when the temperature approaches absolute zero. In this case the finite width of the band must be considered. In fact, the conduction band width provides a natural cutoff of the divergence at 0°K.

We go back to the calculation of $\Sigma'(\omega_n)$. The result up to now is

$$\Sigma_{\mathbf{k}}'(\omega_n)=-N^{-1}cJ^2S(S+1)\left\{\int\frac{N(\omega')d\omega'}{i\omega_n-\omega'}+2N^{-1}J\int N(\omega')d\omega'\int N(\omega'')d\omega''\times\frac{1}{\omega''-\omega'}\frac{1-2f(\omega')}{i\omega_n-\omega''}\right\}. \quad (2.20)$$

We use Eq. (2.16) to compute the contribution of level width by magnetic scattering

$$\Gamma'(\omega)=2\pi N^{-1}cJ^2S(S+1)N(\omega)\times\left\{1-2N^{-1}J\int N(\omega')\frac{1-2f(\omega')}{\omega'-\omega}d\omega'\right\}. \quad (2.21)$$

We may ignore the temperature-independent part of the integral in the above expression because it does not lead to any temperature effect. Then we have

$$\Gamma'(\omega)=\Gamma_0'(\omega)[1+4Jg(\omega)], \quad (2.22)$$

where

$$\Gamma'_0(\omega)=2\pi N^{-1}cJ^2S(S+1)N(\omega) \quad (2.23)$$

and

$$g(\omega)=N^{-1}\int\frac{f(\omega')N(\omega')}{\omega'-\omega}d\omega'. \quad (2.24)$$

The total level width is then

$$\Gamma(\omega)=\Gamma'_0(\omega)[1+4Jg(\omega)]+\Gamma''(\omega). \quad (2.25)$$

The conductivity is related to $\Gamma(\omega)$ by the relation[13]

$$\sigma=-\frac{2ne^2}{m}\int\frac{1}{\Gamma(\omega)}\frac{\partial f(\omega)}{\partial\omega}d\omega$$

$$=\frac{2ne^2}{m}\frac{1}{\Gamma_0'(0)+\Gamma''(0)}\times\left[1+\frac{4J\Gamma_0'(0)}{\Gamma_0'(0)+\Gamma''(0)}\int g(\omega)\frac{\partial f(\omega)}{\partial\omega}d\omega\right], \quad (2.26)$$

[13] G. Rickayzen, in *Lecture Notes on Many-Body Problems from the First Bergen International School of Physics, 1961*, edited by C. Fronsdal (W. A. Benjamin, Inc., New York, 1962), pp. 85–109; K. Baumann and J. Ranninger, Ann. Phys. (N. Y.) **20**, 157 (1962).

where

$$n=(2m\mu)^{3/2}3\pi^2.$$

The integral appearing in Eq. (2.26) was evaluated by Kondo

$$\int g(\omega)\frac{\partial f(\omega)}{\partial\omega}d\omega=-\frac{3z}{4\mu}\left[\ln\frac{kT}{\gamma\mu}-\int\int\ln\beta|\omega'-\omega|\times\frac{\partial f(\omega)}{\partial\omega}\frac{\partial f(\omega')}{\partial\omega'}d\omega d\omega'\right],\quad(2.27)$$

where z is the valence of the metal and γ is of the order unity and its value depends on the band structure and the dependence of the exchange constant J on the energy. The integral is small and independent of T, so it is ignored. The resistance is therefore

$$\rho=\rho_0'(1+(3zJ/\mu)\ln(kT/\gamma\mu))+\rho'',\quad(2.28)$$

where

$$\rho_0'=3\pi xmcJ^2S(S+1)/2e^2n\mu$$

is the resistivity due to normal magnetic scattering, and

$$\rho''=m\Gamma''(0)/2ne^2$$

is due to nonmagnetic scatterings. If $J<0$, ρ tends to rise when T is lowered. When combined with the phonon contribution, the resistance shows a minimum. The result is in complete agreement with Kondo. However, we have shown by this more elaborate calculation that the resistance anomaly is unaffected by other impurities. This is because the anomalous scattering depends not on the sharpness of the Fermi surface but on the sharp change in electron population at the Fermi energy.

It is seen from Eq. (2.28) that the resistance tends to diverge at $T\to 0$. This is not a serious objection to the theory because, as we argued before, when the level width is large the finite width of the conduction band becomes important. It is easy to show that this cutoff actually makes the final result convergent. In practice the cutoff effect shows up only at extremely low temperatures.

