Crossing Relations and Legendre Expansions in Pion-Pion Scattering*

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The direct use of crossing relations for pion-pion scattering amplitudes, outside the triangle bordered by the lines $s=0$, $t=0$, and $u=0$ in the Mandelstam diagram, is not generally possible. This is because the regions of convergence of the usual Legendre expansions for the amplitudes are restricted by cross-channel cuts in appropriate $\cos\theta$ variables. These convergence difficulties may be relieved by suitably decomposing each amplitude into two terms. One term differs in analytic properties from the actual amplitude in that portions of the cross-channel cuts in $\cos\theta$ nearest $\cos\theta=\pm 1$ are absent. The Legendre expansion of this term has a larger region of convergence than that for the actual amplitude. The other term in the decomposition is expressed in terms of the Legendre series for physical scattering in the cross channels. The amplitudes so represented may now be continued from one physical region to another, and crossing relations may, in general, be directly applied outside the triangle. As a simple application of the formalism, the existence and approximate mass and width of the ρ meson are found to be simple consequences of analyticity, unitarity, and crossing symmetry.

I. INTRODUCTION

SINCE the formulation by Chew and Mandelstam[1] of the double dispersion relation approach to pion-pion scattering, various approximation schemes have been used in order to construct the pion-pion scattering amplitude.[2–10]

Although different in many details, these schemes are all more or less similar in their employment of crossing relations. These relations are used (1) to establish connections between the invariant isotopic spin amplitudes inside the triangle bordered by the lines $s=0$, $t=0$, and $u=0$ in the Mandelstam plot and (2) to express the nearby left-hand discontinuities of the partial-wave amplitudes in terms of physical scattering in the crossed channels.

Now as long as we use Legendre expansions for the invariant amplitudes in the three physical regions of the Mandelstam plot, it is not generally possible to use crossing relations directly at points outside the triangle. They cannot be used at all at such points corresponding to physical scattering angles. The reason for this is well known. The Legendre expansion in the physical region of the s channel, for example, fails to converge for $s<0$. The t and u channels give rise to branch cuts along the real axis in the $\cos\theta_s$ plane (s fixed) starting at $\cos\theta_s=\pm(s+4)/(s-4)$. For $s<0$, $(s+4)/(s-4)$ becomes less than unity in magnitude and, consequently, the Legendre series in $\cos\theta_s$ will not converge.[11] Similar results hold for the t and u channels. Thus the triangle enclosed by $s=0$, $t=0$, and $u=0$ is the only common region of convergence of the Legendre series for the invariant amplitudes in the three physical regions.

In this paper, we present a method for using the crossing relations outside the triangle. The convergence difficulties just discussed are eliminated as follows. We expand in a Legendre series, not the actual invariant amplitudes, but "modified amplitudes" in which portions of the cuts in $\cos\theta$ nearest $\cos\theta=\pm 1$ have been removed. The regions of convergence of the Legendre series for the modified amplitudes are thus larger than those for the actual ones. The differences between the original and modified amplitudes are expressible in terms of the physical scattering in the crossed channels. We show in the following section that the actual amplitudes, when decomposed into these modified amplitudes and remainder terms, are easily continued analytically from one physical region to another. The crossing relations may then be used directly even outside the triangle.

The extra supply of "practical" crossing relations thus obtained should prove useful in any program which involves trial amplitudes, partially satisfying unitarity and analyticity requirements and containing parameters to be determined by means of crossing conditions, dispersion relations, etc.[12]

In Sec. II, we describe the construction of the modified amplitudes and remainders and present the crossing

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[12] See, e.g., Refs. 10 for a program of this type.

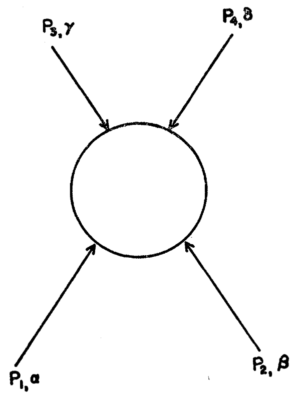

FIG. 1. The pion-pion interaction $\pi+\pi \leftrightarrow \pi+\pi$.

relations in terms of these. Approximate crossing relations for the low-energy region are discussed in Sec. III.

The practical usefulness of the formalism of Secs. II and III can only be determined by detailed calculation. In order to gain some insight into the matter, we describe, in Sec. IV, a crude calculation of low-energy $I=J=1$ pion-pion scattering. The (N/D)-effective range method of Balázs[5] is used to construct a unitary partial-wave amplitude. Two pole terms are used to account for the left-hand singularities and inelastic effects. The pole positions are determined using Balázs' criterion, and the residues are determined by using an approximate form of the crossing relations of Sec. III. This procedure yields values for the residues corresponding to a resonance at a c.m. energy of 575 MeV with a half-width of 120 MeV. These results are about the same as those obtained by Balázs using a fixed s dispersion relation.[5] Unlike the Balázs procedure and other "bootstrap" techniques,[7] the present method does not involve the *a priori* assumption of the existence of a resonance. The resonance, in the present calculation, seems to arise as a natural consequence of analyticity, unitarity, and the crossing relations.[13]

Section V is devoted to a summary and some remarks on work in progress.

II. MODIFIED AMPLITUDES AND CROSSING RELATIONS OUTSIDE THE TRIANGLE

In order to make our argument clear, we first review those aspects of the work of Chew and Mandelstam[1] which are relevant for our purpose.

The pion-pion scattering amplitude is expressed in terms of the three invariant functions A, B, C, as

$$A(stu)\delta_{\alpha\beta}\delta_{\gamma\delta}+B(stu)\delta_{\alpha\gamma}\delta_{\beta\delta}+C(stu)\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (2.1)$$

where α, β, δ, γ are the isotopic spin indices (see Fig. 1) and s, t and u are defined by

$$\begin{aligned} s&=(p_1+p_2)^2,\\ t&=(p_1+p_3)^2,\\ u&=(p_1+p_4)^2. \end{aligned} \quad (2.2)$$

[13] This is consistent with the point of view expounded, e.g., in Refs. 9 and 10.

The pion mass is taken to be unity so that $p_1^2=p_2^2=p_3^2=p_4^2=1$ and $s+t+u=4$.

In the s channel, s is the total c.m. energy squared and t is the c.m. momentum transfer squared. In terms of the square of the center-of-mass momentum ν_s and scattering angle θ_s, we have

$$\begin{aligned} s&=4(\nu_s+1),\\ t&=-2\nu_s(1-\cos\theta_s),\\ u&=-2\nu_s(1+\cos\theta_s). \end{aligned} \quad (2.3)$$

In the t channel, s becomes the c.m. momentum transfer squared and t the square of the total c.m. energy. Thus,

$$\begin{aligned} s&=-2\nu_t(1-\cos\theta_t),\\ t&=4(\nu_t+1),\\ u&=-2\nu_t(1+\cos\theta_t), \end{aligned} \quad (2.4)$$

where ν_t and θ_t are the c.m. momentum squared and scattering angle, respectively.

