Effects of Sample Size and Statistical Weights on Fluctuations in the $C^{12}(C^{12},\alpha)Ne^{20}$ Reactions

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(Received 4 May 1964)

Absolute values of the differential cross sections for the reaction $C^{12}(C^{12},\alpha)Ne^{20}$ to the ground and 1.63-MeV states are given at 100-keV energy intervals between 20.3- and 25.6-MeV bombarding energy at eight equally spaced angles between 3 and 73° in the laboratory, as well as some more detailed angular distributions at selected energies. Both differential cross sections and angle-integrated data are shown to fluctuate strongly with energy. A quantitative comparison between the properties of the fluctuations and the predictions of the statistical theory of the compound nucleus (used in the spirit of Ericson) is presented. The effects of making observations on a sample of finite size are illustrated in detail and are shown in some cases to lead to results that are readily mistaken for evidence of direct interaction. Fluctuations of cross sections involving sums of several incoherent components of different weights are calculated and compared with experiment and with the usual estimates based on equal weights. The proper weight coefficients for the calculation are derived from the treatment of the average cross sections in an accompanying paper. Agreement is found in all cases between observations and the statistical compound-nucleus picture although the finite-sample effects allow a contribution to the cross sections from direct reactions of 30% or less.

1. INTRODUCTION

CONSIDERABLE interest has developed recently in the interpretation of the fluctuations observed in the energy dependence of heavy-ion reaction cross sections when these are studied with the high-resolution ion beams now available from tandem accelerators. Although, in general, the total reaction cross section (with one exception discussed below) increases monotonically[1] with energy, the yield of individual reaction channels in every case has shown prominent resonance-like fluctuations. Examples which are typical for the energy dependence of the differential cross sections are shown in Figs. 1, 2, and 3 for the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction in the range 10.15–12.8-MeV center-of-mass bombarding energy. Similar results for other heavy-ion reactions have been reported by a number of investigators; strong fluctuations in the yield of the reaction $Ne^{20}(\alpha,C^{12})C^{12}$ were observed by Lassen[2] and in the yield of $O^{16}(O^{16},\alpha)Si^{28}$ and $O^{16}(C^{12},\alpha)Mg^{24}$ by Evans *et al.*[3] The last system has been studied in detail by Halbert *et al.*[4] and the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction by Borggreen *et al.*[5] The only exception to the general behavior that has been observed to date is the $C^{12}+C^{12}$ system which, at energies near the Coulomb barrier value (~6-MeV c.m.), shows strong resonance-like fluctuations in the total reaction cross section as well as in individual channels[6] with peaks at the same energy in the various channels. This exceptional behavior led to the suggestion of so-called

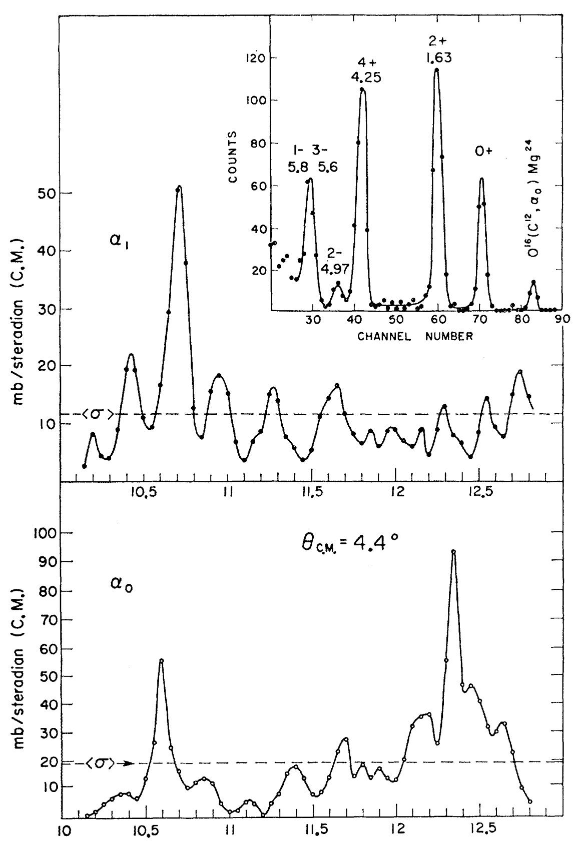

FIG. 1. The energy dependence of the differential cross sections of the reactions $C^{12}(C^{12},\alpha)Ne^{20}$ (ground and 1.63-MeV states) at 4.4°. The inset shows a pulse-height spectrum from the detector at 22.6-MeV bombarding energy. The average of the data over the whole energy interval is denoted by $\langle\sigma\rangle$.

[1] E. Almqvist, D. A. Bromley, and J. A. Kuehner, in *Proceedings of Second Conference on Reactions Between Complex Nuclei*, edited by A. Zucker, F. T. Howard, and E. C. Halbert (J. Wiley & Sons, Inc., New York, 1960), p. 282.

[2] N. O. Lassen, Phys. Letters 1, 65 and 161 (1962); N. O. Lassen and J. S. Olesen, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 33, No. 13 (1963).

[3] J. E. Evans, J. A. Kuehner, A. E. Litherland, and E. Almqvist, Phys. Rev. 131, 818 (1963).

[4] M. L. Halbert, F. E. Durham, C. D. Moak, and A. Zucker, Nucl. Phys. 47, 353 (1963).

[5] J. Borggreen, B. Elbeck, R. B. Leachman, M. C. Olesen, and N.O.R. Poulsen, *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963).

[6] E. Almqvist, D. A. Bromley, J. A. Kuehner, and B. Whalen, Phys. Rev. 130, 1140 (1963); Phys. Rev. Letters 4, 515 (1960).

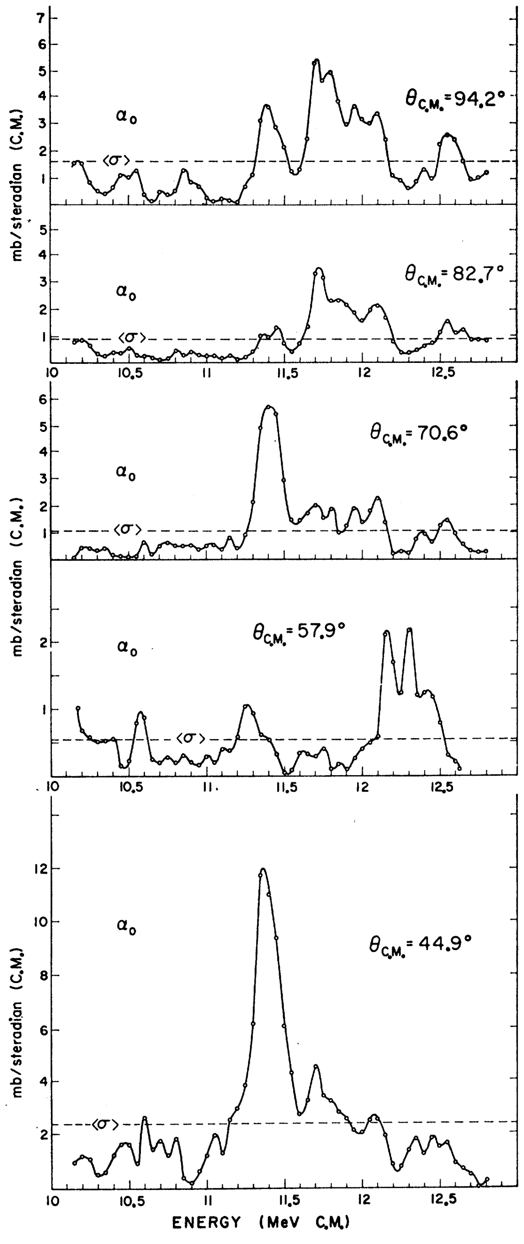

FIG. 2. The energy dependence of the differential cross sections of the reaction $C^{12}(C^{12},\alpha_0)Ne^{20}$ (gd state) at several angles is shown. All the data are in the center-of-mass system. The average of the data over the whole energy interval is denoted by $\langle\sigma\rangle$.

"quasimolecular" states[6,7] with unusually large widths for C^{12} emission for these particular resonances. The behavior of the $C^{12}+C^{12}$ reactions at higher bombarding energies, however, is similar to that of other heavy-ion reactions and is shown in this paper and the accompanying one to be generally consistent with the compound-nucleus picture (statistical theory) of nuclear reactions within the accuracy that the comparison of theory and experiment can be made. It should be noted that this accuracy is ultimately governed by the ratio $\Delta E/\Gamma$, where ΔE is the range of excitation energies over which studies are made and Γ is the average width of the overlapping compound-nucleus states as is discussed more fully later.

The consequences of assuming that the observed structure at higher energies results from unusually strong single resonances have been discussed briefly previously.[8] In the present paper our point of view of the high-energy data differs from that taken in the previous discussion in that the structure is regarded as a manifestation of statistical fluctuations of the type that was pointed out by Ericson[9] and by Brink and Stephen[10] to occur where a large number of compound-nucleus states with

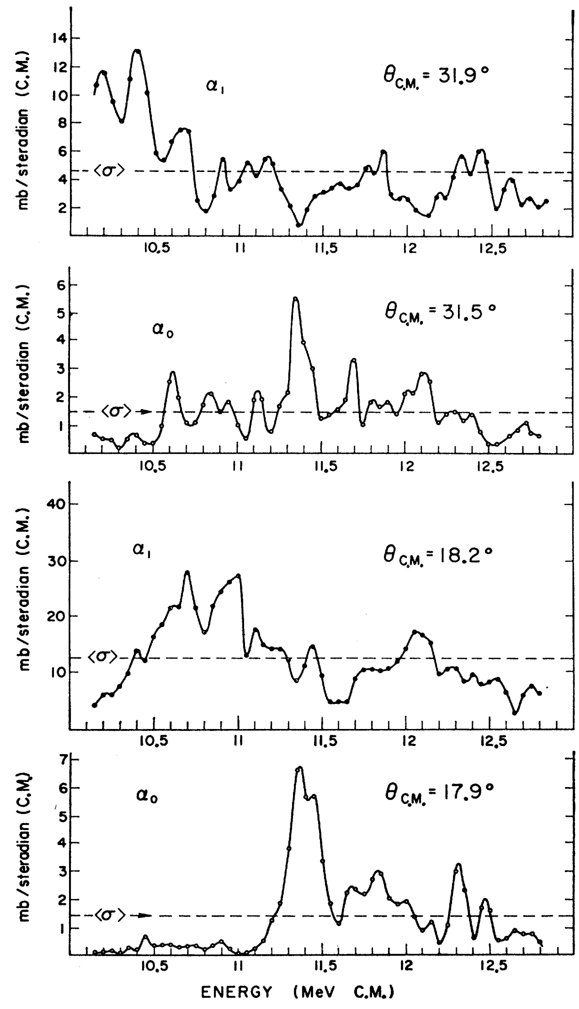

FIG. 3. The energy dependence of the differential cross sections of the reactions $C^{12}(C^{12},\alpha)Ne^{20}$ (gd state and 1.63-MeV state) is shown at several angles. All the data are in the center-of-mass system.