III. MAGNETORESISTANCE

We may carry out the complete analysis of the scattering problem in the presence of a magnetic field. The only change we need to make is to add a Zeeman term

$$-g\mu_BH\sum_j S_j^z\quad(3.1)$$

to the Hamiltonian. Here H is the applied field assumed to be in the z direction, g is the gyromagnetic ratio of the impurity spins, and μ_B is the Bohr magneton. The paramagnetic susceptibility of the conduction electrons may be ignored. The presence of the Zeeman energy term modifies the spin averages, e.g.

$$\begin{aligned}\langle TS^-(\tau)S^+(0)\rangle&=e^{\omega_0\tau}\langle S^-S^+\rangle\quad \tau>0\\&=e^{\omega_0\tau}\langle S^+S^-\rangle\quad \tau<0,\end{aligned}\quad(3.2)$$

where $\omega_0=g\mu_BH$ and

$$\langle S^-S^+\rangle=S(S+1)-\langle(S^z)^2\rangle-\langle S^z\rangle.$$

We illustrate the method by calculating the magnetoresistance in the first Born approximation. The calculation of the next order term is briefly sketched, and the result presented at the end of the section.

To the first Born approximation the self energy of a spin-up electron is

$$\Sigma_{k\uparrow}'(\tau)=cJ\langle S^z\rangle\delta(\tau)+N^{-1}cJ^2\times\sum_{k'}[\langle(S^z)^2\rangle+\langle S^-S^+\rangle e^{\omega_0\tau}]G_{k'}(\tau).\quad(3.3)$$

The difference between $G_{k'\uparrow}(\tau)$ and $G_{k'\downarrow}(\tau)$ is ignored. Putting in the spectral representation of $G_{k'}(\tau)$ and summing over $\mathbf{k}'$, we obtain

$$\Sigma_{k'\uparrow}'(\tau)=cJ\langle S^z\rangle\delta(\tau)+N^{-1}cJ^2\int N(\omega)(1-f(\omega))\times e^{-\omega\tau}[\langle S^z)^2\rangle+\langle S^-S^+\rangle e^{\omega_0\tau}]d\omega.\quad(3.4)$$

We then take the Fourier transform and evaluate the discontinuity of $\Sigma_{k'\uparrow}(\omega_n)$ across the real frequency axis. This gives

$$\Gamma_{k\uparrow}'(\omega)=2\pi N^{-1}cJ^2[N(\omega)\langle(S^z)^2\rangle+N(\omega+\omega_0)\langle S^-S^+\rangle(f(\omega+\omega_0)/f(\omega))e^{\beta\omega_0}].\quad(3.5)$$

Since $\omega_0\ll\mu$, we may take

$$N(\omega+\omega_0)\cong N(\omega).$$

In a similar manner, we find for a spin-down electron

$$\Gamma_{k\downarrow}'(\omega)=2\pi N^{-1}cJ^2N(\omega)[\langle(S^z)^2\rangle+\langle S^+S^-\rangle(f(\omega-\omega_0)/f(\omega))e^{-\beta\omega_0}].\quad(3.6)$$

The conductivity is then

$$\sigma(H)=e^2n/m[(1/\Gamma_{k\uparrow}(0))+(1/\Gamma_{k\downarrow}(0))],\quad(3.7)$$

with

$$\Gamma_{k\uparrow}(0)=\Gamma_{k\uparrow}'(0)+\Gamma_k''(0),$$

etc. It is straightforward to verify that

$$\begin{aligned}\langle(S^z(^2\rangle&=S(S+1)-S\coth\tfrac{1}{2}\beta\omega_0B_s(\beta\omega_0),\\\langle S^-S^+\rangle&=(\coth\tfrac{1}{2}\beta\omega_0-1)SB_s(\beta\omega_0),\\\langle S^+S^-\rangle&=(\coth\tfrac{1}{2}\beta\omega_0+1)SB_s(\beta\omega_0),\end{aligned}\quad(3.8)$$

where $B_s(\beta\omega_0)$ is the Brillouin function. Using these relations we find that

$$\Gamma_{k\uparrow}'(0)=\Gamma_{k\downarrow}'(0)=3\pi\mu^{-1}czJ^2\times[S(S+1)-SB_s(\beta\omega_0)\tanh\tfrac{1}{2}\beta\omega_0].\quad(3.9)$$

Hence the resistance is

$$\rho^{(2)}(H)=\rho_0'[1-(S+1)^{-1}B_s(\beta\omega_0)\times\tanh\tfrac{1}{2}\beta\omega_0]+\rho''(H).\quad(3.10)$$

The superscript on ρ indicates that the result is of the second order in J. The above expression for magnetoresistance is in agreement with that of van Peski-Tinbergen and Dekker[4] except that we ignored the cross term between the spin-independent scattering and the s-d scattering.

