whistler P waves. These whistler P waves represent a superluminous instability associated with the anomalous Doppler effect[20] and there is no excitation of whistler B waves.

B. Generation of VLF Waves in the Magnetodynamic Mode

In accordance with the terminology used in this paper both audible and subaudible frequencies are incuded in the VLF range. Consequently, this range as defined in (3.1) covers frequencies which extend to extremely low values in the neighborhood of $\omega=0$. These extremely low-frequency waves represent magnetodynamic excitation and can be detected at the earth's surface in the form of long period pulsatians of the terrestrial magnetic field. Various mechanisms have been suggested concerning the origin of these pulsations. It is suggested here that at least in some instances the above pulsations have the same origin as the radiations in the whistler mode. Both effects represent an instability produced by the interaction of the helical electron beam trapped in the Van Allen belt with the surrounding exospheric medium.

C. Rate of Growth of Whistler Waves and Magnetodynamic Waves

Except for extremely relativistic transverse beam velocities, waves excited in both the whistler mode and the magnetodynamic mode are characterized by the same rate of growth which is independent of the frequency of the excited wave. The expression for this rate of growth is given by (3.13).

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Generalized Stream Instabilities in Cold Plasmas

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The classical double-stream theory for electrostatic instabilities in a cold plasma is generalized by a more complete treatment of the electrodynamics, whereby an analysis of electromagnetic instabilities is now afforded. Thus, consideration of wave propagation at arbitrary angles to the stream or drift motion leads to a rather involved sixth-degree algebraic dispersion formula which reflects the possible occurrence of coupled growing longitudinal and transverse waves. It is further evidenced that electromagnetic instabilities may arise even in a single-carrier system such as an electron beam. Like the special alignment where the stream and wave vectors are parallel ($\mathbf{U}\|\mathbf{k}$), the orthogonal case ($\mathbf{U}\perp\mathbf{k}$) fails to allow growing transverse waves for the single-carrier stream. However, this latter configuration does admit electromagnetic instabilities for the double-stream plasma; likely unstable regimes are indicated. A more plausible basis for interpreting radio-wave emission from plasmas emerges, and solar noise, for example, may be viewed as a direct consequence of such electromagnetic-wave instabilities.

INTRODUCTION

The research represented herein was responsible for the early recognition that electromagnetic instabilities in plasmas could originate even in the absence of a steady magnetic field if due allowance were made for stream instabilities propagating at arbitrary directions with respect to the flow.[1] This possibility was also pointed out by Neufeld and Doyle[2] and it is worth noting that their analysis formulated for polarization in a moving medium leads to the same findings as the present one founded on individual particle motions in the cold plasma limit.

The two approaches are in essential agreement and overlap somewhat, although different features of the basic dispersion formula are separately discussed. With this background in mind, the actual work performed is next disclosed.

It is the purpose of this report to deal more broadly with the question of stream instabilities in plasmas than has hitherto been the case. Since the pioneer efforts by Haeff,[3] Pierce,[4] Nergaard[5] on the double-stream instability, their cold plasma theory has only recently been extended to "warm" plasmas.[6,7]

The earlier researches have dealt essentially with the electrostatic instability and it has been supposed that electromagnetic instabilities may arise when an external magnetic field is superposed on a plasma. Thus in the absence of a steady magnetic field, such mechanisms as the role of inhomogeneities or boundaries, etc., have

* Based on work performed at the University of Michigan as a visiting member, 1960–1961.

[1] L. Gold, University of Michigan Memo (unpublished).

[2] J. Neufeld and P. H. Doyle, Phys. Rev. **121**, 654 (1961).

[3] A. V. Haeff, Proc. IRE **37**, 4 (1949).

[4] J. R. Pierce, J. Appl. Phys. **19**, 231 (1948).

[5] L. S. Nergaard, RCA Rev. **9**, 585 (1948).

[6] O. Buneman, Phys. Rev. **115**, 503 (1959).

