Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse*

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The Einstein equations for a spherically symmetrical distribution of matter are studied. The matter is described by the stress-energy tensor of an ideal fluid (heat flow and radiation are therefore excluded). In this case, the Einstein equations give a generalization of the Oppenheimer-Volkoff equations of hydrostatic equilibrium so as to include an acceleration term and a contribution to the effective mass of a shell of matter arising from its kinetic energy. A second equation also appears in this time-dependent case; it gives the rate of change of an appropriate "total energy" $m(r,t)$ of each fluid sphere in terms of the work done on this sphere by the fluid surrounding it. These equations would be an appropriate starting point for a study of relativistic gravitational collapse in which an adiabatic equation of state more realistic than the $p=0$ form of Oppenheimer and Snyder could be used.

I. INTRODUCTION AND SUMMARY

THE original discussion of an idealized problem of gravitational collapse due to Oppenheimer and Snyder[1] assumes a spherically symmetric distribution of matter, adiabatic flow (no viscosity, heat conduction, or radiation), the equation of state $p=0$, and simple initial conditions. In this note we maintain the assumptions of spherical symmetry and adiabatic flow, and consider the introduction of pressure gradient forces into the equations. Our purpose is to cast the equations into as simple and physically transparent a form as we can, preliminary to their numerical solution.

Much of the recent interest[2] in gravitational collapse centers about the possibility (in a stage of collapse where the gravitational binding energy GM^2/R becomes comparable to the rest energy Mc^2) of a large *energy output* of a star, a discussion of which falls outside the scope of the equations derived here. Nevertheless, a study of these equations may provide a useful first step in a more realistic analysis of the gravitational collapse of stars—which would presumably include the effects of rotation, departures from spherical symmetry, and radiation—as well as some insight into the issues of principle involved in gravitational collapse.[3]

In the remaining paragraphs of this section we will summarize our results. These are derived in the succeeding sections.

Associated with an ideal fluid is a stress energy given by the tensor

$$T^{\mu\nu}=(p+\epsilon)u^\mu u^\nu+pg^{\mu\nu}, \tag{1.1}$$

where u^μ is the four-velocity field of the fluid, ϵ is the internal energy of the fluid per unit proper rest volume, and p is the pressure. Because this tensor is diagonal in the local rest frame of the fluid, it cannot describe the energy flow associated with heat conduction or radiation. Using Eq. (1.1) in the statement $u_\mu T^{\mu\nu}{}_{;\nu}=0$ of local energy conservation shows that the entropy of each particle in the fluid is constant, $u^\mu s_{,\mu}=0$. We summarize here the equations in the isentropic case, where one further assumes $s_{,\mu}=0$, so that the specific entropy is constant throughout the volume of the fluid.

The metric is chosen to have the diagonal form

$$ds^2=-e^{2\phi}dt^2+e^\lambda dr^2+R^2d\Omega^2, \tag{1.2}$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2. \tag{1.3}$$

Here ϕ, λ, and R are each functions of r and t to be determined by the Einstein field equations. We shall work in a system of coordinates moving at each point with the material located at that point (comoving or Lagrangian coordinates). The components of the four-velocity are thus

$$u^t=e^{-\phi},\quad u^i=0;\quad i=r,\theta,\varphi. \tag{1.4}$$

Then the hydrodynamic equations $T^{\mu\nu}{}_{;\nu}=0$ give the result

$$e^\phi=(-g_{00})^{1/2}=1/h, \tag{1.5}$$

where $h=u+pv=(\epsilon+p)/n$ is the specific enthalpy or heat function for a unit amount of fluid (the amount containing a mole of baryons). [The specific internal energy u and the specific volume v are related to the matter density or baryon number density $n(r,t)$ by $u=\epsilon v$ and $v=1/n$. We choose units of $n(r,t)$ so that $\epsilon\to n$ and $h\to 1$ as $p\to 0$.] In order to compute h, it is sufficient to specify the adiabatic equation of state