[7] E. W. Vogt and H. McManus, Phys. Rev. Letters **4**, 518 (1960).

[8] J. A. Kuehner, J. D. Prentice, and E. Almqvist, Phys. Letters **4**, 332 (1963).

[9] T. Ericson, Ann. Phys. (N. Y.) **23**, 390 (1963).

[10] D. M. Brink and R. O. Stephen, Phys. Letters **5**, 77 (1963).

randomly distributed amplitudes contribute to the cross section in a coherent fashion at each energy. Our discussion also explores the effects of having only a finite number of independent observations in the experimental sample and shows that they can be important in the comparison between experiment and the theoretical predictions. The simplicity of the $C^{12}+C^{12}$ system, which arises from the fact that the two particles are identical spinless bosons, reduces the number of effective quantum numbers and makes this system a particularly favorable one for testing the predictions of the theory of statistical fluctuations of cross sections.

The exact form of the distribution law governing cross-section fluctuations depends on the number of noncoherent partial waves and on the relative weights of their cross-section contributions. Some of the relative weights used in this paper to discuss the observed fluctuations cannot be measured, and are obtained from the average cross-section calculations of the accompanying paper[11] which discusses the application of the statistical compound-nucleus theory to heavy-ion reactions and compares the predicted *average* cross sections and angular distributions with experiment. In discussing the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction data, it has not been found necessary to include any direct reaction component in the theory and all the calculations in both this paper and the companion one assume a pure compound-nucleus reaction process.

2. EXPERIMENTAL ARRANGEMENT

The C^{12} ion beam from the Chalk River tandem was used to bombard a thin self-supporting carbon foil (10–15 μg/cm^2). Reaction alpha particles were detected in eight solid-state counters mounted behind collimating holes that were machined in a curved brass plate so as to be precisely located at 10° intervals about the target spot, and to lie in a plane containing the beam direction. This counter assembly could be rotated about the target to obtain intermediate sets of angles. The angular aperture was $\pm 0.7°$ in all cases except the three most forward holes which subtended $\pm 0.35°$; it is estimated that the angle setting was accurate to 0.1° relative values and 0.3° absolute value. The pulses from each of the eight counters were amplified by a transistorized preamplifier, and the eight outputs recorded simultaneously on the Chalk River 900-channel analyzer operated in the 8×100 channel mode.

The relative efficiencies of the eight counters were checked by mounting an Am^{241} alpha-particle source at the target location and comparing the counting rates in the eight counters. The efficiencies of all the counters were found to be identical within $\pm 2.5\%$ when the counting rates in the three small-aperture ($\pm 0.35°$) counters were multiplied by the normalization factor 4.17.

[11] E. Vogt, D. McPherson, J. A. Kuehner, and E. Almqvist, Phys. Rev. **136**, B99 (1964).

A total of ten equally spaced collimating holes was available of which eight were used with counters thus leaving one hole free at each end of the array. The zero of the angle scale was determined by observing the settings at which the beam passed centrally through the spare holes; then the array would be turned to give the chosen counter angles. The zero degree determinations were found to reproduce to within 0.3°.

The counters at forward angles were covered by sufficient aluminum foil to stop scattered C^{12} beam ions. The alpha-particle counts were observed to vanish when the target was removed, thus demonstrating that the counters, even when at 3°, could not see reaction products from $C^{12}+C^{12}$ reactions being produced by carbon build-up on the slit edges being struck by the beam. The inset in Fig. 1 shows a typical spectrum at 3°.

The spectrum selected for the inset in Fig. 1 clearly shows a peak from the reaction $O^{16}(C^{12},\alpha_0)Mg^{24}$ (ground state) caused by oxygen contamination of the carbon target. In fact this spectrum happens to show the contaminant peak exceptionally strongly; the average intensity of the $O^{16}(C^{12},\alpha_0)Mg^{24}$ reaction was observed to be 3.6% of the average for the $C^{12}(C^{12},\alpha_0)Ne^{20}$ reaction. This measured average combined with computed[11] compound-nucleus values of the cross sections for the two reactions leads to the estimate that the number of O^{16} nuclei in the target is equal to 10% of the number of C^{12} nuclei on the average. This number can then be used together with computed cross sections to estimate the yield of other groups from the reaction $O^{16}(C^{12},\alpha)Mg^{24}$ to various excited states. In particular, alpha particles feeding the doublet of levels in Mg^{24} at 4.12- and 4.23-MeV excitation are not clearly resolved from the alpha group corresponding to the 1.63-MeV state of Ne^{20}. It is therefore important to know whether their intensity makes a noticeable contribution to the observed yield of this group. The estimated intensity of the $O^{16}(C^{12},\alpha)$-Mg^{24} (4.12+4.23 MeV) reactions is 5% of the average $C^{12}(C^{12},\alpha_1)Ne^{20}$ (1.63 MeV) intensity for our conditions. A similar magnitude is obtained using Halbert's[4] measured cross sections. The $C^{12}(C^{12},\alpha_0)Ne^{20}$ (ground-state) reaction is clearly resolved from contaminant groups.

A monitor counter observed the elastic scattering at 45° to the beam. These data allowed an estimate of the rate of increase of the target thickness due to carbon build up and appropriate corrections for this effect to be made. The elastic scattering also shows an O^{16} contamination of a magnitude in agreement with the previous estimate.

The ratios of the alpha-particle cross sections to the Coulomb scattering cross section were determined by removing the foil in front of the one counter and observing the elastic scattering at 45° and 11-MeV bombarding energy. The measured ratios together with the calculated Coulomb cross section gave the absolute value of the alpha-particle cross sections.

3. EXPERIMENTAL RESULTS

Angular distributions of alpha particles leading to the ground state of Ne^{20} at four selected energies are shown in Fig. 4. The error bars shown are the standard deviations reflecting counting statistics; no systematic errors have been included. The curves in Fig. 4 are least-squares fitted angular distribution functions[12] of the form for a reaction involving only spin-zero particles.[13]

$$W(\theta)=\left|\sum_{\text{even}\,l}^{l_{\max}} A_l(2l+1)^{1/2}P_l(\cos\theta)\right|^2, \qquad (1)$$

where A_l are complex numbers and represent the reaction amplitudes. Only even values of $l(=J$ of the compound nucleus) are allowed for a reaction between identical spin-zero particles. For the angular distributions at center-of-mass energies of 10.6, 10.75, and 11.4 MeV, $l_{\max}=8$ was found to provide a satisfactory fit. For the angular distribution at 12.35 MeV, it was necessary to use $l_{\max}=10$ to provide an adequate fit.

The angular distribution function (1) was used to fit

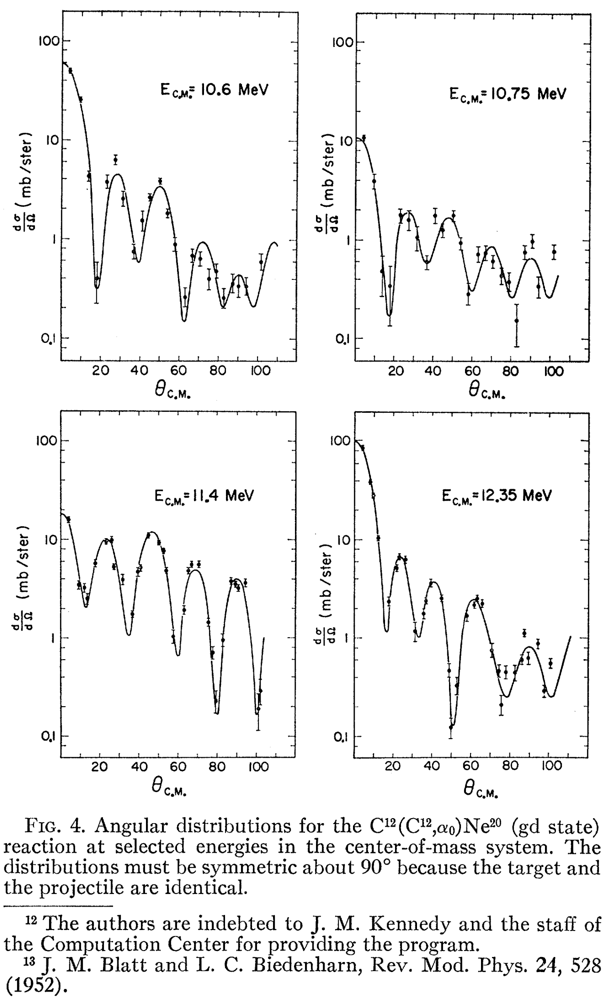

FIG. 4. Angular distributions for the $C^{12}(C^{12},\alpha_0)Ne^{20}$ (gd state) reaction at selected energies in the center-of-mass system. The distributions must be symmetric about 90° because the target and the projectile are identical.

[12] The authors are indebted to J. M. Kennedy and the staff of the Computation Center for providing the program.

[13] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. 24, 528 (1952).