[7] E. A. Jackson, Phys. Fluids **3**, 786 (1960).

been proposed for the conversion of the longitudinal oscillations into radiating transverse waves.[8,9]

A detailed theory for the contribution of an external magnetic field to stream instabilities will be withheld for separate treatment. The present effort concerns analyzing both longitudinal and transverse waves which may propagate at an arbitrary angle with respect to the stream direction; the restriction of coincident stream and wave direction is thus removed. The work to be described is based upon the model introduced by Haeff *et al.* and so pertains to an infinite cold plasma. It is hoped that a Boltzmann formalism may be developed later to determine the effect of finite temperature on the stream behavior.

It may be said at this point that the highlight of the study has been the discovery that transverse or electromagnetic wave instabilities indeed are possible when the wave and stream vectors are oblique to each other. The further possibility prevails that the growth of such waves may occur even for the single-stream situation as represented by the simple electron beam. Much graphical and numerical effort remains to be carried out before a fully satisfactory understanding is to be achieved.

In this paper, attention will be directed toward the development of the electrodynamic fundamentals for dealing with stream or drift effects in a plasma. A generalized dispersion relation, represented by an algebraic equation of sixth degree, is deduced which reflects the growth of plane polarized electromagnetic waves. Several reduced cases are examined with with particular thought given to the circumstance where wave propagation and stream flow are orthogonal to one another; this configuration lends itself to an initial characterization of the transverse wave instability.

ELECTRODYNAMIC FORMALISM FOR WAVE PROPAGATION OBLIQUE TO PLASMA STREAM

In dealing with plane waves in an infinite medium the Cartesian coordinate system indicated in Fig. 1 is adopted. Here **k** is the wave vector taken along the x axis with **U** the stream (or drift) velocity vector taken at an arbitrary angle θ to the direction of propagation

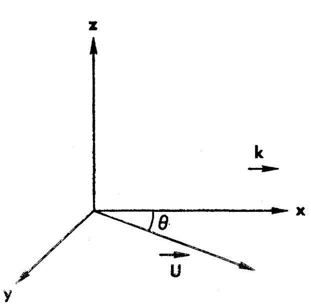

FIG. 1. Coordinate system for plane waves propagating at arbitrary angle θ to stream motion (denoted by **U**).

of the wave. Thus in the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{v}=\frac{q}{m}\left(\mathbf{E}+\frac{\mathbf{U}}{c}\times\mathbf{B}\right), \tag{1}$$

the fluctuating field components **E** and **B** along with the velocity **v** are taken to have a time and spatial dependency $\exp i(\omega t-kx)$. Since $\mathbf{U}=iU_x+jU_y$, the component equation for (1) becomes for the electrons

$$i\omega' v_x^e=\frac{q_e}{m_e}\left(E_x+\frac{U_y}{c}B_z\right),$$

$$i\omega' v_y^e=\frac{q_e}{m_e}\left(E_y-\frac{U_x}{c}B_z\right),\quad \omega'=\omega-U_xk, \tag{2}$$

$$i\omega' v_z^e=\frac{q_e}{m_e}\left(E_z+\frac{U_x}{c}B_y-\frac{U_yB_x}{c}\right),$$

where ω' is the Doppler-shifted frequency associated with stream velocity component along the direction of the wave motion.

Taking the ions in the plasma stream as essentially fixed[10] compared to the electrons ($\mathbf{U}=0$), the component equations may be expressed as

$$i\omega v_{x,y,z}^i=(q_i/m_i)E_{x,y,z}. \tag{3}$$

The foregoing equations have been linearized by neglecting higher order terms. The continuity equation for the charge

$$\partial n/\partial t+\nabla\cdot(N\mathbf{v}+n\mathbf{U})=0 \tag{4}$$

in the linearized approximation leads to the following result

$$n_e=N_ekv_x^e/\omega' \tag{5}$$

for the fluctuating component of the electron charge density whose equilibrium value is N_e; the ion counterpart is taken to be zero.

From the Maxwell relation

$$c\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, \tag{6}$$

it can be shown that the magnetic field components have the values

$$B_x=0,\quad B_y=-\mu E_z,\quad \mu=kc/\omega,\quad B_z=\mu E_y, \tag{7}$$

which accounts for the ultimate nonappearance of the B_x Lorentz force contribution in (2). Thus the determinantal equation of the form

$$\|A_{jk}\|\cdot\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}=0 \tag{8}$$

can be arrived at by introducing (7) and eliminating

[8] G. B. Field, Astrophys. J. 124, 555 (1956).