$$\epsilon=\epsilon(n), \tag{1.6}$$

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[1] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).
[2] W. A. Fowler, Rev. Mod. Phys. **36**, 549 (1964); F. Hoyle, W. A. Fowler, G. R. Burbidge and E. M. Burbidge, Astrophys. J. **139**, 909 (1964); H. Y. Chiu, Ann. Phys. **26**, 364 (1964); F. C. Michel, Astrophys. J. **138**, 1097 (1963); S. A. Colgate and R. H. White, Bull. Am. Phys. Soc. **8**, 306 (1963).
[3] J. A. Wheeler, in *Gravitation and Relativity*, edited by H. Y. Chiu and W. F. Hoffmann (W. A. Benjamin Company, Inc., 1964), Chap. 10.

for then the pressure equation $p(n)$ can be deduced via the thermodynamic relation

$$p=n\left(\frac{\partial\epsilon}{\partial n}\right)_s-\epsilon\,, \tag{1.7}$$

and from $h=(\epsilon+p)/n$ one finds that

$$h=\left(\frac{\partial\epsilon}{\partial n}\right)_s. \tag{1.8}$$

The remaining field equations for this problem take a simple form if one defines a quantity U which gives the relative velocity $Ud\theta$ of adjacent fluid particles on the same sphere of constant r,

$$U=D_tR\equiv e^{-\phi}\dot{R}. \tag{1.9}$$

Here D_t is the comoving proper-time derivative

$$D_t=u^\mu\frac{\partial}{\partial x^\mu}=e^{-\phi}\left(\frac{\partial}{\partial t}\right)_r. \tag{1.10}$$

One also uses in place of $\lambda(r,t)$ a function $m(r,t)$ defined by

$$e^{\lambda(r,t)}=g_{rr}=\left[1+U^2-\frac{2m(r,t)}{R}\right]^{-1}\left(\frac{\partial R}{\partial r}\right)^2. \tag{1.11}$$

The full set of field equations are then the three first-order dynamical equations

$$D_tR=U\,, \tag{1.12-R}$$

$$D_tm=-4\pi R^2pU\,, \tag{1.12-m}$$

$$D_tU=-\left[\frac{1+U^2-2mR^{-1}}{\epsilon+p}\right]\left(\frac{\partial p}{\partial R}\right)_t-\frac{(m+4\pi R^3p)}{R^2}, \tag{1.12-U}$$

two equations free from time derivatives, namely, Eq. (1.5) and the equation

$$\left(\frac{\partial m}{\partial R}\right)_t=4\pi R^2\epsilon\,, \tag{1.13}$$

and the equation of continuity $(nu^\mu)_{;\mu}=0$. The continuity equation can be written in a form

$$\frac{4\pi R^2n}{(1+U^2-2mR^{-1})^{1/2}}\frac{\partial R}{\partial r}=\left(\frac{dA}{dr}\right)_{t=0} \tag{1.14}$$

appropriate to our comoving coordinates, where the amount of matter dA in any spherical shell defined by a fixed coordinate range dr is independent of time.

Solutions to the above system of equations can be obtained by specifying arbitrary initial values for $R(r,0)$, $m(r,0)$, and $U(r,0)$. Equation (1.13) then defines $\epsilon(r,0)$ which gives the values of p, n, and h through an equation of state and thus allows the time derivatives of R, m, and U to be obtained from Eqs. (1.12). One thus obtains a solution for all times without invoking Eq. (1.14); but this equation merely defines dA/dr initially, and it is possible to show that the time derivative of the left member of Eq. (1.14) vanishes as a consequence of Eqs. (1.12), (1.13), (1.5), and (1.7). Thus, Eq. (1.14) is a first integral of this system of equations.

The above system of equations is to be solved subject to the boundary condition that

$$p=0\quad\text{at}\quad r=r_s=\text{constant}\,, \tag{1.15}$$

where r_s defines the outer boundary of the distribution of matter. It is then evident from Eq. (1.12-m) that

$$m(r_s,t)=M \tag{1.16}$$

is a constant and, in fact, the interior metric (1.2) can be joined smoothly at the surface r_s to an exterior Schwarzschild metric whose mass M is given by Eq. (1.16).