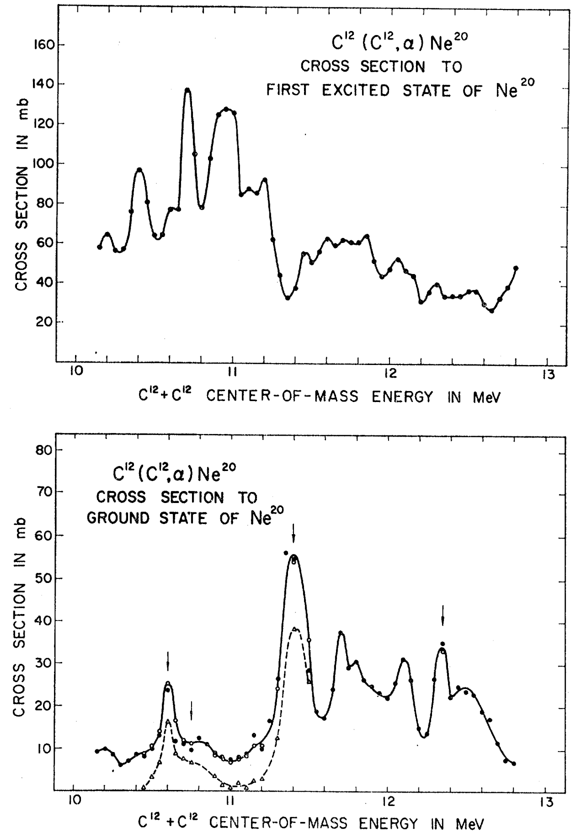

FIG. 5. Angle-integrated cross sections. The dashed curve has been drawn through the data points for the partial cross section resulting from compound states of spin 8 going to the Ne^{20} ground state. The significance of closed and open circles is discussed in the text; the curves have no significance except to guide the eye. Arrows indicate the energies corresponding to the angular distributions of Fig. 4.

the data only at energies where the yield at a large numbers of angles ($\gtrsim 23$) was measured. For these cases the angle-integrated cross sections are the quantities $4\pi\sum_l|A_l|^2$ given by the above fit.

In addition, the angle-integrated cross section was investigated over a wider range of energies for the reactions leading to both the ground state and the first excited state of Ne^{20} using data at only eight angles. These data were fitted at each energy with a Legendre polynamial series of the form

$$W(\theta)=\sum_{\text{even}\,k}^{k_{\max}=12} B_kP_k(\cos\theta). \qquad (2)$$

The cross sections obtained by the use of the function (1) are shown by the open circles and those from function (2) by the solid points in Fig. 5. A comparison at the energies where the number of angular measurements allowed both angular distribution functions to be used suggests that the angle-integrated cross section is reasonably well determined from only eight angular measurements.

TABLE I. Average values of the measured differential cross sections in the center-of-mass system.

$C^{12}(C^{12},\alpha_0)Ne^{20}$ (gd state) θ[a] (deg)	$d\sigma/d\Omega$ (mb/sr)	$C^{12}(C^{12},\alpha_1)Ne^{20}$ (1.63 MeV) θ[a] (deg)	$d\sigma/d\Omega$ (mb/sr)
4.4	19.2	4.4	11.8
17.9(17.8–18.0)	1.41	18.2(18.1–18.2)	12.7
31.5(31.4–31.6)	1.48	31.9(31.8–32.0)	4.63
44.9(44.7–45.1)	2.46	45.5(45.4–45.6)	5.51
57.9(57.7–58.1)	0.55	58.8(58.7–58.9)	3.80
70.5(70.2–70.9)	1.07	71.6(71.4–71.8)	4.58
82.7(82.3–83.0)	0.88	83.9(83.7–84.0)	3.05
93.2(93.8–94.5)	1.62	95.3(95.1–95.6)	3.44

[a] Mean angle in the center-of-mass system: the angle changes through the range in brackets from the lowest to the highest bombarding energy.

It is important to note that the angular distribution function (1) can be shown to yield $2^{(-1+\frac{1}{2}l_{\max})}$ different sets of coefficients, all of which give identical fits to the experimental data. However, the amplitude coefficient for $l_{\max}$ is always determined uniquely. Since an $l_{\max}$ of 8 was found to give an adequate fit at all energies except 12.35 MeV, it was possible to obtain the partial cross section for the ground-state reaction, $\sigma_8=4\pi|A_8|^2$, due to spin-8 states of Mg^{24} over the energy range 10.45 to 11.5 MeV, where sufficient data were available to use expression (1). The cross sections σ_8 are shown by triangles in the bottom half of Fig. 5.

The angle-integrated cross sections shown in Fig. 5 represents the results of measurements at eight different angles at each of fifty-four equally spaced bombarding energies in the range 10.3 to 12.8 MeV. These same data are presented in a different way in Table I. Here the eight differential cross sections shown are the values obtained by averaging the data for each angle over the fifty-four energies that were studied. These average cross sections are compared with the results of a compound-nucleus computation in the accompanying paper.[11]

It should be noted that the measurements were made at eight fixed laboratory angles at 10° intervals in the range 3 to 73°. The corresponding angles in the center-of-mass system change systematically with energy—the maximum resulting spread is shown in brackets in Table I.

TABLE II. Measured values of the absorption cross section for $C^{12}+C^{12}$. Estimated accuracy $\pm 25\%$. The collision energy is given in the center-of-mass system.

$E_{c.m.}$ (MeV)	σ_A (mb)	$E_{c.m.}$ (MeV)	σ_A (mb)
5.0	7.7	7.0	256.0
5.56	27.0	7.5	340.0
5.72	57.0	8.0	370.0
5.80	46.0	9.0	627.0
5.88	39.0	10.0	826.0
5.90	51.0	11.0	865.0
6.00	72.0	12.0	872.0
6.10	65.0	12.5	1020.0
6.15	56.0		
6.20	107.0		

An attempt was also made to determine the total reaction cross section by detecting all charged particles and measuring the angle-integrated yield. The measurements were made at selected energies over the range 5.0–12.5-MeV collision energy in the center-of-mass system. Similar studies which were made for a number of other heavy-ion systems are being published separately.[14] The data for the $C^{12}+C^{12}$ system which are summarized in Table II are compared with the results of an optical-model calculation in the accompanying paper. The measured values of the total cross sections contain a systematic uncertainty that arises (i) from the fact that reactions can occur to unbound excited states resulting in two charged particles per event, (ii) from the fact that some reaction products are not detected because their range is too short to penetrate the foil used to stop scattered beam ions from reaching the detector, and (iii) from the fact that neutrons were not detected. The effect of (i) is in the opposite direction to (ii) and (iii). The combined uncertainties are estimated to yield a probable error of $\pm 25\%$ in the absolute values of the cross sections.

4. COMPARISON WITH FLUCTUATION THEORY

A. General

In the accompanying paper[11] the compound-nucleus model with some plausible assumptions is shown to give a fairly good account of all the *average* properties of the reaction $C^{12}(C^{12},\alpha)Ne^{10}$ for C^{12} energies in the range 10.1–12.8 MeV. In the following discussion the same model is used in the spirit of Ericson[9] and of Brink and Stephen[10] to compute the statistical properties of the cross-section fluctuations which are then compared with the measured properties.[15] The application of the model to the data is straightforward. A cross section in general can be written as a sum of incoherent partial cross sections, each of which has a probability distribution law that is exponential. The number of such incoherent partial cross sections is henceforth referred to as N. The value of N, together with the relative weights C_j of the partial cross sections, determines the distribution law of the observed cross sections. The accuracy of the comparison between the predicted distribution law and the measured fluctuations depends on the "sample size" which is henceforth referred to as S. The value of S is taken to be the ratio $\Delta E/\Gamma$ of the energy interval ΔE, under consideration to the average width Γ of the compound states. A more detailed discussion of the sample size is given in Sec. 4B. It should be noted that

[14] J. A. Kuehner and E. Almqvist, Phys. Rev. **134**, B1229 (1964).

[15] J. P. Bondorf and R. B. Leachman (to be published) have applied fluctuation theory to $C^{12}(C^{12},\alpha)Ne^{20}$ reaction data obtained at Copenhagen and also find essential agreement with the theory.

the level density (or spacing D) has no relevance to the effective sample size provided that $\Gamma/D \gg 1$.

B. Level Widths and Spacings

In published treatments[9,10] of the statistical theory of fluctuations, the total width is assumed to be the same for all levels of the compound nucleus—an assumption which appears justified to a large extent for the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction as is discussed in more detail later, and which is implicit in all the discussions of this paper. Since the theory allows cross-section observations to be counted as independent only if separated in energy by more than this width, its value determines the statistical independence of any two results observed within a finite energy range ΔE; the maximum possible number of *independent* observations within this range being $\Delta E/\Gamma$, where Γ is the level width. Hence, the definition of sample size S equals $\Delta E/\Gamma$. As in the accompanying paper[11] on average cross sections, we shall distinguish sample averages of quantities f from true averages (for infinite S) by denoting the former by $\langle f \rangle_S$ and the latter by either $\bar{f}$ or equivalently $\langle f \rangle$.

The level width can be estimated in several ways from the experimental results: (i) the most direct, but also somewhat subjective way is to equate the directly measured widths of the fluctuations with the level width; (ii) a correlation analysis[9,10] for the same cross section at different energies, as, for example, was carried out by Halbert *et al.*,[4] can yield the level width, and (iii) the average number M of maxima per unit energy interval in the cross-section data, which is directly related to the level width through relations given by Brink and Stephen.[10] These relations contain a factor which depends on the number of independent channels N that contribute to the cross section in question. For the differential cross sections to the ground state, N is 1; for the first excited state, the exact value of N is angle-dependent but, in general, is close to 3 for the angles used here (see Sec. D). The relations of Brink and Stephen then become

$$\Gamma = 0.5/M = 104 \text{ keV from the data for } Ne^{20} \text{ (ground state)},$$

$$\Gamma = 0.375/M = 88 \text{ keV from the data for } Ne^{20} \text{ (1.63 MeV)}.$$

The number of maxima M was obtained from plots of the experimental differential cross sections, such as are illustrated in Figs. 1, 2, and 3. The experimental resolution of 30 keV will not contribute appreciably to the observed width, and no correction for its effect has been made. From the angle-integrated data of Fig. 5, the values

$$\Gamma = 0.375/M = 110 \text{ keV for the data for the } Ne^{20} \text{ (ground state)},$$

$$\Gamma = 0.355/M = 76 \text{ keV for the data for the } Ne^{20} \text{ (1.63-MeV state)},$$

are obtained. Values of $N \approx 3$ and 9, respectively, were assumed in the two cases for reasons that will be discussed more fully in Sec. D. Since angle-integration removes interference between compound states of different spin J, the values of Γ obtained in this case apply to those spin values that contribute predominantly to the cross section. These are shown in Ref. 11 to be spin-6 and -8 states. The situation is somewhat less clear for differential cross sections because here interference between different spin values J plays a very important role in the fluctuations. Nevertheless, the above analysis reveals no significant difference in the results obtained from the two sets of data, and supports the assumption that all levels have nearly equal widths. The mean of the data presented above yields the width Γ equal to 95 keV.