[9] D. A. Tidman, Phys. Rev. **117**, 366 (1960).

[10] Equation (10) anyway points out needless inclusion of ion current, although $n_i=N_ikv_x^i/\omega'$ in keeping with (5).

the velocity terms by invoking the additional Maxwell relation

$$C\nabla\times\mathbf{B}=\mathbf{J}_C+\mathbf{J}_D\,,\quad \mathbf{J}_D=\partial\mathbf{E}/\partial t\,, \tag{9}$$

where the conduction current is given by

$$\mathbf{J}_C=N_iq_i\mathbf{v}^i+N_eq_e\mathbf{v}^e+n_eq_e\mathbf{U}\,. \tag{10}$$

Writing out (9) in component form and making use of (3), (5), and (7), it can be shown that

$$v_x{}^e=\frac{(\omega^2-\omega_{pi}{}^2)(\omega-kU_x)}{iN_eq_e\omega^2}E_x\,,\quad \omega_{pi}{}^2=\frac{N_iq_i{}^2}{m_i}$$

$$v_y{}^e=\frac{i}{N_eq_e}\left[\left(kc\mu-\omega+\frac{\omega_{pi}{}^2}{\omega}\right)E_y+\frac{kU_y}{\omega^2}(\omega^2-\omega_{pi}{}^2)E_x\right], \tag{11}$$

$$v_z{}^e=\frac{i}{N_eq_e}\left(kc\mu-\omega+\frac{\omega_{pi}{}^2}{\omega}\right)E_z\,.$$

Hence the determinantal relation for a nontrivial solution of (8) takes the form

$$\begin{vmatrix} A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & A_{33}\end{vmatrix}=0\,. \tag{12}$$

The E_z component accordingly has the transverse wave dispersion relation

$$A_{33}=0=-\frac{(\omega-kU_x)}{\omega_{pe}{}^2}\left(k\mu c-\omega+\frac{\omega_{pi}{}^2}{\omega}\right)-1+\mu\frac{U_x}{c}, \tag{13}$$

$$\omega_{pe}{}^2=N_eq_e{}^2/m_e\,,$$

which may be recognized to contain the familiar result

$$\omega^2=\omega_{pi}{}^2+\omega_{pe}{}^2+c^2k^2\,. \tag{14}$$

Since no complex ω are compatible with (14), the plasma stream is stable to such transverse waves.

The residual longitudinal (E_x) and transverse (E_y) components are characterized by the coupled dispersion expression

$$A_{11}A_{22}-A_{12}A_{21}=0\,, \tag{15}$$

wherein

$$A_{11}=\frac{(\omega-kU_x)^2(\omega^2-\omega_{pi}{}^2)}{\omega_{pe}{}^2\omega^2}-1\,,$$

$$A_{12}=kU_y/\omega\,,$$

$$A_{21}=-\frac{kU_y}{\omega}\,\frac{(\omega-kU_x)(\omega^2-\omega_{pi}{}^2)}{\omega\omega_{pe}{}^2}\,, \tag{16}$$

$$A_{22}=-\frac{(\omega-kU_x)}{\omega\omega_{pe}{}^2}(k^2c^2+\omega_{pe}{}^2+\omega_{pi}{}^2-\omega^2)\,.$$

Reducing the dispersion formula stated in (15) to a dimensionless form in terms of

$$\Omega=\frac{\omega}{\omega_{pe}}\,,\quad \mathcal{U}=\frac{kU}{\omega_{pe}}\,,\quad V=\frac{kc}{\omega_{pe}}\,,\quad W=\frac{\omega_{pi}{}^2}{\omega_{pe}{}^2}\,, \tag{17}$$

the outcome is

$$\begin{aligned}&\Omega^6-2\mathcal{U}\cos\theta\cdot\Omega^5-(V^2+2+2W-\mathcal{U}^2\cos^2\theta)\Omega^4\\&+2\mathcal{U}(1+V^2+2W)(\cos\theta)\Omega^3-[(1+V^2+W)\\&\times(\mathcal{U}^2\cos^2\theta-1-W)+\mathcal{U}^2(W\cos^2\theta+\sin^2\theta)]\Omega^2\\&-2\mathcal{U}W(1+V^2+W)(\cos\theta)\Omega\\&\qquad+W\mathcal{U}^2[(1+V^2+W)\cos^2\theta+\sin^2\theta]=0\,.\end{aligned} \tag{18}$$

