$\delta(Br^{80})=0.24$ and $\delta(Br^{80m})=0.12$. That these values are so different probably reflects the inadequacy of a strong-coupling approximation.

Unique collective-model configurations can be obtained for both Br^{80} and Br^{80m} if any value of δ between 0.1 and 0.3 is assumed correct. This is accomplished by imposing upon acceptable configurations the following reasonable requirements: (a) they are plausible on the basis of the Nilsson level-filling diagrams,[14] (b) they give the correct spin values when the Gallagher and Moszkowski coupling rules are used,[17] (c) they give the correct nuclear parities, and (d) they account correctly for the relative signs of the μ_I and Q for Br^{80} and the absolute signs of the μ_I and Q for Br^{80m}. The only configurations that satisfy all these requirements are given in Table III.

[17] C. J. Gallagher, Jr., and S. A. Moszkowski, Phys. Rev. **111**, 1282 (1958).

ACKNOWLEDGMENTS

The authors express their appreciation to R. E. McCracken and K. R. Voice of the Lawrence Radiation Laboratory Health Chemistry Division for considerable support during the course of these experiments, and to H. S. Powell and W. A. Berlund of the Glass Shop for their patient help and advice.

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Nuclear Charge Distribution in Calcium from Electron Scattering and Muonic X Rays*†

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A detailed examination is made of the charge distributions predicted for calcium by experiments with electron scattering and muonic x rays. It is shown that, contrary to earlier suggestions, the electron differential cross sections, with both relative and absolute measurements, and the $2p \rightarrow 1s$ x ray energy all predict charge distributions which are in agreement to within experimental error for the three analytic expressions employed. Parameter values for these shapes—Fermi, family II, and modified Gaussian—are given. There is an indication from the electron-scattering relative cross-section analysis, however, that charge distributions with less charge at the extreme edge are favored.

1. INTRODUCTION

THE purpose of this paper is to consider if the information on the size and shape of nuclear charge distributions presently available from electron scattering and from muonic x rays is in agreement. The comparison of these two kinds of experiment can be regarded from various viewpoints. The first question is whether or not the muon-nuclear interaction is entirely electromagnetic. Assuming that this has been settled affirmatively over the large separations involved in these experiments, we would like to know to what extent the two experiments complement each other in determining nuclear charge distributions. The necessity for an investigation of such a well-appreciated question at this late date requires justification. The main aim of our calculational program has been to continue and extend the analysis of electron elastic scattering experiments. Conversations with experimenters at Chicago and Stanford, however, led us to appreciate the following apparent paradox in the presently quoted investigations: Measurement of absolute electron cross sections at small angles by Crannell *et al.*,[1] and of muonic x rays by Anderson *et al.*,[2] and by a CERN-Darmstadt collaboration,[3] seemed to discriminate against one type of charge distribution, the family II, and to agree better with the more commonly used Fermi distribution.[4] Both of these two types of charge distribution are roughly constant inside the nucleus, and drop smoothly to zero at the nuclear edge, but they differ in the functional form assumed for the surface. It was one of the

* Supported in part by the U. S. National Science Foundation.

† Most of this material was presented in Paper I4 of the Stanford American Physical Society Meeting, 28 December 1962.

[1] H. Crannell, R. Helm, H. Kendall, J. Oeser, and M. Yearian, Phys. Rev. **121**, 283 (1961).

[2] C. S. Johnson, E. P. Hincks, and H. L. Anderson, Phys. Rev. **125**, 2102 (1962).

[3] P. Brix, R. Engfer, U. Hegel, D. Quitmann, G. Backenstoss, K. Goebel, and B. Stadler, Phys. Letters **1**, 56 (1962).

[4] This terminology is described fully in Sec. 4.

conclusions of an earlier study of electron scattering,[5] where only relative cross sections at various angles were measured, that any charge distribution of the smoothed uniform type would fit the experiments, provided its radius and skin thickness were adjusted appropriately. Since the small-angle absolute electron cross-section measurement and the muonic x ray are both long-wavelength probes of nuclear size for light nuclei like calcium, it seemed strange that they should tell more about the detailed shape of the nuclear surface than these shorter wavelength experiments. On the other hand, the shape independence of the surface has been demonstrated only for the heavy nucleus gold, $_{79}$Au.[6] It was possible that the situation might be different for a light nucleus like calcium, $_{20}$Ca, where the surface contains a much larger proportion of the charge.[7] It seemed to us necessary to understand this problem, so as to clarify the current status of the questions which we asked at the beginning.