Earlier studies[2,5,6,8] based on direct measurements of peak widths have yielded somewhat larger values of the width Γ. For the purpose of numerical computations in this paper a value of Γ equal to 120 keV has been adopted. This yields the sample size S to be 22 for the present data, and this value of S applies hereafter except when otherwise indicated. Since the statistical effects of the finite sample size depend approximately on $S^{1/2}$, a 25% change in this number does not result in any significant change in the results.

The justification for assuming equal widths for all the participating levels of the compound nucleus in the present case rests on two facts. Firstly, the total width of a state of given angular momentum J and parity π of Mg^{24} at high excitation is a sum of a large number (>100) of independent partial widths, each corresponding to one of the available decay channels. The sum, therefore, is expected to fluctuate very little about its mean value. Secondly, the systematic differences in the average width because of differences in the spin J among levels are not important because the $C^{12}(C^{12},\alpha)$-Ne^{20} reaction in the energy range of interest here involves predominantly compound levels of similar spin—6 or 8 units of angular momentum.[11]

Another assumption implicit in the theory of fluctuations is that a "large" number of compound levels contribute to the reaction amplitude at each energy—i.e., that $\Gamma/D \gg 1$, where D is the average level spacing. Estimates of this ratio made in the accompanying paper[11] are 21 and 7 for spins 6 and 8, respectively. The level spacing for spin-6 and -8 levels can also be estimated from known spacings at lower energies (see, for example, Ref. 5) and combined with the value $\Gamma \approx 120$ keV to yield a value for the number of overlapping levels in the compound system. The result is consistent in magnitude with the estimates already made.

C. Angle-Integrated Partial Cross Sections

In Sec. 3 it was shown that the $J=8$ part of the angle-integrated cross section for the ground-state reaction could be extracted from the experimental data. This

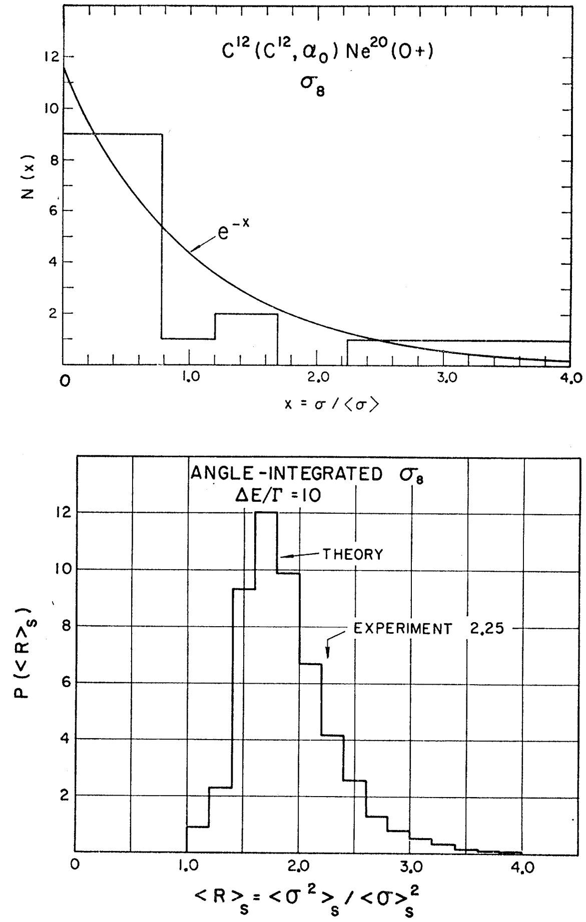

FIG. 6. The upper curve compares the observed and predicted exponential probability distributions for the spin-8 data of Fig. 4 (note that one event occurred at 4.2 on that abscissa which is off the end of the figure). The lower histogram is the frequency distribution of the self-correlation ratio $\langle R\rangle_S$ equals $\langle\sigma^2\rangle_S/\langle\sigma\rangle_S^2$ for this same partial cross section obtained from a Monte Carlo calculation taking 10^4 samples each containing 10 events taken randomly from an exponential distribution. The ordinate has arbitrary normalization. The probability of observing a value equal to or greater than 2.25 is 13%.

partial cross section, which is shown in Fig. 5 represents one of the simplest cases to which the theory of cross-section fluctuations can be applied since it involves no sums over angular momentum quantum numbers and can hence be written as an absolute square. Because the incident particles are identical spin-zero bosons, only even values of spin J of the compound states are allowed and a factor of 2 appears in the expression for the cross section for each partial wave:

$$\sigma_{J=8}=\frac{2\pi}{k^2}(2J+1)\left|\sum_\lambda \frac{\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda\alpha_0}^{1/2}}{E_\lambda-E-(i/2)\Gamma_\lambda}\right|^2 \equiv\frac{2\pi}{k^2}(2J+1)\,|a_J+ib_J|^2=\frac{2\pi}{k^2}(2J+1)(a_J^2+b_J^2)\,, \quad (3)$$

where a_J and ib_J are the real and imaginary parts of the sum over Breit-Wigner amplitudes. If the products $\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda\alpha_0}^{1/2}$ have a Gaussian distribution[16] about a mean-value zero and if

$$\langle(\Gamma_\lambda/D)_J\rangle_{\rm av}\gg 1\,, \quad (4)$$

then a_J and b_J each have independent Gaussian distributions also about the mean-value zero. Moreover, the averaging of Eq. (3) over an energy interval large compared to Γ_λ can be shown to yield[9,11]

$$\langle a_J^2\rangle_{\rm av}=\langle b_J^2\rangle_{\rm av}\,. \quad (5)$$

Therefore, a_J^2 and b_J^2 each have a χ-squared distribution with one degree of freedom and the corresponding probability distribution of σ_J is a χ-squared distribution of two degrees of freedom.[16] The probability P of observing any particular value of the ratio $\sigma_J/\bar\sigma_J$ is then

$$P(\sigma_J/\bar\sigma_J)d\sigma_J=e^{-\sigma_J/\bar\sigma_J}d\sigma_J\,, \quad (6)$$

where $\bar\sigma_J$ is the mean value of the fluctuating cross section. The exponential distribution law Eq. (6) holds for each partial cross section which can be written as a single absolute square as in Eq. (3). The distribution law for more general cross sections which are weighted sums of N such absolute squares is derived in Sec. D.

The exponential law is compared to the observed results in the upper half of Fig. 6. The histogram in the figure was obtained by dividing the cross-section scale (shown in units $x=\sigma/\langle\sigma\rangle_S$) into equal parts or "channels" and counting the number $N(x)$ of observed cross-section values that fell within each channel. For this purpose cross-section values were read off the dashed curve in Fig. 5 at points spaced 50 keV apart in energy. Throughout the paper a "channel" width of 0.46 units of x is used except for the first channel which is 0.31 units wide but is appropriately normalized.

In order to discuss sample-size effects we now introduce the following notation:

$$R_0=\sigma_J^2/\langle\sigma_J\rangle_S^2,$$
$$\langle R_0\rangle_S=\langle\sigma_J^2/\langle\sigma_J\rangle_S^2\rangle_S, \quad (7a)$$
$$\langle\langle R_0\rangle_S\rangle\equiv\langle\langle\sigma_J^2/\langle\sigma_J\rangle_S^2\rangle_S\rangle +1+\langle\langle(\sigma_J-\langle\sigma_J\rangle_S)^2\rangle_S/\langle\sigma_J\rangle_S^2\rangle =1+(S-1)/S\rightarrow 2 \text{ as } S\rightarrow\infty\,, \quad (7b)$$

where S (equal $\Delta E/\Gamma$) is the sample size. The brackets with subscript S indicate a sample average. The result (7b) follows directly from Eq. (6) and definition (7a) if $\langle\sigma_J\rangle$ is a constant.[17]

In general, $\langle\sigma_J\rangle(\approx\langle\sigma_J\rangle_S)$ is a slowly varying function of the bombarding energy because of the $1/k^2$ term in Eq. (3) and the barrier penetrability factors which are implicit in the widths. In this case the mean ratio $\langle\langle R_0\rangle_S\rangle$ given above is related to the measured mean

[16] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).
[17] R. D. Evans, *The Atomic Nucleus* (McGraw-Hill Book Company, Inc., New York, 1955), p. 761.

value, $\langle\langle R\rangle_S\rangle$ as follows:

$$\langle\langle R\rangle_S\rangle=\langle\langle\sigma^2\rangle_S/\langle\sigma\rangle_S{}^2\rangle\cdot\langle\langle R_0\rangle_S\rangle, \qquad (8)$$

in which it is assumed that $\langle\langle R_0\rangle_S\rangle$ is independent of the bombarding energy. In this paper the value of the first term in Eq. (8) was obtained from statistical theory computation[11]; in all cases considered, the effect of the energy dependence of $\bar{\sigma}$ is such as to make this term equal to unity within $\pm 3\%$. Thus, $\langle\langle R\rangle_S\rangle$ is equal to $\langle\langle R_0\rangle_S\rangle$ for the purpose of this paper.

For the small values of sample size discussed in this paper the fluctuations of quantities about their mean value are large. To obtain the probability distribution of $\langle R\rangle_S$ about its mean value, we use a Monte Carlo method and the Chalk River G-20 computer. The quantity calculated is the probability $P(\langle R\rangle_S)$ where

$$\langle R\rangle_S\equiv S\sum_{i=1}^{S}(x_i)^2/(\sum_{i=1}^{S}x_i)^2, \qquad (9)$$

with x_i being numbers chosen randomly from an exponential distribution

$$P(x_i)=e^{-x_i}. \qquad (10)$$

The result for the present case is shown in Fig. 6. It assumes $S=10$ for the 1-MeV interval over which σ_8 was measured (10^4 sets of ten random numbers were found to yield the distribution adequately).