Clearly the complete delineation of the character of the roots of this involved sixth-degree dispersion relation for prescribed values of $\mathcal{U}$, V, and W will entail prohibitive labor. For the time being it will be shown that (18) reduces to a number of familiar specialized expressions and that certain further reductions do in fact indicate electromagnetic wave instabilities.

SOME FAMILIAR DISPERSION FORMULAS CONTAINED IN THE GENERALIZED DISPERSION RELATION (18)

The classic case where the waves propagate along the stream, $\mathbf{U}\|\mathbf{k}$, follows by setting $\theta=0°$ in (18) which then factors into the following:

$$\Omega^2-(1+V^2+W)=0\,, \tag{19a}$$

$$\Omega^4-2\mathcal{U}\cdot\Omega^3+(\mathcal{U}^2-W-1)\Omega^2+2\mathcal{U}W\cdot\Omega-W\mathcal{U}^2=0\,. \tag{19b}$$

It may be seen that (19a) is the dimensionless form for the transverse dispersion relation (14). The quartic (19b) actually is equivalent to the electrostatic double-stream dispersion formula

$$\omega_{pe}{}^2/(\omega-kU)^2+\omega_{pi}{}^2/\omega^2=1\,. \tag{20}$$

Complex Ω are of course contained in (19b) for certain limits of $\mathcal{U}$ and W. Hence we surely would expect (18) to also embody regions of complex Ω.

Inspection of (18) reveals that for $W=\mathcal{U}=0$ the result becomes

$$\Omega^4-(V^2+2)\Omega^2+(1+V^2)=0 \tag{21}$$

or as

$$\begin{aligned}2\Omega^2&=V^2+2\pm[(V^2+2)^2-4(1+V^2)]^{1/2}\\ \Omega^2&=V^2+1\qquad \omega^2=\omega_p{}^2+k^2c^2\\ &=1\qquad\qquad\ \ \omega^2=\omega_p{}^2\,,\end{aligned} \tag{22}$$

which may be recognized by the more familiar expressions indicated for the steady single-component plasma.

GENERAL SINGLE-STREAM BEHAVIOR

It is commonly thought that an essential requirement for growing waves in a current-carrying or streaming plasma is the presence of at least two components of charged carriers which move relative to each other. That this may not always be so for waves oblique to the proper motion of the carriers derives from (18) by setting $W=0$, whereby

$$\Omega^4-2\mathfrak{U}\cos\theta\cdot\Omega^3-(V^2+2-\mathfrak{U}^2\cos^2\theta)\cdot\Omega^2 +2\mathfrak{U}(1+V^2)\cos\theta\cdot\Omega +[(1+V^2)-\mathfrak{U}^2(1+V^2\cos^2\theta)]=0. \quad (23)$$

Comparing it with the quartic for the electrostatic dispersion (19b), a regime for complex Ω is indicated; but now a critical angle is anticipated and the growing waves are also electromagnetic in nature. Thus replacing the contribution of W in the electrostatic dispersion formula, one now has parameters V and $\cos\theta$. The additional degree of freedom introduced by θ makes for greater complication in discussing the stability question.

Without the support of actual detailed calculation, it seems fairly certain nevertheless that growing transverse waves may be expected to arise even for a single stream (as represented by an electron beam) if oblique waves are taken into account.