This paper contains a detailed comparison, *for calcium only*, of the electron relative cross section measurements of Hahn and Hofstadter,[5] the absolute cross-section measurements of Crannel *et al.*,[1] and muonic K_α x-ray measurements of the CERN-Darmstadt collaboration,[3,8] of Anderson *et al.* at Chicago,[2,9] and of a Chicago-Argonne collaboration.[10] Broadly speaking, our results are that the long-wavelength experiments are not in disagreement with the electron relative cross-section experiments for any charge distribution we have tried, although the muonic x ray situation is at present rather unsettled. Our conclusion differs from those drawn previously,[1,2] partly in the inclusion of a reasonable estimate of the errors in the electron relative cross-section analysis. There is also a certain amount of disagreement between our results concerning the family II shape and those of Ford and Wills,[11] which were the basis for all of the previous comparisons. We do find, however, from the relative cross-section analysis alone, a slight shape dependence of the nuclear surface. If substantiated by a more detailed analysis of new experiments, this may lead to a new parameter concerning the nuclear charge distributions.

Time limitations, both personal and computing, prevented us from examining in detail the other light nuclei for which the comparison between electron scattering and muonic x rays showed the same effect. Since the effect was systematic, however, we feel that our conclusions probably would hold for these other nuclei also.

Section 2 of this paper briefly describes some of our computational arrangements. These "experimental details" are necessary in view of our aim, which is to make a detailed and reliable examination and comparison of the present experimental information, and our conclusion, which differs somewhat from those drawn previously. The novelty of our results arises in the details of the electron-scattering analysis, and so, for brevity, we have not made detailed comparison or even reference to the many excellent theoretical discussions of muonic x rays. The charge distributions employed are discussed in Sec. 3, and Sec. 4 describes in detail the results obtained. A summary, and a discussion of the accessibility of additional information about the charge distribution, are given in Sec. 5.

2. THEORY AND METHOD

As has been assumed in previous analyses,[5,12] we represent the nucleus by a static, energy-independent, spherically symmetric charge distribution. The solution of the Dirac equation in the extreme relativistic limit ($mc^2/E=0$) and the calculation of the differential cross section follow very closely the earlier calculation of Yennie, Ravenhall, and Wilson[13] and the analysis will not be described again here. Since any differences in conclusion among various groups must arise from differences in computational technique, we will, as briefly as possible, outline our methods and the checks we have made.

Our calculations were all performed on the IBM-7090 computer at the General Motors Research Laboratory. The code was written entirely in the symbolic language of FORTRAN, and the results presented here were obtained with its single precision, eight decimal digit, floating-point arithmetic. Comparisons have been made at all stages with previous calculations on the Livermore Univac,[5] which used home-made floating point codes, eleven digit for the point–Coulomb phase shifts and wave functions, and nine digit for the rest of the problem. The Coulomb quantities agreed to a few places in the eighth place, which we regard as adequate. The nuclear phase shifts δ_n involve numerical integration of the radial Dirac equation. By careful choice of interval

[5] B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956), hereinafter referred to as HRH. The energy value used in the calculations, 182.5 MeV, allows for a 0.5-MeV average energy loss in the target.

[6] See Table I of Ref. 5.

[7] An investigation very similar in intent to ours has been reported by G. E. Pustovalov, Zh. Eksperim. i Teor. Fiz. **43**, 2170 (1962) [English transl.: Soviet Phys.—JETP **16**, 1534 (1963)]. The parameter values for the family II distribution were taken from Ref. 11, so that a conclusion similar to those of Refs. 1 and 2 was drawn. Calculations of muonic energy levels for the Fermi distribution, with parameter values taken from HRH (Ref. 5) have been reported also by F. J. Bloore, Y. P. Varshni, and J. M. Pearson, Phys. Letters **3**, 303 (1963).

[8] D. Quitmann, R. Engfer, U. Hegel, P. Brix, G. Backenstoss, K. Goebel, and B. Stadler, Nucl. Phys. **51**, 609 (1964).

[9] H. L. Anderson, C. S. Johnson, and E. P. Hincks, Phys. Rev. **130**, 2468 (1963).

[10] Mentioned in C. S. Johnson, H. L. Anderson, E. P. Hincks, S. Raboy, and C. C. Trail, Bull. Am. Phys. Soc. **8**, 324 (1963); S. Raboy (private communication).

[11] K. W. Ford and J. G. Wills, Los Alamos Scientific Laboratory Report 2387, 1960 (unpublished); Nucl. Phys. **35**, 295 (1962).

[12] See R. Hofstadter, *Nuclear and Nucleon Structure* (W. A. Benjamin and Company, Inc., New York, 1963), for a collection of references on this topic.

[13] D. R. Yennie, D. G. Ravenhall, and R. N. Wilson, Phys. Rev. **95**, 500 (1954).

size, the FORTRAN fourth-order Runge-Kutta-Gill subroutine produced values f δ_n accurate to about 10^{-7} rad. The accuracy was ascertained by comparing with hand-calculated phase shifts for a shell charge distribution.[14] Such precision is more than sufficient for the cross sections presented in this paper. The Univac phase shifts[15] were found, by comparison with these results, to be accurate to about 4×10^{-6} rad, but the cross sections were in close agreement over the angular range explored in the Hahn-Hofstadter experiments. For example, the values of χ^2 obtained in the least-squares fitting, usually of order 20, were in agreement to four figures. The improved accuracy, here and in other parts of the calculation, becomes evident at larger angles, however, where by cancellation the cross sections are extremely small ($\sim10^{-33}$ cm^2 per sr).