It should be pointed out that the interval near 11 MeV (see Fig. 5) was selected for study because it was known to contain the strong peak previously reported by Lassen[2] in the inverse reaction. It is therefore not surprising that this selected interval yields the somewhat large value of $\langle R\rangle_S=2.25$. The probability of observing a value of $\langle R\rangle_S$ that exceeds the mean by more than this result is 0.13. The likelihood of the measurement is sufficiently large, under the circumstances, as to constitute reasonable agreement with the predictions of the fluctuation theory.

Note that a partial cross section such as this is perhaps the simplest quantity to which the theory of cross-section fluctuations can be applied since it involves compound states of only a single value of spin J. However, the experimental determination of a partial cross section is possible only in special circumstances and in the remainder of the paper we deal with the more directly measurable cross sections. The simplest of these is the differential cross section to the Ne^{20} ground state which, although it involves coherent contributions from compound states of many values of spin, is also an $N=1$ case. This example is discussed next.

D. Differential Cross Sections

For a nuclear reaction involving only spin-zero particles, the differential cross section has the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\alpha'\alpha}=(\pi/k^2)|\sum(2J+1)^{1/2}Y_{J0}(\theta,\phi)U_{\alpha'\alpha}{}^J|^2, \qquad (11)$$

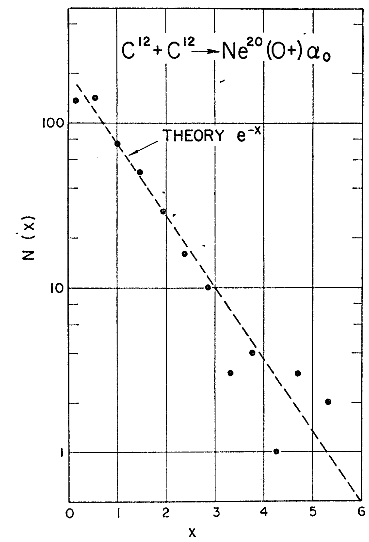

FIG. 7. A comparison of the predicted and observed probability distributions of the differential cross sections for the ground-state reaction. The observed frequency of occurrence per "channel," $N(x)$, is shown as points at the center of each channel. The quantity x equals $\sigma/\langle\sigma\rangle_S$. Each "channel" is 0.46 units wide along the abscissa except the first "channel" which is 0.31 units wide, but has been appropriately normalized. The figure contains data from all eight measured angles.

provided that the reaction channels α and α' are different. For a given set of angles θ and ϕ, Eq. (11) may be written

$$d\sigma/d\Omega_{\alpha'\alpha}=(\pi/k^2)|a+ib|^2, \qquad (12)$$

where a and ib are the real and imaginary parts of the sum on the right side of Eq. (11). In this case the sums, unlike those of the previous section, involve levels of different spin J. The compound-nucleus theory for many overlapping levels assumes that the probability amplitudes $U_{\alpha'\alpha}{}^J$ have a Gaussian distribution about a mean-value zero; from this assumption it follows directly that the discussion of the quantities a and b in Eq. (3) applies equally to Eq. (12), and that the probability distribution of the differential cross section in this example also has the exponential form of Eq. (6).

Defining x to be the ratio $(d\sigma/d\Omega)/\langle d\sigma/d\Omega\rangle$, we have, therefore, that the probability of observing a particular value of x at a given angle is

$$P(x)dx=e^{-x}dx, \qquad (13)$$

for any reaction with all spin-zero particles. This result is plotted in Fig. 7 together with the experimental data. The accuracy of this comparison in terms of sample size will be discussed in Sec. E.

If the intrinsic spins are not all zero, the differential cross section is not a simple square, as in Eq. (12) and the distribution law differs from Eq. (13). For a general differential cross section, sums over channel spins and magnetic substates of the channel spins occur outside the square to yield the cross section

$$\frac{d\sigma_{\alpha\alpha'}}{d\Omega}=\frac{1}{(2I+1)(2i+1)k^2}\sum_{s=(I-i)}^{I+i}\sum_{m_s=-s}^{s}\sum_{s'=(I'-i')}^{I'+i'}\sum_{m_{s'}=-s'}^{s'}\times|\sum_{J,l,l'}\pi^{1/2}(2l+1)^{1/2}(lsm_lm_s|Jm_J)\times(l's'm_{l'}m_{s'}|Jm_J)i^{l'}\times Y_{l'm_{l'}}(\theta,\phi)U_{\alpha sl;\alpha's'l'}{}^J|^2, \quad (14)$$

with

$$m_l=0,\quad m_J=m_s,\quad m_{l'}=m_s-m_{s'}. \quad (15)$$

In Eq. (14) α and α' are, respectively, the incoming and outgoing channel, I and i the intrinsic spins of the two particles in the incoming channel, l the incoming orbital angular momentum, s the incoming channel spin ($\mathbf{s}=\mathbf{I}+\mathbf{i}$) and m_l and m_s the components of $\mathbf{l}$ and $\mathbf{s}$ along the beam direction. Primed quantities refer to the outgoing channel. The $Y_{l'm_{l'}}$ are normalized spherical harmonics and the $U_{\alpha sl;\alpha's'l'}{}^J$ components of the collision matrix. The coefficients $(lsm_lm_s|Jm_J)$ are Clebsch-Gordan coefficients. The value of the differential cross section Eq. (14), averaged over fluctuations of the collision matrix components is according to the statistical theory of nuclear reactions[11]:

$$\left\langle\frac{d\sigma_{\alpha\alpha'}}{d\Omega}\right\rangle=\frac{1}{4k^2(2I+1)(2i+1)}\sum_{s,s',m_s,m_{s'}}\sum_{J,l,l',L}(2l+1)(2l'+1)(lsm_lm_s|Jm_J)^2(l's'm_{l'}m_{s'}|Jm_J)^2(ll'00|L0)$$
$$(l'l'm_{l'}-m_{l'}|L0)P_L(\cos\theta)T_{ls}{}^{J\pi}(\alpha)T_{l's'}{}^{J\pi}(\alpha')/\sum_{\alpha''l''s''}T_{l''s''}{}^{J\pi}(\alpha''), \quad (16)$$

where the magnetic quantum numbers again satisfy Eq. (15) and where $T_{ls}{}^{J\pi}(\alpha)$ are transmission functions discussed in the accompanying paper.[11] Equation (16) differs from the usual form of the average cross section in that the sum over magnetic substates of the channel spins is not carried out and that, consequently, the usual Z coefficients are replaced by products of Clebsch-Gordan coefficients.

For the present case of identical spin-zero bosons in the incident system, only even values of l and J are allowed, and the cross sections Eqs. (14) and (16) are to be multiplied by a factor of 2. Also,

$$I=i=s=m_s=0. \quad (17)$$

For alpha-particle emission we have further

$$i'=0,\quad s'=I'. \quad (18)$$

Because the compound states for the $C^{12}+C^{12}$ reactions are restricted to even values of total angular momentum and parity, the values of l' for any given final state in Ne^{20} must be even if that state has even parity and odd otherwise. As a result it follows that in Eq. (14) the terms for $+m_{I'}$ and $-m_{I'}$ are identical. The cross section may then be written

$$\frac{d\sigma_{\alpha\alpha'}}{d\Omega}=\frac{2}{k^2}(|q_0|^2+2\sum_{m_{I'}=1}^{I'}|q_{m_{I'}}|^2) \quad (19a)$$

$$\equiv\sum_j^N C_jx_j, \quad (19b)$$

where

$$x_j\equiv|q_{m_{I'}}|^2/\langle|q_{m_{I'}}|^2\rangle_{\text{av}}. \quad (19c)$$

The quantity $q_{m_{I'}}$ is the term under the absolute square in Eq. (14). The mean value $\langle|q_{m_{I'}}|^2\rangle_{\text{av}}$ and the weight coefficients C_j are given by the statistical theory calculation as in Eq. (16). Both quantities are angle-dependent.

The probability distribution of $d\sigma_{\alpha\alpha'}/d\Omega$ can be derived from Eqs. (19), and the basic distribution law Eq. (13) of each of the x_j; because the weight coefficients C_j are functions of the angle, the probability distribution of $d\sigma_{\alpha\alpha'}/d\Omega$ is also angle-dependent.

The general solution of the problem is the following. If

$$\frac{d\sigma_{\alpha\alpha'}}{d\Omega}\equiv y=\sum_{j=1}^N C_jx_j, \quad (20)$$

and the probability

$$P(x_j)dx_j=e^{-x_j}dx_j, \quad (21)$$

and the C_j are weight constants, then the probability

$$P(y)dy=\int_A\prod_{j=1}^N e^{-x_j}dA=\sum_{j=1}^N\left[C_j^{N-2}e^{-y/C_j}/\prod_{j=1,j\neq i}^N(C_j-C_i)\right], \quad (22)$$

where A is the surface of constant y in the space of the coordinates x_j. The result Eq. (22) is easily evaluated for any special case such as the reaction to the first excited state in Ne^{20}, where only three terms occur in the sums, Eqs. (20) and (22).

Selection rules arising from angular momentum and parity conservation in some cases lead to simplications of Eq. (22). At 0° the spherical harmonics $Y_{lm}(0,\phi)$ vanish for $m\neq0$ so that at this angle the cross section, Eq. (19a), contains only the first term q_0. Consequently, the probability distribution of the cross sections at 0° has exactly the exponential form of Eq. (13) for any reaction of the type $(0+)+(0-)\rightarrow J\pi+(0+)$. Note in this case also that at 0°, the cross section, Eq. (14), contains the factor $(l'I'00|l0)$ which vanishes unless the sum $I+l'+l$ is even. This leads to the selection rule that at 0° the cross section vanishes unless the parity of the residual state equals $(-)^{I'}$ and only reactions to this type of state can be studied.

At 90° all spherical harmonics of the form $Y_{\text{even,odd}}$ go to zero. This feature reduces the number of terms in the summation of Eq. (19a) by a factor of 2 and the differential cross section at 90° is a sum of $1+(I'/2)$ independently fluctuating terms.

In general the relative weighting of the various terms in either Eqs. (14) or (19) can not be predicted without the use of a model for the reaction mechanism. If a statistical compound-nucleus process such as discussed in Ref. 11 is assumed, the results are those shown in Fig. 8 for the reaction $C^{12}(C^{12}\alpha_1)Ne^{20}(2+)$. These results have been used in Eq. (22) to compute the probability distribution for each angle at which measurements were made, and the average over the eight angles is plotted in Fig. 9, together with the results of averaging all the experimental data for the same eight angles. For comparison the distribution given by assuming three equally weighted partial cross sections is also shown. This assumption is seen in Fig. 9 to approximate the true situation for the angles of concern in this experiment.