THE SPECIAL CASE WHERE STREAM AND WAVE VECTORS ARE ORTHOGONAL ($U\perp k$)

Before dealing with the two-stream circumstance, it will be demonstrated that for $\theta=\pi/2$ in (23) the single stream cannot support growing waves. The dispersion relation becomes

$$\Omega^4-(V^2+2)\cdot\Omega^2+(1+V^2-\mathfrak{U}^2)=0 \quad (24)$$

whence the roots are given by

$$2\Omega^2=V^2+2\pm(V^4+4\mathfrak{U}^2)^{1/2}. \quad (25)$$

Complex Ω are evidently excluded since the expression $V^4+4\mathfrak{U}^2>0$.[11]

Next turning to the double-stream case, the general dispersion relation (18) goes over to a bicubic equation which is more readily amenable to an appraisal of the occurrence of complex roots by inspection of the discriminant Δ.

Thus the bicubic may be written in the form

$$\Omega^6-p\Omega^4+q\Omega^2+r=0, \quad (26)$$

where the symbols p, q, r denote

$$\begin{aligned} p&=V^2+2(W+1),\\ q&=(W+1)^2+V^2(W+1)-\mathfrak{U}^2, \quad (27)\\ r&=W\mathfrak{U}^2. \end{aligned}$$

The discriminant in terms of these quantities becomes

$$\Delta=b^2/4+a^3/27$$

with

$$a=\tfrac{1}{3}(3q-p^2),\quad b=(1/27)(2p^3-9pq+27r).$$

[11] One may detect possible instability for $(V^2+2)^2<V^4+4\mathfrak{U}^2$ which reduces to $V^2+1<\mathfrak{U}^2$, a physically unacceptable result since it implies $c\leqslant\mathfrak{U}$ and so must be rejected.

Since the domain $r\to 0$ reduces (26) to a biquadratic which has only real roots, complex roots must be sought for the regime $W\mathfrak{U}^2\to\infty$. Consider $W\to 0$ and $\mathfrak{U}\to\infty$ with $W\mathfrak{U}^2\to\infty$. This means

$$p\to V^2,\quad q\to-\mathfrak{U}^2$$

with the consequence

$$\begin{aligned} a&\to\tfrac{1}{3}(-3\mathfrak{U}^2-V^4)\\ b&\to(1/27)(2V^6+9V^2\mathfrak{U}^2+27W\mathfrak{U}^2). \end{aligned}$$

In particular where $V\gg\mathfrak{U}$ so that

$$\begin{aligned} &a\to-\tfrac{1}{3}V^4\\ &b\to(1/27)(2V^6+\epsilon),\quad \epsilon=9V^2\mathfrak{U}^2+27W\mathfrak{U}^2\\ &\Delta\cong[1/4(27)^2](4V^6\epsilon+\epsilon^2)>0, \end{aligned}$$

which means complex Ω are possible. Hence growing electromagnetic waves can occur for propagation at right angles to an ion-electron plasma which carries current.

That other regimes may also allow electromagnetic wave instabilities may be seen by posing the limiting behavior $W\to\infty$, $\mathfrak{U}\to 0$ and $W\mathfrak{U}^2$ still finite. Now

$$p\to 2W,\quad q\to W^2$$

whereby

$$a\to-\tfrac{1}{3}W^2,\quad b\to(1/27)W^6$$

and $\Delta\to 0$; the discriminant will actually become positive if $V\to\infty$.

CONCLUDING REMARKS

The foregoing analysis reveals the heretofore incomplete understanding of stream instabilities in plasmas which stemmed from the isolated search for waves propagating along the direction of the stream motion. By allowing for waves traveling in arbitrary directions relative to the stream, it has been shown that growing electromagnetic waves may arise even for the single-carrier plasma.

While much graphical and numerical analysis remains to be carried out in securing a detailed picture of the instabilities, the basic character of the generalized dispersion relation has been delineated to a point where recognition is given to the occurrence of traverse as well as longitudinal growing waves.

Thus the emission of radio-frequency and microwave oscillations from a plasma or electron beam has a perfectly natural explanation. The presence of a steady external magnetic field may be expected to further modify and complicate the picture set forth above; but it should be manifest that it is not an essential ingredient for the onset of transverse instabilities in current carrying plasmas. Solar radio noise may be thought of as a direct consequence of electromagnetic instabilities in streaming plasmas.