For another application,[16] we have made a double precision FORTRAN code to calculate the point-Coulomb scattering amplitude, so that the slowly convergent part of the Legendre series may be summed for small angles. The actual shape-dependent amplitude may then be calculated by combining this with the rapidly convergent Legendre series for the nuclear phase shifts. The scattering amplitudes so calculated can be used as a check, at small angles, of the single-precision calculation we usually use. Thus, loss of significance in the summation method used for the Legendre series has been checked.

Numerical values of cross sections, with completely specified parameters to allow comparison with other calculations, have been given elsewhere.[16] The only correction made before comparing with experimental cross sections is for angular resolution of the experiments, as described in Ref. 5. Radiative corrections to the scattering had already been applied to the experimental data by Hahn and Hofstadter,[5] and by Crannell *et al.*[1]

The analytical part of the calculation of muonic energy levels follows closely the work of Ford and Wills.[11] The computational approach we used for this eigenvalue problem which avoids loss of significance at large distances, however, is to integrate the radial Dirac equations outward from the origin, and inward from some radius at which the point-Coulomb wave functions can be calculated by an asymptotic series. The comparison of the ratio of the two components at a radius equal to 1 Bohr radius (in the case of calcium) from each of these integrations enables us to select a new trial eigenvalue, and an accurate match announces that we have found the correct one. Starting from an initial trial value given by the Sommerfeld formula for point-Coulomb energy levels, the iteration converges for the 1s level in calcium after four or five trials.

Our experience with the same integration methods in the electron-scattering code leads us to believe that the eigenvalues are accurate to a few parts in 10^7 for nuclei in the region of calcium. The most direct check on this was a repetition of the numerical integration, for the special case of a uniform charge distribution, on the Illinois computer Illiac I. A fixed-point integration routine with controlled error, due to Nordsieck,[17] was employed, and our belief was confirmed.[18]

Our confidence in the accuracy of our eigenvalues was somewhat disturbed when calculations with light nuclei (e.g., oxygen) failed to agree to better than 10% with the often quoted results of first-order perturbation theory for the relative shifts of the 1s level due to the finite size[19]:

$$\Delta E_{1s}/E_{1s}=(\tfrac{4}{3})\langle r^2\rangle/a_B{}^2.$$

Here $\langle r^2\rangle$ is the mean-square radius of the charge distribution, and $a_B=\hbar^2/(mZe^2)$ is the radius of the first Bohr orbit for this nucleus. The reason for such a disagreement became clear, however; not only are there succeeding terms of the above expression which decrease only as $\langle r^2\rangle^{\frac{1}{2}}/a_B$ ($\sim10\%$ for oxygen) but the effect of second-order perturbation theory is also of the same order of magnitude as the first correction term to $\Delta E/E$.[20] With these extra contributions there was agreement between our numerically computed eigenvalues and those of perturbation theory to a few parts in 10^6, which was the limit of accuracy of our estimate of the latter.

Such accuracy is of course considerably beyond present experimental techniques, and for the most part beyond the expected theoretical corrections to the simple model we have used. There is, however, an advantage in further work of knowing that such unsophisticated computational methods as we have used are more than adequate. A further benefit is that the fine structure of the higher levels, $n=2, 3, 4, \cdots$ of possible interest in future experiments, may be calcu-

[14] As a reference point, the exact result may be useful to other workers. For $\gamma=Ze^2/\hbar c=0.5765$, the shell charge distribution with radius given by $kR=8$ has for the finite-size part of the first phase shift the value given by $\tan\delta_1=-7.11133(2)$. The above dimensionless numbers, given here to their full significance, completely describe the case. With regard to the program value, we note that under the conditions quoted, with only eight decimal digits, the wave functions themselves cannot be accurate to better than about 10^{-5}. The phase of the wave function (in this case, the ratio of the two components) is accurate to about 10^{-7}, however.

[15] Examples were given in D. G. Ravenhall and D. R. Yennie, Proc. Phys. Soc. (London) **A70**, 857 (1957).

[16] R. Herman, B. C. Clark, and D. G. Ravenhall, Phys. Rev. **132**, 414, (1963).

[17] A. Nordsieck, Math. Computation **16**, 22 (1962).

[18] For calcium, with $\gamma=Ze^2/\hbar c$ taken as 0.1459, a uniform charge density of radius $R/\lambda=2.4$ has the eigenvalue $E_{1s}/mc^2=-0.010037961(2)$.

[19] See, for example, L. Cooper and E. Henley, Phys. Rev. **92**, 801 (1953), or, more fully, D. West, Rept. Progr. Phys. **17**, 271 (1958).