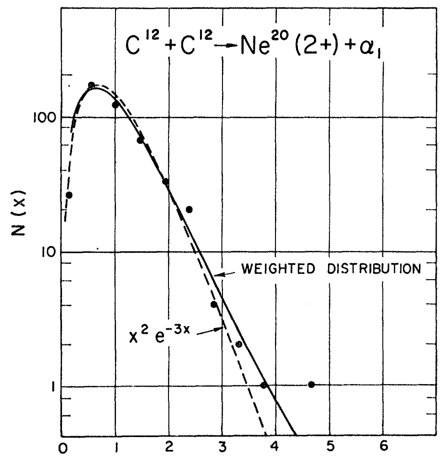

FIG. 9. A comparison of the predicted and observed probability distribution of the differential cross section for the 1.63-MeV state in Ne^{20}. See the caption of Fig. 7 for the description of the method of presenting the data. The dashed curve is the prediction assuming that three equally weighted partial cross sections contribute to the observed differential cross sections. The solid line assumes the weights of Fig. 8.

E. Correlation Ratios

(i) *General*

In the discussion of the angle-integrated cross section σ_8 (Sec. C), the quantity $\langle R\rangle_S$ equal to $\langle\sigma^2\rangle_S/\langle\sigma\rangle_S^2$ was introduced. It forms a convenient parameter for comparing the experimental data with the predictions of fluctuation theories. Any theory that yields the form of the cross-section probability distributions also, in principle, allows the value of $\langle\langle R\rangle_S\rangle$ and the probability distribution $P(\langle R\rangle_S)$ to be calculated. The probability distributions of $\langle R\rangle_S$ were obtained through Monte Carlo calculations from the cross-section distributions $P(x)$ already discussed, and taking account of the finite number S of independent observations in the experimental sample. More precisely, the quantity whose distribution is calculated by the Monte Carlo method is

$$\langle R\rangle_S=S\sum_{i=1}^{S}(\sum_{j}^{N}C_jx_{ij})^2/(\sum_{i=1}^{S}\sum_{j}^{N}C_jx_{ij})^2, \qquad (23)$$

where the x_{ij} are numbers chosen at random from an exponential distribution; the C_j are the weighting factors defined in the previous section.

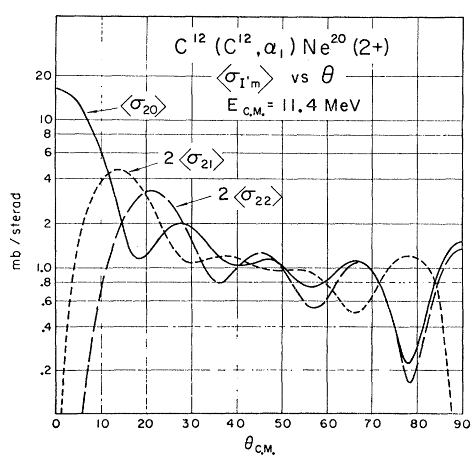

FIG. 8. The predicted angular distributions of the "weight factors" for cross sections to particular magnetic substates of the spin-2 state of Ne^{20} at 1.63 MeV. All quantities are in the center-of-mass system. The "weight factors," $C_m=\langle\sigma_{I'm}\rangle$, are defined in the text.

In the discussion that follows, a definition of the correlation ratio which allows cross-correlations as well as self-correlations is used.

$$\langle R\rangle_S=\langle\sigma_a\cdot\sigma_b\rangle_S/\langle\sigma_a\rangle_S\langle\sigma_b\rangle_S, \qquad (24)$$

where σ may be either a differential cross section or an angle-integrated cross section; the subscripts a and b may refer either to different angles or to different reactions or to both. If a and b are the same a self-correlation results such as was already discussed for σ_8 in Sec. C.

(ii) *The Reaction* $C^{12}(C^{12},\alpha_0)Ne^{20}(0+)$

The differential cross sections at any given angle for a reaction involving only spin-zero particles obeys the distribution law of Eq. (13), and hence, is predicted to yield a mean value for the self-correlation ratio equal to that given by Eq. (7). This prediction is compared with experimental results for the reaction $C^{12}(C^{12},\alpha_0)Ne^{20}$ ground state in Fig. 10. The results fluctuate about a mean value of 1.85 to be compared with the prediction

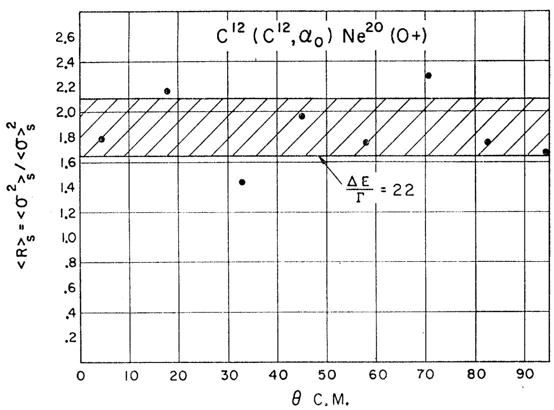

FIG. 10. The measured self-correlation ratio $\langle\sigma^2\rangle_S/\langle\sigma\rangle_S^2$ is shown as a point at each angle. The theory predicts 50% probability of finding values within the shaded band, 25% above and 25% below. The theory assumes that the experimental samples each contain 22 *independent* observations. The results are for the reaction to the Ne^{20} ground state.

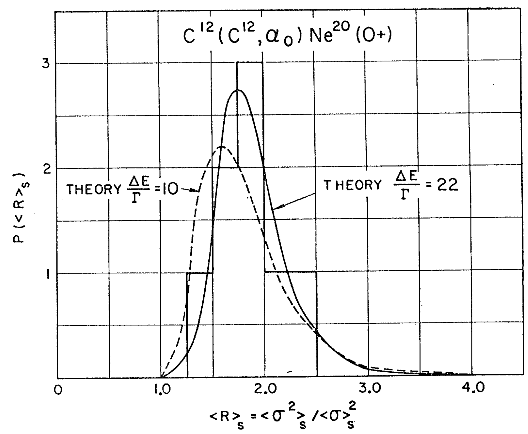

FIG. 11. The data of Fig. 10 are here presented as a histogram of eight results (one for each angle) which is compared with the theoretical probability distributions for a sample size of 22 (solid curve) and 10 (dashed curve) independent observations.

1.95 given by the theory of fluctuations; the theory predicts 50% probability of observing values within the shaded band for our sample. A more meaningful comparison is shown in Fig. 11. Here the individual result for eight angles where measurements were made are shown as a histogram to be compared with the calculated probability distribution of R. The observed frequencies of values of the correlation ratio are entirely consistent with the predictions.

By itself, the plot shown in Fig. 12 for the 31.5° data, if taken at its face value, would suggest that the reaction at this angle proceeded about 75% via direct interaction. The solid curve was computed using the expression including a direct reaction amplitude given by Ericson.[9] However, we have seen that the likelihood of this result under the circumstances is sufficiently large to constitute agreement with the statistical theory without direct interaction. The discussion illustrates the importance of computing the probability distribution of $\langle R\rangle_S$ before drawing any conclusions from the measured values. The conclusion to be drawn from the self-correlation data for the ground-state reaction is that the results are entirely consistent with the statistical theory of fluctuations of a compound-nucleus reaction, but that the statistical accuracy set by the values of $\Delta E/\Gamma$ is such that one can not rule out a contribution up to ~30% of the reaction cross section from some nonfluctuating process.

We turn now to the cross correlation between the differential cross sections at different angles. For a reaction involving only spin-zero particles, $d\sigma/d\Omega$ has the form of Eq. (1), with

$$\bar{A}_J=0,\quad \langle|A_J|^2\rangle_{\rm av}=\bar{\sigma}_J/4\pi,\quad \langle|A_J|^4\rangle_{\rm av}=\langle 2|A_J|^2\rangle_{\rm av}, \tag{25}$$

where $\bar{\sigma}_J$ is the average angle-integrated cross section for formation of compound states of spin J. The first

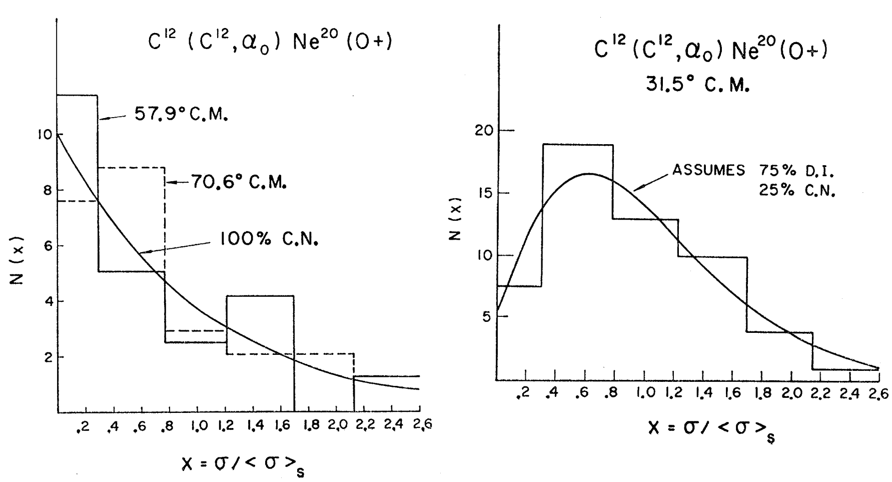

FIG. 12. The histograms are examples of some observed probability distributions at several different angles. The data are for the reaction to the ground state of Ne^{20} and were all included in Fig. 7. The solid curves are theoretical. No significant evidence of direct interaction is observed (see text).