The anomalous part of the resistivity is calculated in a similar way, but the algebra is much more complicated. We show here the calculation of one of the terms in $\Sigma_{\mathbf{k}'\uparrow}(\tau)$, namely,

$$N^{-2}cJ^3 \sum_{\mathbf{k}'\mathbf{k}''} \int_0^\beta \langle S^-(\tau)S^+(\tau')S^z(0)\rangle G_{\mathbf{k}''\downarrow}(\tau-\tau')G_{\mathbf{k}'\uparrow}(\tau')d\tau'$$

$$=N^{-2}cJ^3 \sum_{\mathbf{k}'\mathbf{k}''} \left\{\langle S^-S^+S^z\rangle \int_0^\tau e^{\omega_0(\tau-\tau')}G_{\mathbf{k}'}(\tau')G_{\mathbf{k}''}(\tau-\tau')d\tau' + \langle S^+S^-S^z\rangle \int_\tau^\beta e^{\omega_0(\tau-\tau')}G_{\mathbf{k}'}(\tau')G_{\mathbf{k}''}(\tau-\tau')d\tau'\right\}. \quad (3.11)$$

There are altogether four terms like this in $\Sigma_{\mathbf{k}\uparrow}'(\tau)$. The contribution of this term to $\Gamma_{\mathbf{k}}'(\omega)$ is

$$N^{-2}cJ^3 \int N(\omega')d\omega' \left\{\langle S^-S^+S^z\rangle \left[\frac{1-f(\omega')}{\omega'-\omega-\omega_0} + \frac{f(\omega+\omega_0)e^{-\beta\omega_0}}{f(\omega)}\frac{1-f(\omega')}{\omega'-\omega}\right] + \langle S^+S^-S^z\rangle \left[\frac{f(\omega')}{\omega'-\omega-\omega_0} + \frac{f(\omega+\omega_0)}{f(\omega)}\frac{f(\omega')}{\omega'-\omega}\right]\right\}. \quad (3.12)$$

The terms containing $f(\omega')$ are the anomalous terms we want to find. In the resistance calculation this term is integrated over ω with $-\partial f(\omega)/\partial\omega$ as the weighting factor. Hence, we need to evaluate the integrals

$$-N^{-1}\int \frac{\partial f(\omega)}{\partial\omega}d\omega \int \frac{f(\omega')N(\omega')d\omega'}{\omega'-\omega-\omega_0} \cong \frac{3z}{4\mu}\left[\ln\frac{kT}{\gamma\mu} + \int\int \ln\beta|\omega'-\omega-\omega_0| \frac{\partial f(\omega')}{\partial\omega'}\frac{\partial f(\omega)}{\partial\omega}d\omega' d\omega\right]$$

$$-N^{-1}\int \frac{\partial f(\omega)}{\partial\omega}d\omega \int \frac{f(\omega')N(\omega')d\omega'}{\omega'-\omega} \cong \frac{3z}{4\mu}\left[\frac{2}{e^{\beta\omega_0}+1}\ln\frac{kT}{\gamma\mu} - \beta\int\int \ln\beta|\omega'-\omega| \frac{\partial f(\omega')}{\partial\omega'} f(\omega+\omega_0)(1-f(\omega))d\omega' d\omega\right]. \quad (3.13)$$

It is particularly interesting to consider the first integral when $\beta\omega_0 \gg 1$. In this case it equals approximately $(3z/4\mu)\ln(\omega_0/\gamma\mu)$. So the anomalous resistence contains a term which goes like $\ln H$.