[20] If $\langle r^2\rangle^{1/2}\ll\hbar/mc$, the intermediate states of importance have the muon moving with speed comparable to c, and the contribution can become of the same order in $\langle r^2\rangle^{1/2}/a_B$ as the first-order term itself. The poor convergence of the perturbation series originates in the r^{-1} singularity occurring when all of the finite size effect is treated as a perturbation. Calculations such as those of Ref. 7, in which only differences in nuclear shape are treated as a perturbation, are presumably free from this trouble.

TABLE I. Parameters of best fits for calcium. Parameter values for the best fits of the shapes are examined. The top and bottom rows are values obtained in other investigations, and the three middle rows are our present results. The values of c and z are the coordinates of the centers of the ellipses in Figs. 1, 3, and 5. They, and the associated lengths t, the 90–10% surface thickness, and $R=[5\langle r^2\rangle/3]^{1/2}$, the equivalent radius, are given in F. The energy of the muonic $2p\rightarrow 1s$ transition E is in keV. The errors on all of these quantities can be estimated from Figs. 1, 3, and 5. On c, z (or t), and R, the relative errors are respectively of order 1, 20, and 7%, and on E, about 7 keV.

	Shape	c	z	$c/A^{1/3}$	R	$R/A^{1/3}$	t	$E(2p\rightarrow 1s)$
	Fermi (HRH) (Ref. 5)	3.64	0.57	1.06	4.54	1.32	2.48	···
(1)	Fermi	3.64	0.56	1.06	4.51	1.32	2.44	782.3
(3)	Modified Gaussian	3.49	2.20	1.03	4.39	1.28	2.69	784.8
(2)	Family II	3.69	0.75	1.08	4.71	1.38	2.44	777.6
	Family II[a]	3.64	0.89	1.063	5.06	1.47	2.87	772.6

[a] See Ref. 11.

lated in the same manner, rather than by more tedious perturbation methods.

The known theoretical corrections which have been applied are for nuclear recoil and for electron-pair vacuum polarization.

$$E=E(\text{program})+\Delta E_{\text{vp}}.$$

In E (program) the length and energy scale factors are obtained from the reduced muon mass $m_\mu M_{\text{Ca}}/(m_\mu+m_{\text{Ca}})$. For ΔE_{vp}, a correction of order $\frac{1}{2}$% or 5 keV, for calcium, we use the usual lowest order perturbation expression as described, for example, in Ref. 7.

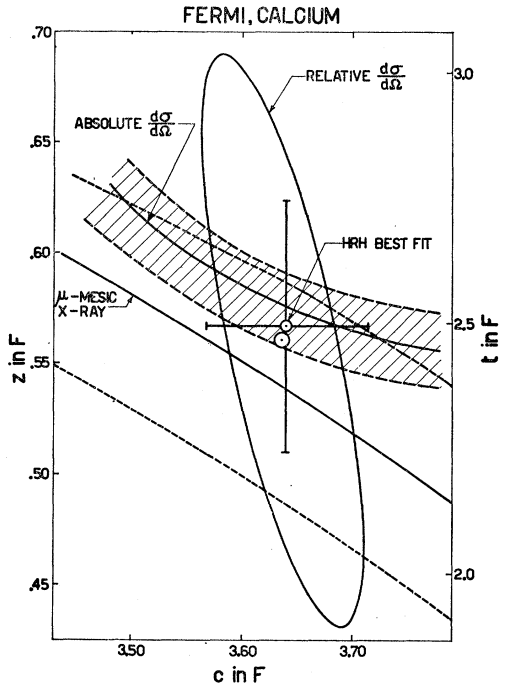

FIG. 1. A comparison of the electron-scattering experiments and the muonic x-ray result for the Fermi shape (1). With c and z, the adjustable parameters of this shape, as coordinates, the ellipse represents shapes which are an equally good fit to the relative cross-section measurements (Ref. 5). The central point, whose parameters are quoted in Table I, is indicated together with the earlier result of HRH (Ref. 5). The shaded band represents the absolute cross-section measurements (Ref. 1) with an assumed relative error of ±10%. The unshaded band is for one of the muonic x-ray measurements (Ref. 3), 784±3 keV. The differential cross section for the best fit and the electron experimental results are shown in the next figure. A comparison of the calculated x-ray energy with all of those experiments (Refs. 2, 3, 8–10) is displayed in Fig. 8.

We have not included any estimate of the level shift due to nuclear polarization. From the most complete calculation we know of, due to Lakin,[21] the contribution from virtual nuclear excitation to the giant dipole resonance, presumably the most important effect in a light nucleus like calcium, can be estimated to be about one keV. This omission is appropriate since we have not included the corresponding effect in electron scattering, which is not calculable in terms of presently known nuclear properties.[22]

3. CALCULATIONS

The discussion will be confined to the two charge distributions between which the question of shape dependence arose, and a third, of some interest in comparison with the Hahn, Ravenhall, and Hofstadter (HRH) analysis.