We denote by $A^I(s;tu)$ or $A^I(\nu_s,\cos\theta_s)$ the amplitude for isotopic spin-I scattering in the s channel. Note that we put the energy variable to the left of the semicolon in the former expression. Similarly, $A^I(t;su)$ or $A^I(\nu_t,\cos\theta_t)$ denotes the isotopic spin I amplitude in the t channel, etc. The A^I for the s channel are related to A, B, and C by

$$\begin{aligned} A^0&=3A+B+C,\\ A^1&=B-C,\\ A^2&=B+C. \end{aligned} \quad (2.5)$$

The crossing relations implied by generalized Pauli statistics are

$$\begin{aligned} A(stu)&=A(sut),\\ B(stu)&=C(sut), \end{aligned} \quad (2.6)$$

$$\begin{aligned} A(stu)&=C(uts),\\ B(stu)&=B(uts), \quad \text{etc.} \end{aligned} \quad (2.7)$$

By inserting these relations into (2.5), we obtain the crossing relations for A^I:

$$A^I(s;tu)=\sum_{I'}\alpha_{II'}A^{I'}(t;su), \quad (2.8)$$

$$A^I(s;tu)=\sum_{I'}\beta_{II'}A^{I'}(u;ts), \quad (2.9)$$

where

$$I,\ I'=0,\ 1,\ \text{and}\ 2$$

and

$$\alpha_{II'}=\begin{bmatrix} 1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6 \end{bmatrix}, \quad (2.10)$$

$$\beta_{II'}=\begin{bmatrix} 1/3 & -1 & 5/3\\ -1/3 & 1/2 & 5/6\\ 1/3 & 1/2 & 1/6 \end{bmatrix}. \quad (2.11)$$

The $A^I(s;tu)$ satisfy the dispersion relations

$$A^I(s;tu)=\frac{1}{\pi}\int_4^\infty dt'\frac{A_t^I(st')}{t'-t}+\frac{1}{\pi}\int_0^\infty du'\frac{A_u^I(su')}{u'-u}, \quad (2.12)$$

where subtractions are ignored. $A_t^I(st)$ and $A_u^I(su)$ are the absorptive parts of $A^I(s;tu)$ in the t and u channels, respectively.

The crossing relations in terms of partial-wave amplitudes are easily obtained by expanding both sides of (2.8) [or (2.9)] in Legendre series.[14]

$$A^I(s;tu)=A^I(\nu_s,\cos\theta_s)$$
$$=\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)A_l^I(\nu_s)P_l(\cos\theta_s), \quad (2.13)$$

$$A^I(t;su)=A^I(\nu_t,\cos\theta_t)$$
$$=\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)A_l^I(\nu_t)P_l(\cos\theta_t). \quad (2.14)$$

The variables on the left- and right-hand sides of (2.13) and (2.14) are related through (2.3) and (2.4), respectively. Thus ν_t, $\cos\theta_t$, ν_s, and $\cos\theta_s$ satisfy the relations

$$\nu_t=\tfrac{1}{2}\nu_s(1+\cos\theta_s)-(\nu_s+1), \quad (2.15)$$

$$\cos\theta_t=\frac{\nu_s(1+\cos\theta_s)+2(\nu_s+1)}{\nu_s(1+\cos\theta_s)-2(\nu_s+1)}. \quad (2.16)$$

The partial-wave amplitudes $A_l^I(\nu_s)$ and $A_l^I(\nu_t)$ are, of course, the same functions of their respective variables ν_s and ν_t. We have

$$A_l^I(\nu)=\frac{1}{2}\int_{-1}^1 d\cos\theta P_l(\cos\theta)A^I(\nu,\cos\theta). \quad (2.17)$$

Although this procedure leads to simple crossing relations, we encounter the inconvenience mentioned in the Introduction. Namely, the common domain of convergence of the Legendre series for both sides of (2.8) and (2.9) is limited to the inside of the small triangle bordered by the lines $s=0$, $t=0$, and $u=0$ in the Mandelstam plot (see Fig. 2). The Legendre series (2.13), for example, converges in the inside of a certain ellipse with foci $(\pm1,0)$ in the complex $\cos\theta_s$ plane (ν_s fixed), if, and only if, the original function $A^I(\nu_s,\cos\theta_s)$ is analytic in the same ellipse.[11] It is, however, well known[1] that $A(\nu_s,\cos\theta_s)$ has branch points at $\cos\theta_s=\pm(1+2/\nu_s)$. Thus, when $\nu_s<-1$ [or $s<0$], these branch points are in the interval $-1\leqslant\cos\theta_s\leqslant+1$ on the real axis, and we have no region where the convergence of the series is guaranteed. These circumstances are very clearly seen on the Mandelstam plot. We see that when $s<0$, the two crossed cuts starting

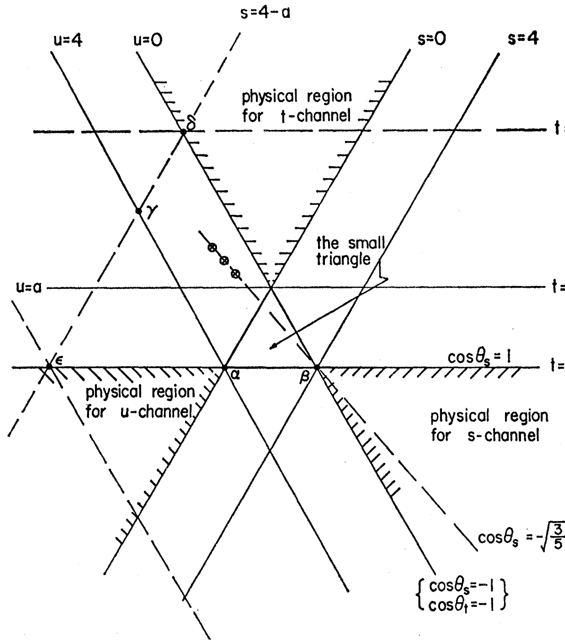

FIG. 2. The Mandelstam diagram for π-π scattering. The region of convergence of the expansion (2.20) (with $\cos\theta_s$ physical) and $s<4$ is given by the triangle $\epsilon\beta\delta$. The common region of convergence of the expansions (2.20) (with $\cos\theta_s$ physical) and (2.14) is given by the trapezoid $\alpha\beta\gamma\delta$. The matching points (4.17) are indicated by the symbol $\oplus$. For $a=16$, the regions of nonvanishing double spectral functions are beyond the region indicated in the figure.

from $t=4$ and $u=4$ extend into the region $-1\leqslant\cos\theta_s\leqslant+1$. The thresholds for these branch cuts correspond to the above-mentioned branch points in the $\cos\theta_s$ plane. Similar results hold for the expansion (2.14). In particular, the expansion fails to converge for $t<0$. Finally, the Legendre expansion for physical scattering in the u channel fails to converge for $u<0$, and we thus verify our previous statement concerning the common domain of convergence.

In order to overcome this difficulty, we remove from $A^I(s;tu)$ the contributions due to nearby singularities of the crossed cuts. This results in a modified amplitude $\tilde{A}^I$ given explicitly by

$$\tilde{A}^I(s;tu)=A^I(s;tu)-F^I(s;tu), \quad (2.18)$$

where

$$F^I(s;tu)=\frac{1}{\pi}\int_4^a dt'\frac{A_t^I(st')}{t'-t}+\frac{1}{\pi}\int_4^a du'\frac{A_u^I(su')}{u'-u}. \quad (2.19)$$

a is an arbitrary constant ($a>4$), which is to be chosen conveniently in the course of numerical calculations.