It is also necessary to require that at $r=0$ the functions R, m, and U all vanish.

II. THERMODYNAMIC PRELIMINARIES

Local properties of a fluid such as pressure, temperature, specific entropy, internal energy density, etc., which are scalars in nonrelativistic physics can all be defined in special and general relativity so that they are again scalars. For, to be scalars, they need merely have a well-defined value at any event, independent of every arbitrary choice of a coordinate system. One achieves this by defining these quantities to have (in any coordinate system) the values measured by an observer who is at rest relative to the chosen small piece of fluid at the time in question.

The basic law of thermodynamics[4]

$$du=Tds-pdv \tag{2.1}$$

applies to a fixed amount of matter which, for convenience, we take to be a unit amount. The fact that the amount of matter does not change can be expressed by introducing the particle number density $n=(1/v)$ and requiring it to satisfy a continuity equation:

$$(nu^\mu)_{;\mu}=0. \tag{2.2}$$

This law of conservation of matter in hydrodynamics can be derived from the microscopic law of conservation of baryons.

[4] The discussion in Secs. II and III is based on that of L. Landau and E. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chap. XV, especially in its emphasis on the continuity equation (2.2). For more complete discussions of the subject see: (i) A. Lichnerowicz, *Theories Relativistes de La Gravitation et de L'Electromagnetisme* (Masson et Cie, Paris, 1955); (ii) J. L. Synge, Relativistic Hydrodynamics, Proc. London Math. Soc. **43**, 376 (1937).

When Eq. (2.1) is rewritten in terms of the energy density $\epsilon = u/v$ and particle number density n, it reads

$$d\epsilon = nTds + (\epsilon + p)n^{-1}dn \qquad (2.3)$$

and gives Eq. (1.7) in the case $ds=0$. Thus, $\epsilon=\epsilon(s,n)$ is a convenient fundamental thermodynamic relationship for describing a fluid; it immediately gives the pressure equation $p(s,n)$ via Eq. (1.7). By differentiating the definition of specific enthalpy, $h=u+pv=(\epsilon+p)/n$, and employing Eq. (2.1) or (2.3), one obtains a relation

$$\frac{dh}{h} = \frac{dp}{(\epsilon+p)} + \frac{Tds}{h} \qquad (2.4)$$

which will be useful later.

III. HYDRODYNAMICS REVIEW

The equations of motion of a fluid described by the stress-energy tensor[4] $T^{\mu\nu}$ of Eq. (1.1) and an equation of state $\epsilon=\epsilon(s,n)$ are $T^{\mu\nu}{}_{;\nu}=0$. One of these four equations, namely, $u_\mu T^{\mu\nu}{}_{;\nu}=0$, reduces as a consequence of Eqs. (2.2) and (2.3) to the heat transfer equation for an ideal fluid, which is the condition of adiabatic flow

$$u^\mu s_{,\mu} = 0 . \qquad (3.1)$$

The remaining equations can be reduced to the form of relativistic Euler equations:

$$u^\mu{}_{;\nu} u^\nu = -(g^{\mu\nu} + u^\mu u^\nu) \frac{p_{,\nu}}{\epsilon + p} . \qquad (3.2)$$

In the special case of *isentropic flow*, where one assumes the specific entropy s to be constant throughout the fluid $s_{,\mu}=0$, the Euler equation can be rewritten as

$$u^\mu{}_{;\nu} u^\nu = -(g^{\mu\nu} + u^\mu u^\nu)(\ln h)_{;\nu} \qquad (3.3)$$

by use of Eq. (2.4).

It is evident that in the isentropic case we may consider ϵ, p, and h as functions of the particle number density n alone, i.e., $\epsilon=\epsilon(n)$, $p=p(n)$, $h=h(n)$.