Fermi:

$$\rho(r)=\rho_1\{\exp[(r-c_1)/z_1]+1\}^{-1}; \tag{1}$$

Family II:

$$\rho(r)=\rho_2\{1-\tfrac{1}{2}\exp[(r-c_2)/z_2]\},\ r\leqslant c_2, \tag{2}$$
$$=\rho_2\,\tfrac{1}{2}\exp[-(r-c_2)/z_2],\ r>c_2;$$

Modified Gaussian (M.G.):

$$\rho(r)=\rho_3\{\exp[(r^2-c_3^2)/z_3^2]+1\}^{-1}. \tag{3}$$

We use a notation for the family II shape which displays its similarity to the others. Each of these shapes, for reasonable values of the adjustable parameters c and z, represents a charge distribution roughly constant in its interior, dropping to half of its central value at

[21] W. Lakin, Technical Report No. 2, OOR No. 116-53, Carnegie Institute of Technology (unpublished). See also, W. Lakin and W. Kohn, Phys. Rev. 94, 787 (1954) (abstract). We thank L. Wolfenstein for help with this reference.

[22] L. I. Schiff, Phys. Rev. 98, 756 (1955), has expressed the effect on the scattering amplitude in terms of the proton-proton correlation function inside the nucleus.

$r=c$, and having a falloff distance characterized by z. The relationships between the three z parameters and the roughly model-independent 90%–10% distance t are approximately $t\simeq 4.40z_1$, $t\simeq 3.22z_2$, $t\simeq 2.20z_3^2/c_3$. More accurate values for t, and also for the "equivalent radius" $R=[5\langle r^2\rangle/3]^{\frac{1}{2}}$, obtained in the present work are given in Table I. In the present calculation, quantities needed in the partial-wave calculation such as ρ_i are obtained automatically in the numerical integration of Poisson's equation which leads to the Coulomb potential.

In comparing the predictions of any one of these charge distributions with the experiments, the procedure was as follows: For some chosen pair of values, c, z, the differential cross section for electron scattering was calculated with the methods outlined in Sec. 2. To allow for experimental angular resolution, the cross section was folded in angle over a Gaussian shape $\exp[-\theta^2/\Delta^2]$ with $\Delta=1°$. A least-squares fit was then made to the experimental data of Hahn and Hofstadter.[5] At the time of those experiments, absolute values of the cross section were not available. To obtain a good fit with theory it is thus necessary to adjust the absolute scale of each of the two experimental runs arbitrarily, by factors which are irrelevant to the fit. In comparing by least squares to the experimental cross sections of Crannell *et al.*,[1] for which absolute values were measured, it is the scale factor which expresses the goodness of fit. For the same shape, after the above procedures have been completed, a calculation is made of the $2p\rightarrow 1s$ x-ray energy, incorporating the corrections mentioned in the previous section.

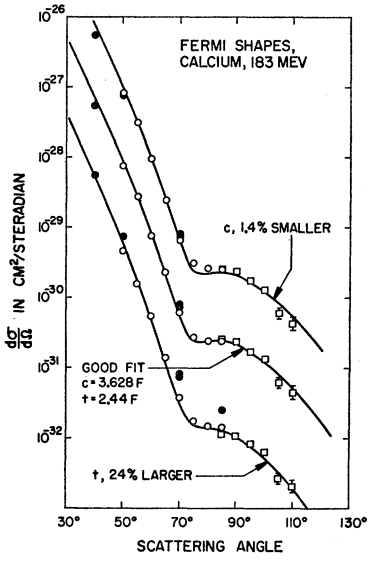

FIG. 2. Differential cross sections for the Fermi shape (1), together with the measurements of relative cross sections (Ref. 5) (open points) and absolute values (Ref. 1) (shaded points). The central curve, the best fit corresponding to the center of the ellipse in the previous figure is described fully from the computational point of view by the following values: $\gamma=Ze^2/\hbar c=0.1459$; $k=0.9250$ F^{-1} (182.5 MeV); $kc=3.365$; $kz=0.515$. The other curves, with vertical scale displaced by a factor 10, correspond to c, z values on the bottom and at the side of the ellipse of the previous figure. They allow a visual judgement of our error estimate (size of the ellipse).

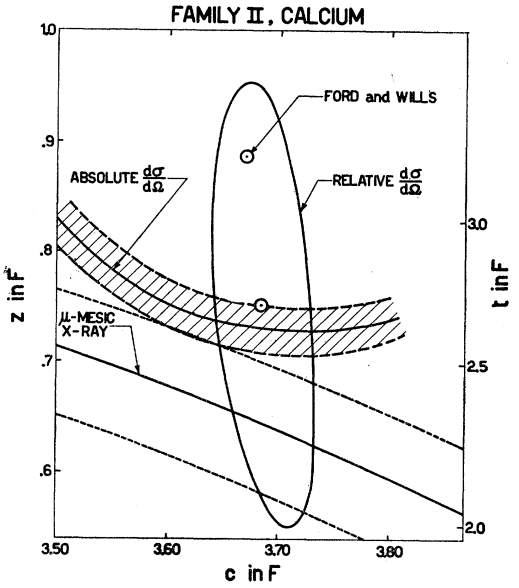

FIG. 3. The same as for Fig. 1, for the family II shape (3). The additional point towards the top of the ellipse is the result of Ford and Wills (Ref. 11).