The crossed cuts of $\tilde{A}^I(s;tu)$ now start from $t=a$ and $u=a$ [see Eq. (2.12)]. Thus the Legendre series expansion of $\tilde{A}^I(s;tu)$, i.e.,

$$\tilde{A}^I(s;tu)=\tilde{A}^I(\nu_s,\cos\theta_s)$$
$$=\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)P_l(\cos\theta_s)\tilde{A}_l^I(\nu_s) \quad (2.20)$$

[14] The restriction of $\binom{\text{even}}{\text{odd}}l$ for $\binom{\text{even}}{\text{odd}}I$ follows from (2.5) and (2.6), which imply $A^I(stu)=(-1)^IA^I(sut)$.

and

$$\tilde{A}_l^I(\nu_s)=\frac{1}{2}\int_{-1}^{1} d\cos\theta_s P_l(\cos\theta_s)\tilde{A}^I(\nu_s,\cos\theta_s) \qquad (2.21)$$

converges for $\nu_s>-\frac{1}{4}a$. By inserting (2.18) into (2.21) we obtain, for each l, the relations

$$\tilde{A}_l^I(\nu_s)=A_l^I(\nu_s)-\frac{1}{2}\int_{-1}^{1} d\cos\theta_s P_l(\cos\theta_s)\times F^I(\nu_s\cos\theta_s)$$
$$\equiv A_l^I(\nu_s)-F_l^I(\nu_s). \qquad (2.22)$$

Thus, $A^I(s;tu)$ has the following modified expansion for $\nu_s>-\frac{1}{4}a$:

$$A^I(s;tu)=A^I(\nu_s,\cos\theta_s)$$
$$=\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)P_l(\cos\theta_s)\times[A_l^I(\nu_s)-F_l^I(\nu_s)]+F^I(s;tu). \qquad (2.23)$$

Note that the summations on the right-hand side of (2.23) cannot be taken separately, since each diverges individually. In Fig. 2, the domain of convergence of the expansion (2.23), corresponding to $-1\leqslant\cos\theta_s\leqslant 1$, is shown. This domain does not contain any of the physical region for the t channel. However, since $A^I(t;su)$ is analytic in the strip $0\leqslant u\leqslant 4$, this function can be easily continued into this strip in terms of the Legendre series (2.14). Thus we have a common domain where both expansions (2.23) and (2.14) converge (see Fig. 2).

Our final task in this section is to show that $F^I(s;tu)$ and $F_l^I(\nu_s)$ in (2.23) can also be expressed in terms of the Legendre expansions of the physical amplitudes. The function $A_t^I(st)$ in (2.19) and (2.12) is the discontinuity of the t cut of $A^I(s;tu)$. By using the crossing relation (2.8), we can express this discontinuity in terms of the imaginary part of $A^I(t;su)$, as long as we remain outside the region of nonvanishing double spectral functions ($s>-32$).[1] After expressing this imaginary part in terms of its Legendre series, we obtain

$$A_t^I(st)=\sum_{I'}\alpha_{II'}\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)P_l\left(1+\frac{2s}{t-4}\right)\times\operatorname{Im}A_l^{I'}(\tfrac{1}{4}t-1). \qquad (2.24)$$

We have used the relations $\cos\theta_t=[2s/(t-4)]+1$ and $\nu_t=(\frac{1}{4}t)-1$. Similarly, we have

$$A_u^I(su)=\sum_{I'}\beta_{II'}\sum_{\substack{l\text{ even }(I=0,2)\\ l\text{ odd }(I=1)}}(2l+1)\times P_l\left(-1-\frac{2s}{u-4}\right)\operatorname{Im}A_l^{I'}(\tfrac{1}{4}u-1). \qquad (2.25)$$

These expansions converge for $s>-32$.[1] Substitution of (2.24) and (2.25) into (2.19) gives

$$F^I(s;tu)=\frac{1}{\pi}\int_4^a dx\left(\frac{1}{[x+2\nu_s(1-\cos\theta_s)]}+(-)^I\frac{1}{[x+2\nu_s(1+\cos\theta_s)]}\right)\times\sum\alpha_{II'}\sum_{\substack{l\text{ even }(I'=0,2)\\ l\text{ odd }(I'=1)}}(2l+1)\times\operatorname{Im}A_l^I(\tfrac{1}{4}x-1)P_l\left(1+\frac{2s}{x-4}\right). \qquad (2.26)$$

In obtaining (2.26), we have used

$$P_l(-z)=(-)^lP_l(z), \qquad (2.27)$$

and

$$\alpha_{II'}=(-)^{I+I'}\beta_{II'}. \qquad (2.28)$$

Finally, $F_l^I(\nu_s)$ is given by

$$F_l^I(\nu_s)=\frac{1}{2}\int_{-1}^{1}d\cos\theta_s P_l(\cos\theta_s)F^I(\nu_s,\cos\theta_s)$$
$$=\frac{1}{\pi\nu_s}\int_4^a dx Q_l\left(1+\frac{x}{2\nu_s}\right)\times\sum_{I'}\sum_{\substack{l'\text{ even }(I'=0,2)\\ l'\text{ odd }(I'=1)}}\alpha_{II'}(2l'+1)\times P_{l'}\left(1+\frac{2s}{x-4}\right)\operatorname{Im}A_{l'}^{I'}(\tfrac{1}{4}x-1), \qquad (2.29)$$

where $Q_l(z)$ is the Legendre function of the second type defined by[15]

$$Q_l(z)=\frac{1}{2}\int_{-1}^{1}\frac{P_l(y)}{z-y}dy. \qquad (2.30)$$

Substitution of $A(s;tu)$, given by (2.23), (2.26), and (2.29), into the left-hand side of (2.8) and (2.9), thus yields crossing relations, expressed entirely in terms of partial-wave Legendre expansions, which are valid for a region outside the small triangle.

III. APPROXIMATE CROSSING RELATIONS AT LOW ENERGY AND THE CONSTRUCTION OF UNITARY PARTIAL-WAVE AMPLITUDES

In Sec. II, we set up exact crossing relations in terms of Legendre expansions. In this section, we propose a low-energy approximation scheme based, in part, on these relations.

A basic assumption in our approach is that for sufficiently small ν_t, only s-wave ($I=0$, 2) and p-wave ($I=1$) terms need be retained in the expansion (2.14) for the right-hand side of the crossing relation (2.8). This assumption, which is based on the ν^l threshold

[15] Reference 11, p. 316.

behavior of A_l^I imposed by the Mandelstam representation[1] (or in less precise terms, the finite range of interaction), is more or less common to all previous approaches.[1–10]

For reasons discussed below, it is convenient to consider $\cos\theta_s$ on the left-hand side of (2.8) to be physical ($|\cos\theta_s|\leqslant 1$). Therefore, when using (2.8), we must continue the Legendre series (2.14) into the region $\cos\theta_t<-1$ (see Fig. 2). The region where the above approximation for (2.14) remains valid must then be carefully studied. We discuss this point in the next section.

Now consider the left-hand side of the crossing relation (2.8). We expand this as (2.18) and (2.20) and retain only the s and d waves for $I=0$, 2 and the p and f waves for $I=1$. In this case, the expansion coefficient is not $A_l^I(\nu_s)$ but $\tilde{A}_l^I(\nu_s)$. For points outside the triangle in the unphysical region with $|\cos\theta_s|<1$, we have $\nu_s<-1$. Now, for ν_s less than -1 but considerably greater than $-\frac{1}{4}a$, we may justify our retention of only the first few partial-wave terms of (2.20) as follows.