TABLE III. Values of the correlation ratio $R_{ab}=\langle\sigma_a\cdot\sigma_b\rangle/\langle\sigma_a\rangle\langle\sigma_b\rangle$, where σ_a and σ_b are differential cross sections measured at the angles θ_a and θ_b to the beam. Theoretical predictions are shown in brackets.[a]

	$C^{12}(C^{12},\alpha_0)Ne^{10}$ (gd state)							
θ_b \ θ_a	4.4°	17.9°	31.5°	44.9°	57.9°	70.6°	82.7°	94.2°
4.4°	*1.78* (1.95)	1.11 (1.08)	1.05 (1.80)	0.97 (1.43)	1.50 (1.00)	1.00 (1.20)	1.08 (1.10)	1.06 (1.12)
17.9°		*2.16* (1.95)	1.48 (1.02)	1.94 (1.43)	1.18 (1.16)	2.04 (1.07)	1.35 (1.09)	1.51 (1.06)
31.5°			*1.43* (1.95)	1.48 (1.29)	0.93 (1.25)	1.54 (1.31)	1.16 (1.08)	1.24 (1.16)
44.9°				*1.96* (1.95)	1.07 (1.04)	2.01 (1.62)	1.18 (1.35)	1.35 (1.44)
57.9°					*1.75* (1.95)	1.09 (1.00)	0.90 (1.10)	0.95 (1.03)
70.6°						*2.28* (1.95)	1.37 (1.52)	1.56 (1.74)
82.7°							*1.75* (1.95)	1.66 (1.94)
94.2°								*1.67* (1.95)

[a] Mean value of diagonal terms =1.85 (1.95). Mean value of off-diagonal terms =1.31 (1.26).

and last of Eqs. (25) follow from the statistical theory (compare Sec. C), and the second is simply a result of integrating Eq. (1) over 4π sr. From Eq. (1) for $d\sigma/d\Omega$ and Eqs. (25) it follows at once that the cross-correlation ratio between different angles has the form

$$\bar{R}=1+\frac{(\sum_J \bar{\sigma}_J Y_{J0}(\theta_1)Y_{J0}(\theta_2))^2}{(\sum_J \bar{\sigma}_J Y^2{}_{J0}(\theta_1))(\sum_J \bar{\sigma}_J Y^2{}_{J0}(\theta_2))}, \quad (26)$$

where Y_{J0} are normalized spherical harmonics and $\bar{\sigma}_J$ is the average angle-integrated cross section for the reactions through compound states with spin J. Using $\bar{\sigma}_J$ obtained in the compound-nucleus computations[11] Eq. (26) was evaluated for each pair of angles that was studied. These results are compared with the experimental set of values in Table III.

The mean value of the experimental cross correlations (i.e., off-diagonal entries in Table III) is 1.31 to be compared with the predicted mean of 1.26. Individual values show fluctuations, but the average agreement is satisfactory and consistent with the statistical theory.

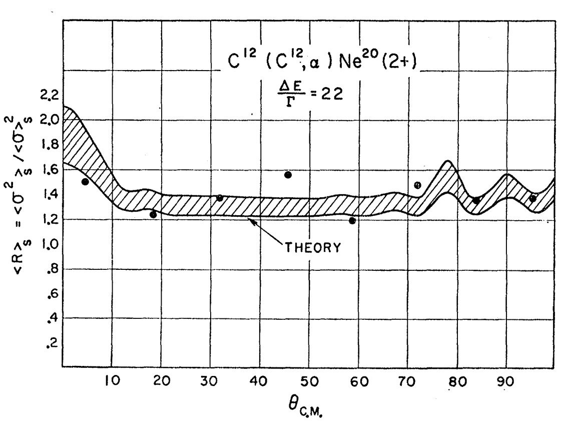

FIG. 13. The measured self-correlation ratio $\langle\sigma^2\rangle_S/\langle\sigma\rangle_S^2$ is shown as a point at each angle. The theory predicts 50% probability of finding values within the shaded band and 25% each above and below. The results are for the reaction to the 1.63-MeV level in Ne^{20}.

(iii) *The Reaction* $C^{12}(C^{12},\alpha_1)Ne^{20}(2+)$

The differential cross sections for the reaction to the first excited state is predicted within the framework of the compound-nucleus theory as used in Ref. 11 to obey the probability distribution law of Eq. (22). Since this law is angle-dependent, the resulting mean values of the self-correlation ratios will also vary with angle in the way shown in Fig. 13. The probable errors (50% chance of lying within the limits shown as a shaded band) were estimated from the probability distribution $P(R)$ of the self-correlation ratio obtained by a Monte Carlo calculation.

Table IV summarizes all the correlation ratios for the first-excited-state reaction. Although again the cross-correlation ratios could, in principle, be computed, no attempt was made to do so in view of the complexity of the expressions that arise from the angle dependence of several independent degrees of freedom. The mean value of the self-correlation ratio (diagonal terms in Table IV) is 1.38 to be compared with the predicted value of 1.42. The off-diagonal cross-correlation terms have a mean value of 1.14. The smaller values of both self- and cross-correlation ratios for the first-excited-state reaction, as compared with the ground-state reaction, is a direct result of the cross section for the former being composed of a sum of several independently fluctuating cross sections which yields a generally smoother energy dependence than that of individual terms in the sum.

TABLE IV. Values of the cross-correlation ratio $R_{ab}=\langle\sigma_a\cdot\sigma_b\rangle/\langle\sigma_a\rangle\langle\sigma_b\rangle$, where σ_a and σ_b are differential cross sections measured at the angles θ_a and θ_b to the beam.[a]

	$C^{12}(C^{12},\alpha_1)Ne^{20}$ (1.63 MeV)							
θ_b \ θ_a	4.4	18.2	31.9	45.5	58.8	71.6	83.9	95.3
4.4	*1.50*	1.17	1.05	1.22	1.03	1.27	1.04	1.05
18.2		*1.23*	1.00	1.23	0.99	1.24	1.14	1.15
31.9			*1.37*	1.12	1.16	1.03	1.00	1.01
45.5				*1.56*	1.13	1.34	1.13	1.19
58.8					*1.19*	1.06	1.04	1.07
71.6						*1.48*	1.28	1.31
83.9							*1.36*	1.35
95.3								*1.37*

[a] Mean value of diagonal terms is 1.38. Mean value of off-diagonal terms is 1.14.

(iv) *Angle-Integrated Cross Sections*

Because all the particles have spin-zero, the angle-integrated cross section for the ground-state reaction has the simple form

$$\sigma_0=\frac{2\pi}{k^2}\sum_J(2J+1)|U_{\alpha\alpha'}{}^J|^2 = \sum_J \sigma_J. \quad (27)$$

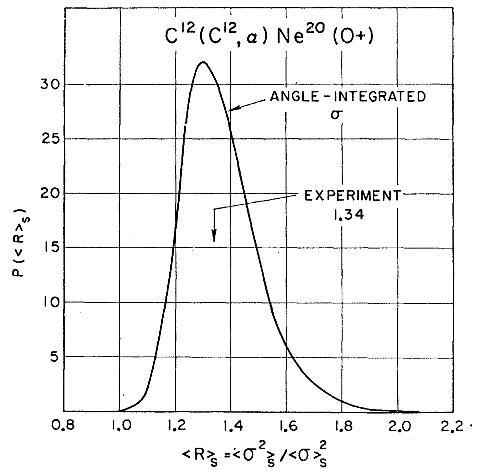

FIG. 14. The predicted probability distribution of the self-correlation ratio $\langle R\rangle_S$ for the angle-integrated cross section to Ne^{20} (gd state). The measured value is indicated. A sample size of 22 independent observations is assumed.

The identity of the incident particles restricts the sum to even values of J and introduces the factor 2 in front of Eq. (27). Equation (27) has the form of Eq. (20), with each term in the sum having the exponential distribution of Eq. (21). The appropriate weight factors C_j obtained from the statistical-model calculation of Ref. 11 were used in Eq. (23) to predict the probability distribution, $P(\langle R\rangle_S)$, for the self-correlation function $\langle R\rangle_S$ when the sample size S equals 22. This distribution is shown in Fig. 14. The measured value $\langle R\rangle_S$ equal to 1.34 is not significantly different from the predicted mean 1.37.

For the reaction $C^{12}(C^{12},\alpha_1)Ne^{20}(2+)$, the angle-integrated cross section is a sum over a larger number of independent terms than for the ground-state reaction, and hence, a smaller value of the correlation ratio is expected. The cross section has the form

$$\sigma_1=\frac{2\pi}{k^2}\sum_J\sum_{l'}(2J+1)|U_{\alpha\alpha' l'}{}^J|^2, \tag{28}$$

where l' has the values $J-2$, J, and $J+2$, only. The restriction to even values arises in this particular case because the compound states are restricted to even-spin, even-parity types for reactions between indistinguishable particles. The dominant contribution to σ_1 is shown by the compound-nucleus computations[11] to be from states of spin 4, 6, and 8; consequently, the main contribution to the sums in Eq. (28) are from 9 independently fluctuating terms. This fact leads to a rough estimate for the mean value of the autocorrelation ratio of 1.11 as compared with the experimental value of 1.15. Again, the cross section may be written in the form of Eq. (20) with weight factors C_i calculated by the statistical theory.[11] The corresponding Monte Carlo calculation using Eq. (23) yielded the probability distribution $P(\langle R\rangle_S)$ shown in Fig. 15. The predicted mean value of $\langle R_S\rangle$ obtained in this way equals $1.13(\pm0.03)$ in excellent agreement with the experimental value 1.15.