4. RESULTS

The detailed information about diffraction structure in the Hahn-Hofstadter experimental cross section[5] gives a sensitivity of the fit to both c and z. Roughly speaking, c determines where in the angular scale the diffraction dip occurs, and z moderates the over-all slope and the height of the succeeding plateau. The goodness of fit must thus be displayed, as in Fig. 1, on a two-dimensional plot against c and z. The best fits occupy the area circumscribed by the ellipse in the center.

The absolute cross section of Crannell *et al.*,[1] containing measurements at fewer angles, are not enough by themselves to determine both c and z. If there were a measurement at one angle only, for example, it would always be possible to move from one good fit to an adjacent one by changing c arbitratily, then adjusting z to move the cross section up or down appropriately. Thus, this latter experiment determines not a point (area) on the c, z plot, but a line (band). The fit to the energy of the muonic x ray, which is one number, also displays itself in this way on the c, z plot. As has been emphasized in earlier work on this subject, this energy is sensitive mainly to the mean-square radius of the charge distribution, i.e., to R, so one expects the lines of constant x ray energy to correspond to the lines of constant R. In calcium, over a range of shapes which includes the uniform distribution, we find this corre-

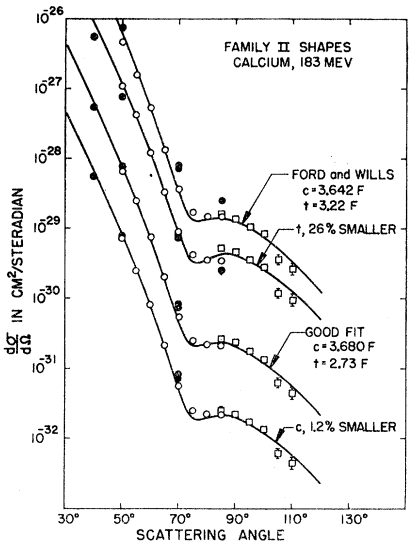

FIG. 4. The same as for Fig. 2 for the family II shape (3). The central value, our best fit, has the following parameter values: γ and k, as in Fig. 2; $kc=3.413$; $kz=0.697$. The Ford-Wills cross section, as calculated by us, uses the values $kc=3.378$, $kz=0.824$. It is seen to be a poor fit to the absolute cross-section measurements (shaded circles).

spondence to be good to about 1 keV, which is rather smaller than the quoted error on the experimental energy.

The question of agreement among these three sets of experiments is then settled by the extent to which the two bands, from absolute cross section and muonic x ray, overlap with each other, and with the ellipse of the relative cross sections. In Figs. 1, 3, and 5 the plots for the three shapes are displayed. The size of the ellipses (for M.G. a more general shape) was chosen to make shapes at the edge a noticeably poorer fit than the best fit, at the center of the ellipse.[23] The actual differential cross sections are displayed in Figs. 2, 4, and 6, to allow visual judgment of our criterion. The width of the absolute cross-section band corresponds to an error in the cross section of $\pm$ 10%. For the muonic x-ray band, the first result of the CERN-Darmstadt collaboration[3] is used,[24] i.e., 784$\pm$3 keV. Because the shapes from relative cross sections are rather elongated in the z direction, producing the previously noted wide error limits on the surface thickness,[25] there seems to be no evidence of disagreement among the three experiments. This is true for each of the shapes considered, and not only for the Fermi shape, as has been hitherto suggested.[1,2]

At this stage it is clear that, with errors from each experiment as indicated here, the absolute cross section and muonic x-ray measurements contribute in a very useful way to a precise knowledge of the nuclear surface thickness.

In Fig. 1 is also shown the HRH result for calcium, with the suggested errors.[25] The slight difference between that result and the present best fit is due entirely to a slightly different treatment of the statistical errors on the individual experimental points. To allow for errors other than those of statistics, the number of counts (see HRH, p. 1135) was assumed to be always less than 150.

The electron-scattering analysis of Ford and Wills,[11] using the family II shape, produced the value shown in Fig. 3. It corresponds to a surface thickness about 18% higher than our best fit, but lies within our error ellipse. This small disagreement could be due to a somewhat different treatment by them of the experimental data. Their result gave a value for R about 7% higher than our value *for this shape*, and thus predicted an x-ray energy about 10 keV lower than that obtained experimentally. It also gave an apparent disagreement with the absolute cross-section experiment, as is seen in Fig. 4. This led to the suggestion of an incompatibility between electron scattering and muonic x rays.[2] The alternative suggestion was also made[1,2] that the lack of agreement for this shape, compared with the good agreement for the Fermi shape, allowed a choice to be made