The complete expression for the magnetoresistance is as follows:

$$\rho(H)=\rho_0'(1+(3zJ/\mu)\ln(kT/\gamma\mu))[1-(S+1)^{-1}B_s(\beta\omega_0)\tanh\tfrac{1}{2}\beta\omega_0] - \rho_0'[S(S+1)]^{-1}\frac{3zJ}{\mu}\int_{-\infty}^{\infty} G(u,\beta\omega_0)du + \rho''(H), \quad (3.14)$$

where

$$G(u,v)=\langle (S^z)^2\rangle \operatorname{csch}^2\tfrac{1}{2}u[\tfrac{1}{2}u\coth\tfrac{1}{2}u-1]\ln|u+v|$$
$$+\tfrac{1}{8}\langle S^z\rangle \operatorname{csch}^2\tfrac{1}{2}v[u\operatorname{csch}^2\tfrac{1}{2}u-(u+v)\operatorname{csch}^2\tfrac{1}{2}(\mathrm{u}+\mathrm{v})-\tfrac{1}{2}\sinh\tfrac{1}{2}v\operatorname{csch}\tfrac{1}{2}u\operatorname{csch}\tfrac{1}{2}(u+v)][\ln|u|+\ln|u+v|]. \quad (3.15)$$

When $\beta\omega_0 \gg 1$, it is easy to verify that

$$\int_{-\infty}^{\infty} G(u,\beta\omega_0)du \cong 2\langle (S^z)^2\rangle \ln\beta\omega_0.$$

Hence the second term in Eq. (3.14) is

$$-\rho_0'(6zJ/\mu)(\langle (S^z)^2\rangle/S(S+1))\ln\beta\omega_0. \quad (3.16)$$

We estimate the size of this term by taking $z=1$, $J=0.2$ eV, $\mu=5$ eV, and $g=2$. Then at 1°K and under 20 kG of field, the term amounts to roughly 5% of ρ_0'. This is of the same order of magnitude as the resistance anomaly. For alloys that show resistance minimum, this term gives rise to a reduction in negative magnetoresistance at high fields and low temperatures.

IV. EFFECT ON SUPERCONDUCTIVITY

In this section we consider the effect of anomalous scattering on the critical temperature and energy gap of a superconductor containing paramagnetic impurities. The calculation proceeds in very much the same way as in the classic paper of Abrikosov and Gor'kov.[9] We shall refer many details to their paper. One defines a matrix Green function

$$\hat{G}_s(\mathbf{r},\mathbf{r}';\tau)=\begin{pmatrix} \langle T\psi_s(\mathbf{r},\tau)\psi_s^*(\mathbf{r}',0)\rangle & -\langle T\psi_s(\mathbf{r},\tau)\psi_{-s}(\mathbf{r}',0)\rangle \\ \langle T\psi_{-s}^*(\mathbf{r},\tau)\psi_s^*(\mathbf{r}',0)\rangle & -\langle T\psi_{-s}^*(\mathbf{r},\tau)\psi_{-s}(\mathbf{r}',0)\rangle \end{pmatrix}. \tag{4.1}$$

This Green function may be Fourier expanded both in space and in time

$$\hat{G}_s(\mathbf{r},\mathbf{r}';\tau)=\frac{1}{\beta}\sum_n\sum_{\mathbf{k}}\hat{G}_{\mathbf{k}s}(\omega_n)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\omega_n\tau}, \tag{4.2}$$

and $\hat{G}_{\mathbf{k}s}(\omega_n)$ has the form

$$\hat{G}_\kappa(\omega_n)=\begin{pmatrix} G_\kappa(\omega_n) & -F_\kappa(\omega_n) \\ F_\kappa^*(\omega_n) & G_\kappa(-\omega_n)\end{pmatrix}, \tag{4.3}$$

with $G_{\mathbf{k}\uparrow}(\omega_n)=G_{\mathbf{k}\downarrow}(\omega_n)$, $F_{\mathbf{k}\uparrow}(\omega_n)=-F_{\mathbf{k}\downarrow}(\omega_n)$. The self energy is also a matrix

$$\hat{\Sigma}_\kappa(\omega_n)=\begin{pmatrix} \Sigma_\kappa^{(1)}(\omega_n) & -\Sigma_\kappa^{(2)}(\omega_n) \\ \Sigma_\kappa^{(2)*}(\omega_n) & \Sigma_\kappa^{(1)}(-\omega_n)\end{pmatrix}. \tag{4.4}$$