(v) *Cross Correlations Between α_0 and α_1*

The cross correlation between the ground-state and first-excited-state angle-integrated cross sections is experimentally found to be 0.89, which corresponds to a slight anticorrelation between these sets of data. Inspection of Fig. 5 shows that the σ_0 and σ_1 curves both have a broad structure that indeed is obviously anticorrelated. The statistical theory predicts zero correlation between different reaction channels, i.e., $\bar{R}$ equal to 1. Because the sample size is finite, the measured value of $\langle R\rangle_S$ has a probability distribution about its mean value; this probability distribution was computed by a Monte Carlo method using

$$\langle R_{ab}\rangle_S=\frac{S\sum_{i=1}^{S}\sum_j C_jx_{ij}\sum_k C_ky_{ik}}{(\sum_{i=1}^{S}\sum_j C_jx_{ij})(\sum_{i=1}^{S}\sum_k C_ky_{ik})}, \tag{29}$$

where the x_{ij} and y_{ik} are numbers chosen at random from an exponential distribution and C_j and C_k are the appropriate weighting factors [compare Eq. (19b)] for the ground-state and first-excited-state cross sections, respectively. The sample size S was taken to be 22. The predicted probability distribution for the cross correlation between the angle-integrated values is shown in Fig. 15. The chance of observing a value equal to or less than the measured result 0.89 is computed to be 2.1%. This observation is therefore a fairly unlikely result on the basis of the simple statistical compound-nucleus

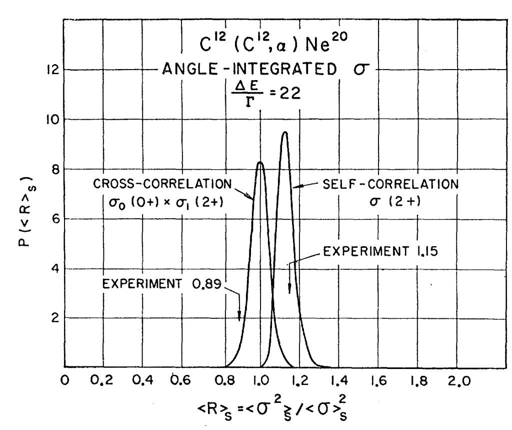

FIG. 15. The right-hand curve is the same as that described in the caption of Fig. 14 except that the calculation is for the reaction to the 1.63-MeV state of Ne^{20}. The left-hand curve is the result of a similar computation of the cross correlation between the angle-integrated cross sections for the reactions to the ground state and 1.63-MeV state of Ne^{20}. The probability of observing a self-correlation ratio equal to or greater than 1.15 is 30%; the probability of a cross-correlation ratio equal to or less than 0.89 is only 2.1%.

model. On the other hand the discussion of Sec. (*iv*) showed that good agreement exists in individual cases between the measured self-correlation functions and the predicted values. It is suggested that the broad structure is entirely consistent with random fluctuations of the average cross sections, and that the observed anticorrelation is a somewhat improbable statistical accident.

Since the angle-integrated cross sections σ_0 and σ_1 anticorrelate slightly, it is to be expected that a similar result will be obtained in the cross correlations of the individual differential cross sections which are comprised in the integral result. The values given in Table V indeed show a tendency for an anticorrelation, i.e., correlation ratios, in general, smaller than unity. The average is 0.87 to be compared with the value 0.89 for the angle-integrated data. The predicted cross-correlation distribution for a sample size of 22 is shown as the curve marked "theory" in Fig. 16. This curve was computed using Eq. (29) with the appropriate weights C_j and C_k for the differential cross sections. The weights change with angle, but the effect is small and the resulting probability distributions are essentially the same for all the 64 angle combinations in Table V. The distribution of the data which is plotted as a histogram can therefore be compared directly with the theoretical curve. The width of the experimental curve is that expected for a sample size of 22 but the mean value of $\langle R_{01}\rangle_S$ equal to 0.87 is somewhat small in accord with the result just given for the angle-integrated data.

5. SUMMARY AND DISCUSSION

General agreement was obtained in the previous sections between the experimental data and the predictions of the statistical theory of cross-section fluctuations of Ericson[9] and of Brink and Stephen.[10] The results did not require the addition of any direct reaction component and are consistent with a predominantly compound-nucleus mechanism for the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction in the range 10–13-MeV center-of-mass bombarding energy. A quantitative discussion of the effects of finite sample size is given and the predictions are

TABLE V. Values of the cross-correlation ratio $R_{01}=\langle\sigma_0\cdot\sigma_1\rangle/\langle\sigma_0\rangle\langle\sigma_1\rangle$, where σ is the differential cross at an angle θ to the beam. The subscripts refer to the ground state and first excited state (1.63 MeV), respectively.[a]

$\theta_0 \backslash \theta_1$	4.4	18.2	31.9	45.5	58.8	71.6	83.9	95.3
4.4	*0.956*	0.946	0.906	0.750	0.903	0.729	0.736	0.724
17.9	0.810	*0.873*	0.743	0.771	0.812	0.788	0.882	0.856
31.5	0.950	1.054	*0.820*	0.957	0.877	1.003	1.026	1.020
44.9	0.872	0.946	0.808	*0.870*	0.816	0.943	0.982	0.950
57.9	0.842	0.864	1.038	0.789	*0.934*	0.706	0.745	0.722
70.6	0.793	0.951	0.708	0.802	0.772	*0.866*	0.957	0.928
82.7	0.821	0.871	0.849	0.796	1.035	0.707	*0.939*	0.950
94.2	0.820	0.884	0.836	0.823	0.996	0.730	0.912	*0.920*

[a] Mean value of all the terms is 0.866.

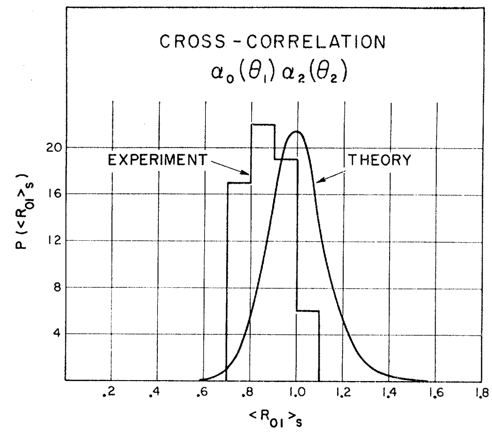

FIG. 16. This histogram shows the distribution of the measured cross-correlation ratios given in Table V. The solid curve is the calculated probability distribution assuming that each sample contains 22 independent observations. The probability of observing a deviation from the statistical theory equal to or larger than that shown is 2%.

shown to be in good agreement with the observed distributions of measured average quantities.

The theory of cross-section fluctuations is based on the compound-nucleus picture together with the assumption that all the reduced-width amplitudes of the compound states have Gaussian distributions about a mean value, zero. In the case of a large number of overlapping levels in the compound nucleus, the collision matrix components then can be shown to have the form[9,11]

$$U_{cc'}\approx u_{cc'}+iv_{cc'}\,,$$

where $u_{cc'}$ and $v_{cc'}$ are both real and have Gaussian distributions with a mean value of zero.

$$\langle u_{cc'}\rangle=\langle v_{cc'}\rangle=0\,,$$

and with equal widths, (30)

$$\langle u_{cc'}{}^2\rangle=\langle v_{cc'}{}^2\rangle\,.$$

The basis of these assumptions in nuclear reaction theory and some questions about their application to heavy-ion reactions are discussed in the parallel paper on average cross sections.[11] In the present paper we discuss the extent to which the above form of the collision-matrix components yields predictions that fit the observed fluctuations of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions to the ground state and first excited state of Ne^{20}.

Each measured cross section $\sigma_{\alpha\alpha'}$ (whether an integrated cross section or a differential cross section) can be written

$$\sigma_{cc'}=\textstyle\sum_i C_i\left|\sum_j A_j(u_{cc'}+iv_{cc'})\right|^2\,, \qquad (31)$$

where i and j are some of the quantum numbers of the channels c, c'; C_i, and A_j are known weight coefficients, such as those already discussed in Sec. 4D. With the above assumptions about collision-matrix components each absolute square in Eq. (31) has an independently fluctuating exponential probability distribution. The probability distribution of the cross section is then determined by the number of terms in the sum over i and by the magnitude of the weight coefficients C_i.

In the experiment two simple cases (see Secs. 4C and D) involving only one term in the sum, Eq. (31), were observed to have the predicted exponential probability distributions. In other cases where the number of terms N exceeds unity (Secs. 4D and E), the probability distributions were found to be modified from the exponential law in the way predicted by the fluctuation theory using weight coefficients C_i obtained from a statistical model computation.

The degree of independence of a cross section at one energy from that at another is determined in theory by the extent to which the energy separation exceeds the width Γ of the underlying compound states. Estimates of Γ for the reactions under discussion were made in Sec. 4B and are also discussed in the accompanying paper. The findings suggest that the widths pertinent to the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction at our energies ($E^* \approx 25$ MeV in Mg^{24}) are nearly the same ($\Gamma \approx 120$ keV) for all compound states even though several spins are involved. This width results in an estimated sample size (energy interval/width) of 22 independent observations. The effects of the finite sample size has been investigated and has been found to account quite generally for the distribution of the measured quantities about the values predicted for an infinite sample. Typical effects of the sample size S are as follows:

(1) reduction of the average correlation ratio $\langle\langle R\rangle_S\rangle \equiv \langle\langle\sigma^2\rangle_S/\langle\sigma\rangle_S{}^2\rangle$ from its mean value for an infinite sample by a factor $(S-1)/S$ and the introduction of a distribution law for $\langle R\rangle_S$. For large S the width of the distribution of $\langle R\rangle_S$ about its mean value is roughly proportional to $S^{-1/2}$;

(2) spreading of the cross-section distribution law so that extraction of a direct-reaction component becomes difficult from individual samples. The discussion of Fig. 12 (Sec. E) illustrates a sample with a small value of the correlation ratio $\langle R\rangle_S$ resulting from a low number of small cross sections. These features in themselves suggest a large direct reaction component, but are shown to be consistent with an accidental deviation from the statistical mean of the fluctuation theory arising from the finite sample size.

(3) The introduction of a probability distribution law for such quantities as polarization, average cross sections, symmetry about 90°, etc. Any measurements of these quantities based on a finite sample will have a predictable probability of differing from the corresponding statistical model prediction for an infinite sample. A quantitative estimate of the finite sample effects for the average cross sections of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions is given in the accompanying paper.[11]

In conclusion it can be said that in this paper a large body of experimental data for the reaction $C^{12}(C^{12},\alpha)Ne^{20}$ is shown to agree in considerable detail with the assumption that the collision matrix components are randomly distributed in the sense discussed at the beginning of this section. This may well be the result of a statistical compound-nucleus reaction mechanism, but it is not at all certain that other fairly simple models will not yield similar results for the fluctuations. For example, in the theory of neutron reduced widths, a fairly simple model of the compound state such as that resulting in the "doorway states" of Feshbach and Shakin[18] leads to distributions of reduced widths very much like the Porter-Thomas distribution which corresponds to the opposite extreme in models—completely random reduced width amplitudes. The final conclusion is therefore that the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction at bombarding energies in the range 10–13 MeV proceeds predominantly through some compound-nucleus mechanism with compound states of average width ≈ 120 keV. In this energy range compound states of 8 units of angular momentum make the most important contributions to the reaction yield.

[18] C. Shakin, Ann. Phys. (N. Y.) 22, 373 (1963).