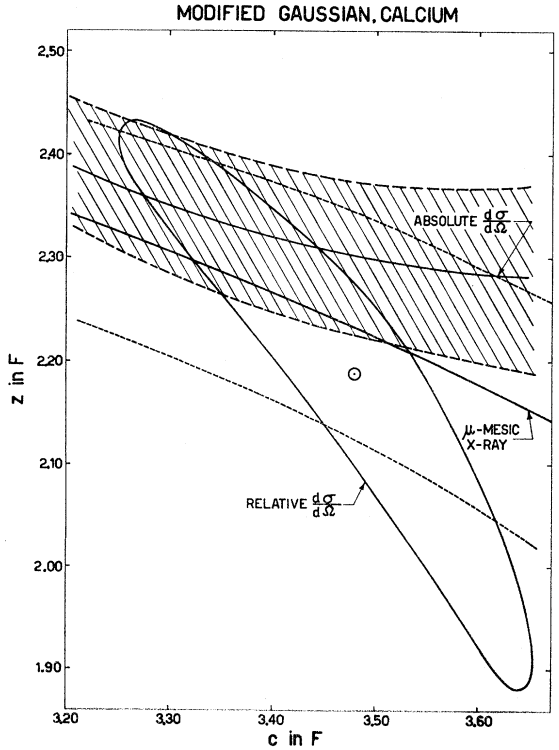

FIG. 5. The same as for Fig. 1, for the modified Gaussian shape (3). The ellipse has become distorted, a reflection of the fact that it is the quantity z_3^2/c_3 rather than z_3 which is proportional to the usual surface thickness t.

[23] The χ^2 of shapes on the ellipse is greater than the χ^2 at the center by 5.

[24] To avoid confusion on these figures, we have plotted the data contained in Refs. 2, 3, 8, 9, and 10 only on the simpler Fig. 8.

[25] See Ref. 5, Sec. V.

between the Fermi and family II shapes. Our present results show that in fact for each shape there is agreement among the three experimental methods analyzed here, provided due allowance is made for the errors in each method.

5. DISCUSSION

A numerical summary of the electron-scattering results quoted in this paper is contained in Table I. The parameter values given relate to the best fits shown in Figs. 2, 4, and 6, and are the coordinates of the centers of the ellipses in Figs. 1, 3, and 5. There are errors on these values corresponding to the sizes of the ellipses, e.g., for the Fermi shape, the relative errors on c and z are, respectively, of order 1.3 and 20%. Of the derived parameters given, we note that t, the 90–10% thickness, varies somewhat among the shapes, as does the equivalent radius R. In the actual normalized charge distributions, shown in Fig. 7, the feature which is a common property of all of the shapes is the charge density in the region of the surface, i.e., from $r \simeq 3F$ to $r \simeq 5F$. It is this, rather than t, R, or c, that the electron experiments may be said to determine. This is the same conclusion as was reached previously with gold.[5]

The comparison with the muonic x ray experiments is summarized in Fig. 8. This figure displays a present rather large spread in energy among the various x-ray measurements, considerably bigger than the quoted errors. This prevents at present our using them to sharpen the electron-scattering results as regards the measurements of the surface thickness. It is to be emphasized, however, that once a precise value is agreed upon for the x-ray energy, it will be very valuable in this respect. This assumes that remaining theoretical corrections to both electron scattering and muonic x rays do not turn out to be important, and that the muon interaction is entirely electromagnetic. On the first point, it may be that effects like nuclear polarization affect the two processes in corresponding ways, and are already allowed for by regarding $\rho(r)$ as a phenomenological quantity rather than a property of the nuclear ground state. The second seems to be well taken care of in the range of recoil momenta of interest here by muon-scattering experiments reported recently.[26]

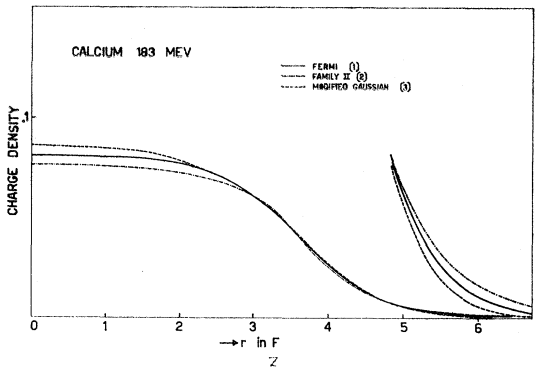

FIG. 7. The normalized charge densities for our best fits with the Fermi shape (1), family II (2), and modified Gaussian (3). The parameter values are given in Table I. At the edge, the densities are shown times a factor 5.

A feature of the relative cross-section analysis which has not been stressed, but which may have significance for future work, is that among three shapes examined, the value of χ^2 at the minimum differs markedly. The absolute value is not too significant, because of our somewhat arbitrary assumption about the errors on each point. But for the family II, Fermi, and M.G. shapes, χ^2 has the minimum values 15, 9, and 6.5, respectively. In Fig. 9, the corresponding cross sections, after being scaled by the factors indicated to display the agreement at small angles, show a systematic trend at large angles, the best fit (modified Gaussian) falling off most rapidly. As is seen in Fig. 7, a feature of $\rho(r)$ which also changes systematically in progressing through these shapes is the amount of charge at large distances, i.e., beyond the 10% radius. The present results favor the M.G. charge distribution, which, of the three types, has least charge at large distances. It is to be noted that the variation is found after the two variable parameters in each shape have been adjusted to give a best fit.