The quantities $\hat{G}$ and $\hat{\Sigma}$ are related by

$$[\hat{G}_\kappa(\omega_n)]^{-1}=\begin{pmatrix} \epsilon_k-\mu-i\omega_n-\Sigma_\kappa^{(1)}(\omega_n) & \Delta_\kappa+\Sigma_\kappa^{(2)}(\omega_n) \\ -\Delta_\kappa-\Sigma_\kappa^{(2)*}(\omega_n) & \epsilon_k-\mu+i\omega_n-\Sigma_\kappa^{(1)}(-\omega_n)\end{pmatrix}, \tag{4.5}$$

where

$$\Delta_\kappa=g\langle\psi_s(\mathbf{r})\psi_{-s}(\mathbf{r})\rangle=\frac{g}{\beta}\sum_n\sum_{\mathbf{k}}F_\kappa(\omega_n). \tag{4.6}$$

As before the self energy contains a magnetic scattering part $\hat{\Sigma}_\kappa'(\omega_n)$ and a nonmagnetic scattering part $\hat{\Sigma}_\kappa''(\omega_n)$. For simplicity we consider H'' to be due to nonmagnetic scattering centers. The diagram for $\hat{\Sigma}_\kappa''(\tau)$ is in Fig. 2 and the diagrams for $\hat{\Sigma}_\kappa'(\tau)$ are in Fig. 1. Explicitly

$$\hat{\Sigma}_{\mathbf{k}s}''(\tau)=N^{-1}c'u^2\sum_{\mathbf{k}'}G_{\mathbf{k}'s}(\tau) \tag{4.7}$$

and

$$\hat{\Sigma}_\kappa'(\tau)=\sum_{\kappa'}\langle TH_{\kappa\kappa'}'(\tau)H_{\kappa'\kappa}'(0)\rangle\hat{G}_{\kappa'}(\tau)-\sum_{\kappa'\kappa''}\int_0^\beta\langle TH_{\kappa\kappa''}'(\tau)H_{\kappa''\kappa'}'(\tau')H_{\kappa'\kappa}'(0)\rangle\hat{G}_{\kappa''}(\tau-\tau')\hat{G}_{\kappa'}(\tau')d\tau', \tag{4.8}$$

where c' is the density of nonmagnetic impurities and u is the strength of the electron-impurity scattering potential. Resolving $\hat{G}_{\kappa'}(\tau)$ and $\hat{G}_{\kappa''}(\tau-\tau')$ into Fourier series, we can evaluate the τ' integration and subsequently take the Fourier components on both sides. The result is

$$\hat{\Sigma}_{\mathbf{k}s}''(\omega_n)=N^{-1}c'u^2\sum_{\mathbf{k}'}\hat{G}_{\mathbf{k}'s}(\omega_n), \tag{4.9}$$

$$\hat{\Sigma}_\kappa'(\omega_n)=\sum_{\kappa'}\langle H_{\kappa\kappa'}'H_{\kappa'\kappa}'\rangle\hat{G}_{\kappa'}(\omega_n)-\frac{1}{\beta}\sum_m\sum_{\kappa'\kappa''}L(\kappa,\kappa',\kappa'')\frac{\hat{G}_{\kappa'}(\omega_n)\hat{G}_{\kappa''}(\omega_m)}{i(\omega_m-\omega_n)}, \tag{4.10}$$

where

$$L(\kappa,\kappa',\kappa'')=\langle[H_{\kappa\kappa''}',H_{\kappa''\kappa'}']H_{\kappa'\kappa}'\rangle+\langle[H_{\kappa\kappa'}',H_{\kappa'\kappa''}']H_{\kappa''\kappa}'\rangle. \tag{4.11}$$

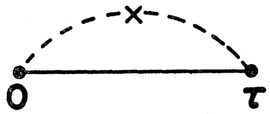
Since the spins are oriented at random, there is no loss of generality by choosing $\kappa=k\uparrow$; then

$$\langle H_{\mathbf{k}\uparrow\mathbf{k}'\uparrow}'H_{\mathbf{k}'\uparrow\mathbf{k}\uparrow}'\rangle=\tfrac{4}{3}N^{-1}cJ^2S(S+1)\quad\langle H_{\mathbf{k}\uparrow\mathbf{k}'\downarrow}'H_{\mathbf{k}'\downarrow\mathbf{k}\uparrow}'\rangle=\tfrac{2}{3}N^{-1}cJ^2S(S+1)$$
$$L(\mathbf{k}\uparrow,\mathbf{k}'\uparrow,\mathbf{k}''\uparrow)=0\quad L(\mathbf{k}\uparrow,\mathbf{k}'\uparrow,\mathbf{k}''\downarrow)=L(\mathbf{k}\uparrow,\mathbf{k}'\downarrow,\mathbf{k}''\uparrow)=L(\mathbf{k}\uparrow,\mathbf{k}'\downarrow,\mathbf{k}''\downarrow)=\tfrac{4}{3}N^{-2}cJ^3S(S+1). \tag{4.12}$$