First, we see from (2.12), (2.18), and (2.20) that $\tilde{A}_l^I(\nu_s)$ is analytic in the interval $(-\frac{1}{4}a<\nu_s<0)$ on the real axis. Note that $A_l^I(\nu_s)$ is only analytic in the interval $(-1<\nu_s<0)$ on the real axis. Furthermore, $\tilde{A}_l^I(\nu_s)$ and $A_l^I(\nu_s)$ have the same $\nu_s{}^l$ threshold behavior. Since $\tilde{A}_l^I(\nu_s)$ has its left-hand cut starting further out than that of $A_l^I(\nu_s)$ [this corresponds to a shorter range "effective" interaction for $\tilde{A}_l^I(\nu_s)$], the $\tilde{A}_l^I$ should be more suppressed at large l and small ν_s than the corresponding A_l^I.

The situation may perhaps be clarified by considering $\tilde{A}_l^I$ as a function of $k=\sqrt{\nu_s}$. Then $\tilde{A}_l^I(k)$ has the analytic structure shown in Fig. 3. $\tilde{A}_l^I(k)$ has a Taylor expansion about $k=0$ and we thus expect that $\tilde{A}_l^I(k)$ for large l is still small on the imaginary axis as long as $|k|$ is considerably less than the smaller of $\frac{1}{2}\sqrt{a}$ or $\sqrt{3}$, the radius of convergence of the Taylor expansion. The situation is quite different for $A_l^I(k)$, where $\frac{1}{2}\sqrt{a}$ in Fig. 3 is effectively 1. In this case, the radius of convergence is one and we would expect $A_l^I(k)$ to be small only for $|k|$ considerably less than unity.

In (2.20), the d-wave contribution for $I=0$, 2 and the f-wave contribution for $I=1$ can easily be eliminated by choosing the matching point for the crossing relation (2.8) on the line $\cos\theta_s=-\sqrt{\frac{1}{3}}$ and $\cos\theta_s=-\sqrt{\frac{3}{5}}$, respectively. On these lines $P_2(\cos\theta_s)$ and $P_3(\cos\theta_s)$, respectively, vanish. We now see the reason for restricting $\cos\theta_s$ to physical values.

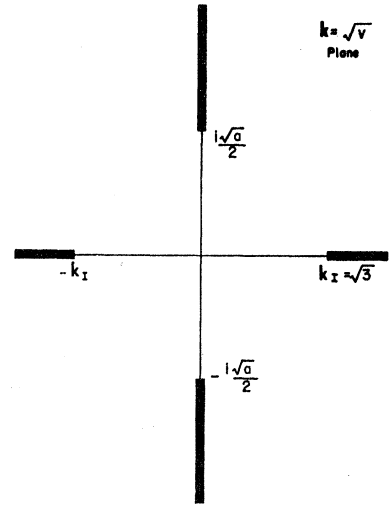

FIG. 3. The analytic structure of $A_l^I(k)$ where $k=\sqrt{\nu}$; the values $\pm k_I=\pm\sqrt{3}$ correspond to the inelastic threshold.

Finally, in $F^I(\nu_s,\cos\theta_s)$ and $F_l^I(\nu_s)$ given by (2.26) and (2.29), respectively, we retain only the s- and p-wave contributions. This approximation should be fairly good as long as $(\frac{1}{4}a-1)$ is $\gtrsim 3$.

In summary, the approximate expressions for (2.8), appropriate for the low-energy region, are

$$A_0{}^0(\nu_s)-F_0{}^0(\nu_s)+F^0(\nu_s,-\sqrt{\tfrac{1}{3}})\\=\tfrac{1}{3}A_0{}^0(\nu_t)+3A_1{}^1(\nu_t)\cos\theta_t+(\tfrac{5}{3})A_0{}^2(\nu_t)\,,\quad(3.1)$$

$$-3\sqrt{\tfrac{3}{5}}[A_1{}^1(\nu_s)-F_1{}^1(\nu_s)]+F^1(\nu_s,-\sqrt{\tfrac{3}{5}})\\=\tfrac{1}{3}A_0{}^0(\nu_t)+\tfrac{3}{2}A_1{}^1(\nu_t)\cos\theta_t-\tfrac{5}{6}A_0{}^2(\nu_t)\,,\quad(3.2)$$

$$A_0{}^2(\nu_s)-F_0{}^2(\nu_s)+F^2(\nu_s,-\sqrt{\tfrac{3}{5}})\\=\tfrac{1}{3}A_0{}^0(\nu_t)-\tfrac{3}{2}A_1{}^1(\nu_t)\cos\theta_t+\tfrac{1}{6}A_0{}^2(\nu_t)\,,\quad(3.3)$$

where ν_t and $\cos\theta_t$ are given by (2.15) and (2.16), respectively, with $\cos\theta_s=-\sqrt{\frac{1}{3}}$ in (2.26) and $\cos\theta_s=-\sqrt{\frac{3}{5}}$ in (2.27). Also,

$$F^0(\nu_s,\cos\theta_s)=\frac{1}{\pi}\int_4^a dx\left[\frac{1}{x+2\nu_s(1-\cos\theta_s)}+\frac{1}{x+2\nu_s(1+\cos\nu_s)}\right]\left\{\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)+3\left(1+\frac{8(\nu_s+1)}{x-4}\right)\right.\\\left.\times\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)+(\tfrac{5}{3})\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right\},\quad(3.4)$$

$$F^1(\nu_s,\cos\theta_s)=\frac{1}{\pi}\int_4^a dx\left[\frac{1}{x+2\nu_s(1-\cos\theta_s)}-\frac{1}{x+2\nu_s(1+\cos\theta_s)}\right]\left\{\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)+\frac{3}{2}\left(1+\frac{8(\nu_s+1)}{x-4}\right)\right.\\\left.\times\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)-\tfrac{5}{6}\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right\},\quad(3.5)$$

$$F^2(\nu_s,\cos\theta_s)=\frac{1}{\pi}\int_4^a dx\left[\frac{1}{x+2\nu_s(1-\cos\theta_s)}+\frac{1}{x+2\nu_s(1+\cos\theta_s)}\right]\left\{\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)-\frac{3}{2}\left(1+\frac{8(\nu_s+1)}{x-4}\right)\right.$$
$$\left.\times\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)+\tfrac{1}{6}\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right\},\quad(3.6)$$

$$F_0{}^0(\nu_s)=\frac{1}{\pi\nu_s}\int_4^a dxQ_0\left(1+\frac{x}{2\nu_s}\right)\left[\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)+3\left(1+\frac{8(\nu_s+1)}{x-4}\right)\right.$$
$$\left.\times\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)+(\tfrac{5}{3})\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right],\quad(3.7)$$

$$F_1{}^1(\nu_s)=\frac{1}{\pi\nu_s}\int_4^a dxQ_1\left(1+\frac{x}{2\nu_s}\right)\left[\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)+\frac{3}{2}\left(1+\frac{8(\nu_s+1)}{x-4}\right)\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)-\tfrac{5}{6}\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right],\quad(3.8)$$

$$F_0{}^2(\nu_s)=\frac{1}{\pi\nu_s}\int_4^a dxQ_0\left(1+\frac{x}{2\nu_s}\right)\left[\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\tfrac{1}{4}x-1)-\frac{3}{2}\left(1+\frac{8(\nu_s+1)}{x-4}\right)\mathrm{Im}A_1{}^1(\tfrac{1}{4}x-1)+\tfrac{1}{6}\,\mathrm{Im}A_0{}^2(\tfrac{1}{4}x-1)\right].\quad(3.9)$$

These relations should be approximately valid for points along the line $\cos\theta_s=-\sqrt{\frac{1}{3}}$ (or $-\sqrt{\frac{3}{5}}$) and ν_s in, but not too close to, the end points of the interval $(-\frac{1}{4}a<\nu_s<0)$.