Less detailed calculations have been made with other shapes, including the trapezoidal shape[5] and the HRH three-parameter shape.[5] They tend to bear out the above suggestion, in that the trapezoidal shape gives the best fit found yet. Detailed numerical results will be reported in future work.

We realize, of course, that there is an uncertainty in the charge distribution predicted by each shape which

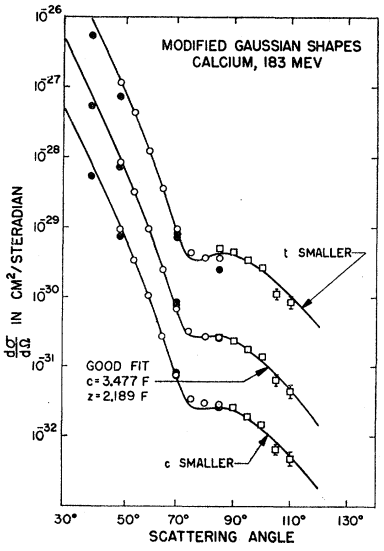

FIG. 6. The same as for Fig. 2 for the modified Gaussian shape (3). The central value, our best fit, has the following parameter values: γ and k, as in Fig. 2; $kc=3.225$, $kz=2.03$.

[26] A. Citron, C. Delorme, D. Fries, L. Goldzahl, J. Heintze, E. G. Michaelis, C. Richard, and H. Øveras, Phys. Letters **1**, 175 (1962); H. F. Davis, T. E. Ewart, G. E. Masek, E. D. Platner, J. P. Toutonghi, and R. W. Williams, Phys. Rev. **131**, 2192 (1963).

smears out the results of Fig. 7. There are also uncertainties arising from our somewhat arbitrary and restricted choices of trial functions for the charge density. We wish at this time only to suggest that, based on the early data of Hahn and Hofstadter, there is more information to be obtained from the elastic-scattering measurements than the hitherto quoted radius and surface thickness. An investigation is in progress now to analyze more recent electron-scattering data,[27] and to try to find what model-independent feature, be it the tail of the charge distribution or the possibility of a central depression, is the next piece of information to be discovered.

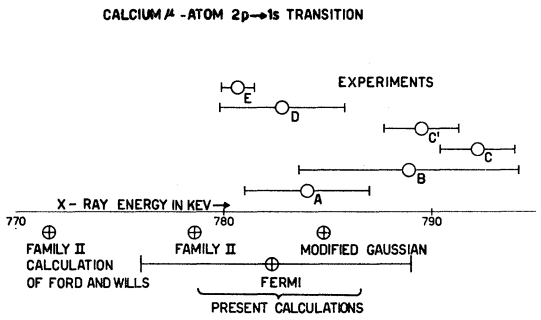

FIG. 8. A one-dimensional display of the muonic x-ray energies for the $2p \rightarrow 1s$ transition. The points above the line are the following experimental values: A, CERN-Darmstadt (Ref. 3), 784±3 keV; B, Chicago (Ref. 2), 788.9±5.3 keV; C, C′, Chicago (Ref. 9), 792.2±1.8 keV, 789.5±1.8 keV; D, CERN-Darmstadt (Ref. 8), 782.8±3 keV; E, Chicago-Argonne (Ref. 10), 780.7±0.8. The points below the line are the prediction of Ford and Wills (Ref. 11), and/or present calculations (see Table I) with an estimated error taken from the ellipse in Fig. 1.

[27] A. E. Walker, M. R. Yearian, R. Hofstadter, M. Croissiaux, D. G. Ravenhall, R. Herman, and B. C. Clark (unpublished).

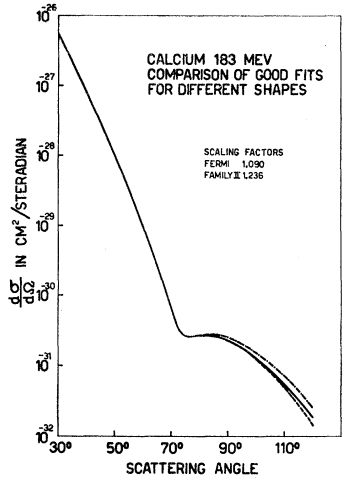

FIG. 9. The three cross sections corresponding to the best fits with the three shapes (for parameters, see Table I and the captions to Figs. 2, 4, and 6), with the same line convention as in Fig. 7. They have been scaled by the indicated factors which came from the scale factors of the least-squares fitting. The agreement over most of the angular range is seen, together with a systematic difference at the largest angles.

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