Separating the various components of $\hat{\Sigma}$, we obtain

$$\hat{\Sigma}_{\mathbf{k}}^{(1)}(\omega_n)=N^{-1}[c'u^2+cJ^2S(S+1)]\sum_{\mathbf{k}'}G_{\mathbf{k}}(\omega_n)\tfrac{4}{3}N^{-2}cJ^3S(S+1)\sum_{\mathbf{k}'\mathbf{k}''}(1/\beta)\sum_m(1/i(\omega_m-\omega_n))$$
$$\times[3G_{\mathbf{k}}(\omega_n)G_{\mathbf{k}''}(\omega_m)+F_{\mathbf{k}'}(\omega_n)F_{\mathbf{k}''}^*(\omega_n)]$$
$$\hat{\Sigma}_{\mathbf{k}}^{(2)}(\omega_n)=N^{-1}[c'u^2-\tfrac{1}{3}cJ^2S(S+1)]\sum_{\mathbf{k}'}F_{\mathbf{k}'}(\bar{B}_n)+\tfrac{4}{3}N^{-2}cJ^3S(S+1)\sum_{\mathbf{k}'\mathbf{k}''}(1/\beta)\sum_m(1/i(\omega_m-\omega_n))$$
$$\times[G_{\mathbf{k}'}(\omega_n)F_{\mathbf{k}''}(\omega_m)+F_{\mathbf{k}'}(\omega_n)G_{\mathbf{k}''}(-\omega_m)], \tag{4.13}$$

FIG. 1. The diagram for the nonmagnetic impurity scattering contribution to the self energy of conduction electrons.



where the spin indices on the Green functions are dropped, and it is understood that all the functions refer to the same spin state. These equations are to be solved self-consistently to determine the effect of scattering on superconductivity.

To approximately solve these equations we employ the following trick.[14] Since the self energy is a function of the frequency only, we may make the following substitution:

$$\Sigma_{\mathbf{k}}^{(1)}(\omega_n)=-i\omega_n(1-Z(\omega_n)),\qquad \Sigma_{\mathbf{k}}^{(2)}(\omega_n)=-\Delta+\Delta(\omega_n)Z(\omega_n), \tag{4.14}$$

where $\Delta=\Delta_{\mathbf{k}\uparrow}$ and $Z(\omega_n)$ are to be solved. With these substitutions we find from Eq. (4.5)

$$[G_k(\omega_n)]^{-1}=Z(\omega_n)\begin{pmatrix}\tilde{\epsilon}_k-i\omega_n & \Delta(\omega_n)\\ -\Delta^*(\omega_n) & \tilde{\epsilon}_k+i\omega_n\end{pmatrix}, \tag{4.15}$$

where

$$\tilde{\epsilon}_k=[Z(\omega_n)]^{-1}(\epsilon_k-\mu). \tag{4.15}$$

Inverting this matrix we find

$$G_k(\omega_n)=\frac{1}{Z(\omega_n)}\frac{\tilde{\epsilon}_k+i\omega_n}{\tilde{\epsilon}_k^2+\omega_n^2+|\Delta(\omega_n)|^2},\qquad F_k(\omega_n)=\frac{1}{Z(\omega_n)}\frac{\Delta(\omega_n)}{\tilde{\epsilon}_k^2+\omega_n^2+|\Delta(\omega_n)|^2}, \tag{4.16}$$