Of course, (3.1)–(3.9) represent only one possible way of using the crossing relations at low energy. These approximate relations are in accord with the original "low-energy philosophy" of Chew and Mandelstam.[1]

We have not as yet discussed the actual construction of partial-wave amplitudes with the required unitarity and analyticity properties. With regard to these, there are a number of possible approaches. One could, for example, use a simple pole approximation for left-hand cuts in the N/D method.[2,3,5] A more accurate method would be to use crossing relations to express the nearby left hand cuts in terms of, say, s- and p-wave scattering in the crossed channels and to use pole terms to simulate the more distant portions of the cuts.[4] One could also use the boundary condition method[8] or the inverse amplitude method.[3,9] Alternatively, one could parametrize the partial-wave amplitudes in such a way as to be able to sum them explicitly to obtain the total amplitude and, in addition, to account for cross cuts and some inelastic effects.[10] Inelastic effects can also be simulated in the boundary condition model.[16]

In any case, the crossing relations (2.8), in the form with (2.23) on the left and (2.14) on the right, which are generally directly usable outside the triangle, should prove helpful in determining the parameters appearing in any of these approaches.

In future notes, we will discuss in detail the practical aspects of using our formulation of crossing relations in connection with these schemes.

IV. APPLICATION TO LOW-ENERGY PION-PION SCATTERING IN THE $I=J=1$ STATE

In order to indicate the potential usefulness of the crossing relations, (3.1)–(3.3), we describe in this section a crude calculation of low-energy pion-pion scattering in the $I=J=1$ state. The calculation will be based on a simplified version of (3.2). This crossing relation will be used for the points $(-\frac{1}{4}a<\nu_s<-1;\ \nu_t>0)$ outside the triangle. In Appendix 1, it is shown that the main features of the low-energy p-wave amplitude (e.g., resonance behavior) probably do not depend strongly on the s-wave scattering in the crossed channel. In particular, it is shown that $F^1(\nu_s,-\sqrt{\frac{3}{5}})$ in (3.2) tends to cancel the s-wave terms on the right-hand side of (3.2). If we assume that the cancellation is exact, we are left with the approximate p-wave crossing relation

$$\cos\theta_s[A_1{}^1(\nu_s)-F_1{}^1(\nu_s)]\approx\tfrac{1}{2}A_1{}^1(\nu_t)\cos\theta_t,\quad(4.1)$$
$$\cos\theta_s=-\sqrt{\tfrac{3}{5}},$$

where the variables are related according to (2.15) and (2.16). It is also shown in Appendix 1 that $A_1{}^1(\nu)-F_1{}^1(\nu)$ for $-\frac{1}{4}a<\nu\leqslant-1$, and $A_1{}^1(\nu)$ for ν small and greater than zero, are both represented approximately by

$$\hat{A}_1{}^1(\nu)=A_1{}^1(\nu)-\frac{\nu}{\pi}\int_{-\frac{1}{4}a}^{-1}\frac{\mathrm{Im}A_1{}^1(\nu')d\nu'}{\nu'(\nu'-\nu)},\quad(4.2)$$

provided there are no resonances in pion-pion scattering for $\nu\gtrsim\frac{1}{4}a-1$. In other words, (4.1) becomes

$$\cos\theta_s\hat{A}_1{}^1(\nu_s)\approx\tfrac{1}{2}\hat{A}_1{}^1(\nu_t)\cos\theta_t,\quad(4.3)$$

where $\hat{A}_1{}^1(\nu)$ has the following properties: (1) it has a left-hand cut starting, not at $\nu=-1$, but at $\nu=-\frac{1}{4}a$; (2) its left-hand discontinuity for $\nu<-\frac{1}{4}a$ is the same as that of $A_1{}^1(\nu)$; it varies as ν at threshold; (4) it is approximately unitary and coincides approximately with $A_1{}^1(\nu)-F_1{}^1(\nu)$ for $(-\frac{1}{4}a<\nu\leqslant-1)$ and $A_1{}^1(\nu)$ in the physical region. For simplicity, we assume exact unitarity in the following analysis.

A word should be said at this point concerning the validity of retaining only p waves on the right-hand side of (4.1). F waves have, of course, been eliminated from the left-hand side of (4.1) by suitably choosing

[16] H. Goldberg and E. L. Lomon, Phys. Rev. **131**, 1290 (1963); **134**, B659 (1964).

$\cos\theta_s$. In Appendix 2, we make a rough estimate of the higher partial-wave contributions to $A^1(\nu_t, \cos\theta_t)$ on the basis of a simple $I=J=1$ resonance exchange model. We find there that, for the region of interest in this calculation, the contribution is practically negligible.

Our task now is simply (1) to write $\hat{A}_1{}^1(\nu_s)$ in a suitably parametrized form which is unitary and has a left-hand cut starting at $\nu=-\frac{1}{4}a$, and (2) to determine the parameters by using (4.3).

The N/D method[1] furnishes the simplest means for constructing a unitary amplitude with the prescribed left-hand cut. We write

$$\hat{A}_1{}^1(\nu)=N(\nu)/D(\nu) \tag{4.4}$$

$$N(\nu)=\frac{\nu}{\pi}\int_{-\infty}^{-a/4}\frac{D(\nu')\,\mathrm{Im}A_1{}^1(\nu')d\nu'}{\nu'(\nu'-\nu)}, \tag{4.5}$$

$$D(\nu)=1-\frac{\nu-\nu_0}{\pi}\int_0^{\infty}\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N(\nu')R_1{}^1(\nu')d\nu'}{(\nu'-\nu_0)(\nu'-\nu)}, \tag{4.6}$$

where the threshold behavior of $\hat{A}_1{}^1(\nu)$ has been explicitly exhibited. $R_1{}^1(\nu)$ is the ratio of the total to elastic scattering cross section in the $I=J=1$ state.

Now we are only interested in the amplitude for $-\frac{1}{4}a\lesssim\nu\gtrsim 4.5$. In this case, we may use a parametrization procedure suggested by Balázs,[17] which is briefly outlined below. The reader is referred to Balázs' papers[5,17] for details.