We substitute these results into Eq. (4.13). The sums over $\mathbf{k}'$, $\mathbf{k}''$ are carried out by use of the relations

$$\sum_{\mathbf{k}}\frac{1}{Z(\omega_n)}\frac{1}{\tilde{\epsilon}_k^2+\omega_n^2+|\Delta(\omega_n)|^2}\cong\frac{\pi N(0)}{(\omega_n^2+|\Delta(\omega_n)|^2)^{1/2}},\qquad \sum_k\frac{1}{Z(\omega_n)}\frac{\tilde{\epsilon}_k}{\tilde{\epsilon}_k^2+\omega_n^2+|\Delta(\omega_n)|^2}\cong 0, \tag{4.17}$$

where $N(\omega)$ was defined in Eq. (2.19). If we define the following quantities

$$\begin{aligned}1/\tau_1&=\pi N^{-1}N(0)[c'u^2+cJ^2S(S+1)],\\ 1/\tau_2&=\pi N^{-1}N(0)[c'u^2-\tfrac{1}{3}cJ^2S(S+1)],\\ 1/\tau_s&=(1/\tau_1)-(1/\tau_2)=\tfrac{4}{3}\pi N^{-1}N(0)cJ^2S(S+1),\end{aligned} \tag{4.18}$$

then we can write the result of the above manipulations as

$$\begin{aligned}1-Z(\omega_n)&=-\frac{1}{\tau_1}\frac{1}{(\omega_n^2+\Delta^2(\omega_n))^{1/2}}+\frac{3\pi zJ}{4\tau_s\mu}\sum_m\frac{1}{\omega_n(\omega_m-\omega_n)}\frac{3\omega_m\omega_n-\Delta(\omega_m)\Delta(\omega_n)}{([\omega_n^2+\Delta^2(\omega_n)][\omega_m^2+\Delta^2(\omega_m)])^{1/2}},\\ \Delta-\Delta(\omega_n)Z(\omega_n)&=-\frac{1}{\tau_2}\frac{\Delta(\omega_n)}{[\omega_n^2+\Delta^2(\omega_n)]^{1/2}}+\frac{3\pi zJ}{4\tau_s\mu}\sum_m\frac{1}{\omega_m-\omega_n}\frac{\omega_n\Delta(\omega_m)-\omega_m\Delta(\omega_n)}{([\omega_n^2+\Delta^2(\omega_n)][\omega_m^2+\Delta^2(\omega_m)])^{1/2}}.\end{aligned} \tag{4.19}$$

We eliminate $Z(\omega_n)$ from these equations to obtain

$$\Delta=\Delta(\omega_n)\left[1+\frac{1}{\tau_s[\omega_n^2+\Delta^2(\omega_n)]^{1/2}}\right]-\frac{3\pi zJ}{4\tau_s\mu}\sum_m\frac{1}{(\omega_m-\omega_n)\omega_n}\frac{4\omega_m\omega_n\Delta(\omega_n)+\Delta(\omega_m)\Delta^2(\omega_n)-\omega_n^2\Delta(\omega_m)}{([\omega_m^2+\Delta^2(\omega_m)][\omega_n^2+\Delta^2(\omega_n)])^{1/2}}. \tag{4.20}$$

One can see from this result that the nonmagnetic scattering effect does not enter the energy-gap equation. Therefore, even when the anomalous scattering term is in cluded, the nonmagnetic scattering still has no effect on the gap and the critical temperature. We can find an iterative solution for Eq. (4.20). For the zero order, we ignore all terms proportional to $\Delta(\omega_m)$. This gives

$$\Delta\cong\Delta(\omega_n)\left\{1+\frac{1}{\tau_s[\omega_n^2+\Delta^2(\omega_n)]^{1/2}}\times\left[1-\frac{3\pi zJ}{\mu\beta}\sum_m\frac{\omega_m}{(\omega_m-\omega_n)[\omega_m^2+\Delta^2(\omega_m)]^{1/2}}\right]\right\}. \tag{4.21}$$

We discuss two limiting cases, namely near absolute zero and near the critical temperature.

Near the critical temperature Eq. (4.21) reduces to

$$\Delta=\Delta(\omega_n)\left\{1+\frac{1}{\tau_s|\omega_n|}\left(1-\frac{3\pi zJ}{\mu\beta}\sum_m\frac{\omega_m}{|\omega_m|(\omega_m-\omega_n)}\right)\right\}.$$

The sum in the parentheses may be evaluated by transforming it into an integral

$$\frac{1}{\beta}\sum_m\frac{\omega_m}{|\omega_m|(\omega_m-\omega_n)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-2f(z)}{z+i\omega_n}dz.$$

As before we ignore the part of the numerator which does not depend on the temperature. The lower limit of the integral should be cutoff at $-\mu$, the bottom of the

[14] D. Markowitz and L. P. Kadanoff, Phys. Rev. **131**, 563 (1963).

band, so

$$\int_{-\mu}^{\infty}\frac{f(z)}{z+i\omega_n}dz=-\ln\mu+\int_{-\infty}^{\infty}\ln(z+i\omega_n)\frac{\partial f(z)}{\partial z}dz$$
$$\cong\ln(kT/\mu).$$

Thus

$$\Delta=\Delta(\omega_n)\left\{1+\frac{1}{\tau_s|\omega_n|}\left(1+\frac{3zJ}{\mu}\ln\frac{kT}{\mu}\right)\right\}. \quad (4.22)$$

We solve this equation for $\Delta(\omega_n)$, put it in Eq. (4.16) for $F_{\mathbf{k}}(\omega_n)$, and then substitute it in Eq. (4.6). After summing over $\mathbf{k}$ and cancelling Δ, we find the equation for the critical temperature to be

$$1=\frac{\pi g N(0)}{\beta_c}\sum_n\frac{1}{|\omega_n|+1/\tau_s'}, \quad (4.23)$$

where $1/\tau_s'=1/\tau_s[1+(3zJ/\mu)\ln(kT_c/\mu)]$. This result may be reduced to Eq. (22) of Ref. 9 except that τ_s is replaced by τ_s'. Since τ_s' depends only slightly on the temperature, it may be regarded as a constant. Hence, the inclusion of the anomalous scattering term does not produce any new effect.

Near absolute zero, we estimate the effect of anomalous scattering by putting $\Delta(\omega_m)=\Delta$ in Eq. (4.21). The sum over m may be evaluated by the same procedure as near T_c. The result is that

$$\Delta\cong\Delta(\omega_n)\left\{1+\frac{1}{\tau_s[\omega_n^2+\Delta^2(\omega_n)]^{1/2}}\left(1+\frac{3zJ}{\mu}\ln\frac{\Delta}{\mu}\right)\right\}. \quad (4.24)$$

Since $\Delta\cong kT_c$, we may also write

$$\Delta\cong\Delta(\omega_n)\left\{1+\frac{1}{\tau_s'[\omega_n^2+\Delta^2(\omega_n)]^{1/2}}\right\}. \quad (4.25)$$

Again this is the same result as in Ref. 9 except that τ_s is replaced by τ_s'.

The conclusions of this section are

(1) The effect of anomalous scattering on superconductivity is insensitive to the presence of nonmagnetic impurities. So the mean free path effect in indium-manganese systems as observed by Merriam *et al.* is perhaps not related to magnetic scattering.

(2) The reduction in critical temperature is determined by the same scattering time τ_s' as the reduction in energy gap.

V. DISCUSSION

In rare-earth ions there is a strong spin-orbit coupling in the $4f$ shell. As a result the effective s-f interaction constant is $(g-1)J$ instead of J, where g is the gyromagnetic ratio. It is generally believed that J is not much affected by alloying. Since the factor $(g-1)$ is negative for light rare-earth metals and is positive for heavy rare-earth metals, then depending on the sign of J, we expect the resistance of the dilute alloys of one group to increase with lowering temperature and that of the other group to decrease. Therefore, a systematic study of the resistivity of dilute rare-earth alloys should furnish the most direct experimental test of the anomalous scattering theory. As a by-product of this measurement we can also settle the question about the sign of s-f interaction in rare-earth metals.

Dilute magnetic alloys have another outstanding property, namely their large and negative thermoelectric power.[15] We shall not elaborate on the theories of this effect except to point out that the model we used in this paper does not explain this phenomenon. The normal scattering gives a very small thermoelectric power. The contribution of the anomalous scattering is smaller than the normal scattering term by roughly the same ratio as their contributions to the electrical resistivity. Hence, although the anomalous scattering gives rise to a remarkable effect on the resistivity, its effect on the thermoelectric power is totally insignificant. Recently Kondo explained this effect by invoking a complex s-d scattering potential.[16]

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[15] See J. M. Ziman, *Electrons and Phonons* (Oxford University Press, London, 1960), Chap. 9.

[16] J. Kondo, Proceedings of the Ninth International Conference on Low Temperature Physics, 1964 (unpublished).