After the change of variable $\nu'=-1/x$, (4.5) becomes

$$N(\nu)=\frac{\nu}{\pi}\int_0^{4/a}dx\frac{D(-1/x)\,\mathrm{Im}A_1{}^1(-1/x)}{1+x\nu}. \tag{4.7}$$

In the range $-\frac{1}{4}a<\nu\gtrsim 5$, $0\gtrsim x\gtrsim 4/a$, we may approximate $1/(1+x\nu)$ as[17]

$$\frac{1}{1+x\nu}\approx\sum_{i=1}^{N}\frac{F_i(x)}{1+x_i\nu}, \tag{4.8}$$

with

$$F_i(x)=\prod_{j\neq i}(x-x_j)/\prod_{j\neq i}(x_i-x_j). \tag{4.9}$$

Balázs[17] has made plausible the fact that the relative error in $N(\nu)$, for $(-\frac{1}{4}a\lesssim\nu\lesssim 5)$, due to the approximation (4.8) and (4.9), is of the same order as the error in that approximation. After inserting (4.8) into (4.7), we have

$$N(\nu)\approx\frac{\nu}{\pi}\sum_i\frac{f_i}{\nu-\nu_i};\quad \nu_i=-\frac{1}{x_i}, \tag{4.10}$$

where the f_i, given by

$$f_i=\int_0^{4/a}D\left(-\frac{1}{x}\right)\mathrm{Im}A_1{}^1\left(-\frac{1}{x}\right)\frac{F_i(x)}{x_i}dx, \tag{4.11}$$

are constants. In the following analysis, we take $a\approx 16$. Balázs has shown[5] that for this case, a reasonably accurate approximation of the form (4.8) is obtained by choosing $N=2$ with

$$\begin{aligned}-1/x_1=\nu_1=-6.25,\\ -1/x_2=\nu_2=-50.0.\end{aligned} \tag{4.12}$$

Substitution of (4.10) into (4.6) gives

$$D(\nu)=1-\gamma(\nu)N(\nu)+\frac{\nu}{\pi}\sum_{i=1}^{2}\frac{f_i\gamma(\nu_i)}{\nu-\nu_i}, \tag{4.13}$$

$$\gamma(\nu)=-\frac{2}{\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln[\nu^{1/2}+(\nu+1)^{1/2}]+i\left(\frac{\nu}{\nu+1}\right)^{1/2};\quad 0\lesssim\nu\lesssim\infty \tag{4.14}$$

$$=-\frac{2}{\pi}\left(\frac{|\nu|}{\nu+1}\right)^{1/2}\tan^{-1}\left[\left(\frac{\nu+1}{|\nu|}\right)^{1/2}\right];\quad -1\lesssim\nu\lesssim 0 \tag{4.15}$$

$$=-\frac{2}{\pi}\left|\frac{\nu}{\nu+1}\right|^{1/2}\ln(|\nu+1|^{1/2}+|\nu|^{1/2});\quad -\infty<\nu\lesssim -1, \tag{4.16}$$

where we have chosen the subtraction point ν_0 as zero and have set $R_1{}^1(\nu)=1$.[18]

As was previously stated, we determine the parameters f_1 and f_2 by requiring (4.3) to be satisfied[19] at several matching points. We use the points:

$$\begin{aligned}&\text{(a)}\ \cos\theta_s=-\sqrt{\tfrac{3}{5}},\quad \nu_s=-2.25,\quad \nu_t=1.0,\quad \cos\theta_t=-1.50;\\ &\text{(b)}\ \cos\theta_s=-\sqrt{\tfrac{3}{5}},\quad \nu_s=-2.82,\quad \nu_t=1.5,\quad \cos\theta_t=-1.42;\\ &\text{(c)}\ \cos\theta_s=-\sqrt{\tfrac{3}{5}},\quad \nu_s=-1.69,\quad \nu_t=0.5,\quad \cos\theta_t=-1.76.\end{aligned} \tag{4.17}$$

[17] L. A. P. Balázs, Phys. Rev. 125, 2179 (1962).

[18] Balázs has shown (see the first paper of Refs. 5) that the deviation from unity of the inelastic factor $R_1{}^1(\nu)$ for $\nu\gtrsim 3$ and the inadequacy of the parametrization of $N(\nu)$ for large ν do not change the form of (4.10) and (4.13) for $\nu\gtrsim 4.5$, which is the region of interest in our analysis.

[19] Because of the various assumptions made in obtaining the approximate Relation (4.3), we may only require it to hold for the real parts involved.

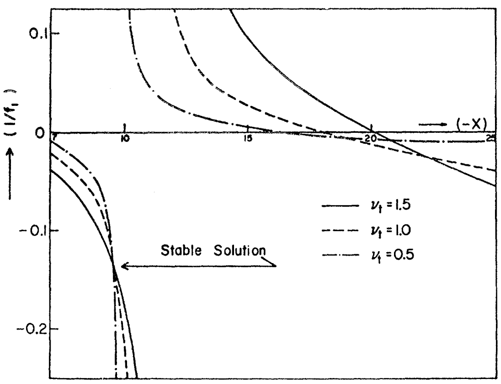

FIG. 4. Stable and unstable solutions for the self-consistent p-wave amplitude equation (4.3). The stable solution, $f_1=-7.7$, $x=f_2/f_1=-9.5$, implies a scattering resonance at a c.m. energy of 575 MeV with a half-width of about 120 MeV.

Our procedure is to determine first f_1 and f_2 using two of the matching points in (4.17). The sensitivity of the solution to changes in the matching points is then examined by seeing how well (4.3), with the previously determined f_1 and f_2, is satisfied at the remaining matching point.

The results of the calculation are illustrated in Fig. 4. There we plot values of $x=(f_2/f_1)$ and $1/f_1$ which satisfy (4.3) at the three matching points of (4.17). We see that there are two solutions, one of which is unstable with respect to the matching points. We discard the unstable solution and assume that the stable one has some connection with reality. It corresponds to $f_1=-7.7$ and $f_2=73$. In Fig. 5, we plot $N(\nu)$ and $\mathrm{Re}D(\nu)$ for the stable solution. The total p-wave cross section is given in Fig. 6. We see that our solution corresponds to an $I=J=1$ resonance at $\nu=\nu_R=3.2$ (575-MeV total c.m. energy) with a half-width of about 120 MeV.

These results are very similar to those obtained by

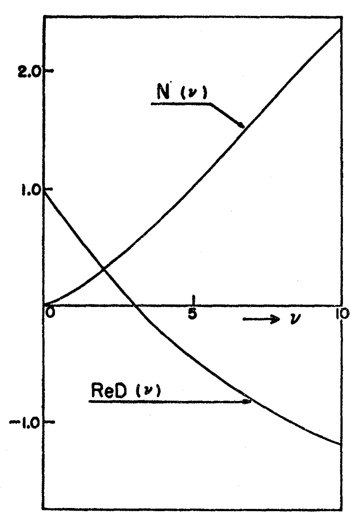

FIG. 5. $N(\nu)$ and $\mathrm{Re}D(\nu)$ for the stable solution of (4.3).

Balázs[5] using a fixed s dispersion relation for the determination of parameters. It is instructive to compare Balázs' method with the one employed here.

In Balázs' approach, we let $a\rightarrow\infty$ in (3.8) [or more generally (2.29) with $I=l=1$] to obtain

$$A_1{}^1(\nu_s)=\frac{4}{\pi\nu_s}\int_0^\infty d\nu Q_1\left(1+\frac{2[\nu+1]}{\nu_s}\right)\times\sum_I\sum_{l\text{ odd}}\alpha_{II}(2l+1)P_l\left(1+2\frac{[\nu_s+1]}{\nu}\right)\times\mathrm{Im}A_l{}^I(\nu). \quad (4.18)$$

Note that as $a\rightarrow\infty$, $F_1{}^1(\nu)\rightarrow A_1{}^1(\nu)$. (4.18) is valid for $-9\leqslant\nu_s\leqslant 0$.[1] Balázs shows, on the basis of a conjectured high-energy Regge behavior, that the integral in (4.18) is convergent. By retaining only the $A_1{}^1$ term in the integral, and requiring $A_1{}^1$ to satisfy (4.18) and the corresponding derivative relation at one point, he

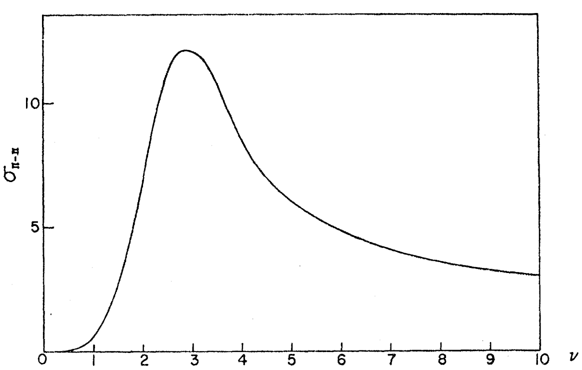

FIG. 6. The total $I=J=1$ partial wave cross section $\{=(12\pi/\nu)\times[\nu/(\nu+1)]\,\mathrm{Im}A_1{}^1(\nu)\}$ calculated from (4.4), (4.10), (4.13), and (4.14) with the parameter values $f_1=-7.7$; $f_2=73$. The unit for the cross section is the square of the pion Compton wavelength.

determined the equivalent of our f_1 and f_2. His determination of the parameters, however, was not as straightforward as ours. In order to make his calculation tractable, he inserted a delta function resonance form for $\mathrm{Im}A_1{}^1(\nu)$ into the integral of (4.18), determined the parameters equivalent to f_1 and f_2 in terms of the initially assumed resonance parameters, and then checked to see whether the amplitude with these parameters exhibited a resonance with the initially assumed characteristics. This process was repeated until a self-consistent solution was obtained. This procedure is similar in spirit to the usual bootstrap calculations,[7] although the Balázs method presumably treats the distant left-hand singularities in $A_1{}^1$ more realistically.

The two main difficulties of Balázs' formalism are (1) the explicit appearance, in his fixed s dispersion

[20] There have, however, been several recent attempts to determine the Regge trajectory parameters in a self-consistent way. See, e.g., M. Bander and G. L. Shaw, Stanford University, 1964 (to be published).

relation, of high-energy scattering contributions which must either be ignored or described, for example, in terms of empirically determined Regge trajectories,[5,20] and (2) the fact that resonances must be assumed *a priori*.

The above features are absent in our approach. We subtract from the amplitudes only the low-energy contribution to the fixed s dispersion relations. The resulting crossing relations, formulated so as to be valid outside the small triangle, involve explicitly only low-energy scattering. The implicit high-energy effects reflected by the singularities of the partial wave amplitudes far from the low-energy region, are determined self-consistently by appropriately parametrizing them and applying the crossing relations. The existence of a resonance arises naturally as a result of direct parameter determination and need not be assumed from the start.

SUMMARY AND CONCLUSIONS

In this paper we have introduced a representation for the pion-pion scattering amplitude which is expressed in terms of the ordinary partial-wave amplitudes, but whose region of validity is considerably greater than that of the usual Legendre expansion. This representation makes possible the direct application of crossing relations in a much larger region than was previously available for such use. A plausible low-energy approximation to the exact crossing relations, which involves only the s- and p-wave amplitudes, was presented.

The availability of useful crossing relations for physical scattering angles, outside the small triangle bordered by lines $s=0$, $t=0$, and $u=0$ in the Mandelstam diagram, should be of value in any program which involves trial amplitudes containing parameters to be determined (at least in part) by crossing relations. The extent to which our representation of the amplitudes will help provide a self-contained scheme for low-energy scattering (i.e., for generation of the low-energy s- and p-wave amplitudes with no, or perhaps one, adjustable parameter) must now be investigated. The crude calculation of low-energy p-wave scattering, in which the energy and width of the $I=J=1$ (ρ) resonance were estimated, gives some indication of the possible usefulness of our representation in more elaborate calculations. It should be stressed again that the existence of the resonance was a direct result of the application of crossing symmetry and unitarity. The *a priori* assumption of its existence was not necessary. Also, high-energy effects entered the crossing relations only through the "distant" singularities of the low partial-wave amplitudes and were determined "self-consistently" (see the last paragraph of Sec. IV).

We are concurrently investigating by means of a high-speed computer the possibility of using our formulation of the crossing relations with trial amplitudes of the type discussed in Refs. 5, 8, and 10 in a self-consistent low-energy program *not* involving the simplifying assumptions of Sec. IV.

ACKNOWLEDGMENTS

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APPENDIX 1

Here we attempt to partially justify the simplifying assumptions made in Sec. IV.

Let us examine carefully the crossing relation (3.2). First consider the modified p-wave amplitude $A_1^1(\nu_s)-F_1^1(\nu_s)$. From (2.29) and (2.30), it follows that $F_1^1(\nu)$ is analytic in ν_s except for a branch cut in the interval $(-\infty<\nu\leqslant-1)$ on the real axis. The imaginary part of $F_1^1(\nu_s)$ along the cut is

$$\mathrm{Im}F_1^1(\nu_s)=\frac{1}{2\nu_s}\int_0^{U(\nu_s)}d\nu P_1\left(1+\frac{2[\nu+1]}{\nu_s}\right)\times\sum_I \alpha_{II}\sum_{\substack{l\text{ odd }(I=1)\\ l\text{ even }(I=0)}}P_l\left(1+\frac{2[\nu_s+1]}{\nu}\right)\times\mathrm{Im}A_l^I(\nu),\quad\text{(A1.1)}$$

$$U(\nu_s)=-(\nu_s+1);\quad -\tfrac{1}{4}a\leqslant\nu_s\leqslant-1$$
$$=-(-\tfrac{1}{4}a+1);\quad -\infty<\nu_s\leqslant-\tfrac{1}{4}a,\quad\text{(A1.2)}$$

where we have made the substitutions $s=4(\nu_s+1)$, $x=4(\nu+1)$ in (2.29). The imaginary part of $F_1^1(\nu_s)$ coincides with that of $A_1^1(\nu_s)$ in the interval $(-\frac{1}{4}a\leqslant\nu_s\leqslant-1)$. For large ν_s, $F_1^1(\nu_s)\propto(\ln\nu_s)/\nu_s$. Also, $F_1^1(\nu_s)\propto\nu_s$ at threshold. Thus, we may represent $F_1^1(\nu_s)$ as

$$F_1^1(\nu_s)=F_1^{1(0)}(\nu_s)+\Delta F_1^1(\nu_s)\quad\text{(A1.3)}$$

$$F_1^{1(0)}(\nu_s)=\frac{\nu_s}{\pi}\int_{-\infty}^{-\frac{1}{4}a}\frac{d\nu}{\nu}\frac{\mathrm{Im}F_1^1(\nu)}{(\nu-\nu_s)};$$
$$\Delta F_1^1(\nu_s)=\frac{\nu_s}{\pi}\int_{-\frac{1}{4}a}^{-1}\frac{d\nu\,\mathrm{Im}A_1^1(\nu)}{\nu(\nu-\nu_s)}.\quad\text{(A1.4)}$$

Note that $A_1^1(\nu_s)$ satisfies[1]

$$A_1^1(\nu_s)=\frac{\nu_s}{\pi}\int_{-\infty}^{-1}\frac{d\nu\,\mathrm{Im}A_1^1(\nu)}{\nu(\nu-\nu_s)}+\frac{\nu_s}{\pi}\int_0^{\infty}\frac{d\nu\,\mathrm{Im}A_1^1(\nu)}{\nu(\nu-\nu_s)}.\quad\text{(A1.5)}$$

Now if there is no resonance behavior in π-π scattering for $\nu\gtrsim\frac{1}{4}a-1$, it is reasonable to assume that $|\mathrm{Im}F_1^1(\nu)|\ll|\mathrm{Im}A_1^1(\nu)|$ for $\nu\ll-\frac{1}{4}a$ [see (A1.1) and

(A1.2)] and also that the effect of the discontinuity of $A_1^1(\nu)$ in the interval $(-\frac{1}{4}a \leqslant \nu \leqslant -1)$ should not be very important in the physical region. In other words, $\hat{A}_1^1(\nu)$ given by

$$\hat{A}_1^1(\nu) = A_1^1(\nu) - \Delta F_1^1(\nu) \qquad \text{(A1.6)}$$

should be approximately the same as $A_1^1(\nu) - F_1^1(\nu)$ for ν considerably greater than $-\frac{1}{4}a$ and less than -1; and should coincide approximately with $A_1^1(\nu)$ in the physical region. We have thus made plausible the statements made in connection with relations (4.2) and (4.3).

We will now consider the term $F^1(\nu_s, \cos\theta_s = -\sqrt{\frac{3}{5}})$ in (3.2) and show that there might be a considerable cancellation between the s-wave terms in F^1 and those on the right-hand side of (3.2). After the change of variables

$$\begin{aligned} x &= 4(\nu+1), \\ -2\nu_s(1-\cos\theta_s) &= t = 4(\nu_t+1), \\ -2\nu_s(1+\cos\theta_s) &= u = 4-s-t = 4(\nu_u+1), \\ \nu_u &= -2-\nu_s-\nu_t, \end{aligned} \qquad \text{(A1.7)}$$

(3.5) becomes

$$\begin{aligned} F^1(\nu_s, \cos\theta_s) = &\frac{1}{\pi}\int_0^{\frac{1}{4}a-1} \frac{d\nu}{\nu-\nu_t}\left[\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\nu) + \frac{3}{2}\left[1+\frac{2(\nu_s+1)}{\nu}\right]\mathrm{Im}A_1{}^1(\nu) - \tfrac{5}{6}\,\mathrm{Im}A_0{}^2(\nu)\right] \\ &+ \frac{1}{\pi}\int_0^{\frac{1}{4}a-1} \frac{d\nu}{\nu-\nu_u}\left[\tfrac{1}{3}\,\mathrm{Im}A_0{}^0(\nu) + \frac{3}{2}\left[1+\frac{2(\nu_s+1)}{\nu}\right]\mathrm{Im}A_1{}^1(\nu) - \tfrac{5}{6}\,\mathrm{Im}A_0{}^2(\nu)\right]. \end{aligned} \qquad \text{(A1.8)}$$

The first term in (A1.8) is simply the Cauchy integral contribution to the right-hand side of (3.2), from the interval $0 \leqslant \nu \leqslant \frac{1}{4}a - 1$. The second term is the Cauchy integral contribution, from the interval $0 \leqslant \nu \leqslant \frac{1}{4}a - 1$, to the right-hand side of (3.2) with ν_t replaced by ν_u, (ν_s fixed). Now for points in the interval $(-\frac{1}{4}a < \nu_s < -1)$ and $\cos\theta_s = -\sqrt{\frac{3}{5}}$, ν_t is greater than zero and ν_u is less than -1 (see Fig. 2). Therefore, in this region, the second term of (A1.8) should be of less importance than the first. The s-wave parts of the first term should partially cancel the s-wave contributions to the left-hand side of (3.2). Similar cancellations should occur for p-wave contributions. However, the Cauchy integral contribution, from the low-energy scattering region, should be much more important for s waves than for p waves (we assume no p-wave resonances for $\nu < \frac{1}{4}a - 1$). Therefore, we shall only assume important cancellations for s waves. Exact cancellation for s waves was assumed in Sec. IV.

APPENDIX 2

We discuss here the contributions of higher partial waves ($l > 1$) to $A^1(\nu_t, \cos\theta_t)$ which were dropped in obtaining the approximate relation (4.3). The $I = J = 1$ (ρ) resonance-exchange model will be used to estimate the neglected terms. This model corresponds to the lowest order scattering given by the effective Hamiltonian

$$H = f_{\rho\pi\pi}\boldsymbol{\rho}_\mu \cdot (\boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi}); \qquad \text{(A2.1)}$$

or, alternatively, to the insertion of the zero-width approximation

$$\mathrm{Im}A_l{}^I(\nu) = \pi\Gamma\nu_R\delta(\nu-\nu_R)\delta_{I1}\delta_{l1} \qquad \text{(A2.2)}$$

into the integrands, (2.24), (2.25) of the dispersion relation (2.12). ν_R is the c.m. momentum squared at resonance and the parameter Γ is related to the half-width, $W_{1/2}$, of the ρ resonance by[5]

$$\Gamma = \frac{\nu_R+1}{2\nu_R{}^{3/2}} W_{1/2},$$

where $W_{1/2}$ is in units of the pion rest energy. Experimentally[21] $\nu_R \approx 5.7$ and $W_{1/2} \approx 1.0$ so that $\Gamma \approx 0.25$.

The ρ exchange contribution to $A^1(\nu_t, \cos\theta_t)$ is

$$\begin{aligned} A^1(\nu_t, \cos\theta_t) &\approx \frac{3\Gamma}{2}\left[\frac{2(\nu_t+1)+\nu_t(1+\cos\theta_t)}{2(\nu_R+1)+\nu_t(1-\cos\theta_t)} - \frac{2(\nu_t+1)+\nu_t(1-\cos\theta_t)}{2(\nu_R+1)+\nu_t(1+\cos\theta_t)}\right] \\ &= \sum_{l\,\text{(odd)}} (2l+1)P_l(\cos\theta_t)A_l{}^1(\nu_t) \end{aligned} \qquad \text{(A2.3)}$$

$$A_l{}^1(\nu_t) = 3\Gamma\left(4+\frac{2(\nu_R+2)}{\nu_t}\right)Q_l\left(1+\frac{2(\nu_R+1)}{\nu_t}\right) \qquad \text{(A2.4)}$$

$$\underset{\nu_t\to 0}{\longrightarrow} 3\Gamma\left(4+\frac{2(\nu_R+2)}{\nu_t}\right)\frac{2^l(l!)^2}{(2l+1)!}\left(\frac{\nu_t}{2(\nu_R+1)}\right)^{l+1}. \qquad \text{(A2.5)}$$

Now at the second matching point (b) of (4.17), we find that the stable solution of (4.3) corresponds to (see Fig. 5)

$$\mathrm{Re}[3\cos\theta_t A_1{}^1(\nu_t)] \approx -0.80. \qquad \text{(A2.6)}$$

The higher partial-wave contributions [(A2.3) minus the p-wave contribution] to $A^1(\nu_t, \cos\theta_t)$ are found to be ≈ 0.022, which is small compared to (A2.6).

[21] A. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961).