IV. COORDINATES AND METRIC

The metric (1.2) must satisfy certain conditions at the origin $r=0$ to assure regularity there. The first is

$$R(0,t) = 0 . \qquad (4.1)$$

Next, in order for the usual Lorentz-Minkowski geometry to be valid in an infinitesimal neighborhood of the origin, we must require that the circumference $2\pi R$ of an infinitesimal sphere about the origin be just 2π times its proper radius $e^{\lambda/2}dr$, or

$$e^\lambda = (\partial R/\partial r)^2 \quad \text{at} \quad r=0 . \qquad (4.2)$$

Other conditions must hold at the interface between the region occupied by matter (defined by a certain constant coordinate value $r=r_s$ for the interior solution) and the surrounding empty space in order that the interior metric (1.2) can be joined smoothly to the exterior Schwarzschild metric

$$ds^2 = -(1-2MR^{-1})dt^2 + \frac{dR^2}{1-2MR^{-1}} + R^2 d\Omega^2 . \qquad (4.3)$$

These conditions will serve to relate the exterior coordinates R and t to the interior t coordinate and interior metric component $g_{\theta\theta}=R^2(r,t)$. Assume that in the exterior R,t coordinates the interface is described by an equation

$$R = R_s(t) . \qquad (4.4)$$

The metric on the interface is obtained by inserting this in Eq. (4.3), or alternatively by setting $r=r_s=\text{const}$ in Eq. (1.2). By equating these two expressions,

$$(ds^2)_{\text{surf}} = -\left(1-\frac{2M}{R_s}\right)dt^2 + \frac{\dot{R}_s{}^2 dt^2}{1-2MR_s{}^{-1}} + R_s{}^2 d\Omega^2$$
$$= -(e^{2\phi})_s dt^2 + R^2(r_s,t) d\Omega^2 , \qquad (4.5)$$

we find an equation for the interface in the exterior coordinates. It reads

$$R = R_s(t) = R(r_s,t) , \qquad (4.6)$$

provided that we insist that the interior and exterior time coordinates agree on the surface. This boundary condition on t then leads to one on e^ϕ, namely,

$$(e^\phi)_{r=r_s} = [1-2MR_s{}^{-1}][1+U_s{}^2-2MR_s{}^{-1}]^{-1/2} , \qquad (4.7)$$

where we have defined

$$U_s = (e^{-\phi})_s \dot{R}_s \equiv \left(e^{-\phi}\frac{\partial R}{\partial t}\right)_s . \qquad (4.8)$$

The function $U_s(t)$ is the rate of change of R_s with respect to the proper time of a comoving observer. The conditions derived from the continuity of the derivatives of the metric can best be considered later.

V. EULER EQUATION

In the comoving coordinates defined by Eq. (1.4), one obtains from Eq. (3.2) only one nontrivial Euler equation, which reads:

$$\partial\phi/\partial r = -[1/(\epsilon+p)]\partial p/\partial r . \qquad (5.1)$$

In the isentropic case we may use Eq. (2.4) to integrate Eq. (5.1). With the boundary condition (4.7), one finds

$$e^\phi = \frac{1}{h(r,t)} \frac{1-[2M/R_s(t)]}{\{1+U_s{}^2(t)-[2M/R_s(t)]\}^{1/2}} \qquad (5.2)$$

when h is normalized so that $h=1$ at the surface $r=r_s$. However, the coordinate conditions (1.4) and the diagonal form of the metric (1.2) are preserved by transformations of the interior time coordinate t of the form $t \rightarrow f(t)$, so it is possible to change e^{ϕ} by a *factor* which is an arbitrary function of time. Consequently, the solution (1.5) is also acceptable. Use of Eq. (1.5) synchronizes the interior time coordinate with the proper time of a comoving observer at the interface $r=r_s$, and prevents the interior time coordinate from inheriting the singularities of the exterior time coordinate when the surface falls through the Schwarzschild "singularity" $(R_s(t)-2M) \rightarrow 0$.

When different layers in the body are allowed to have different adiabatic equations of state $\epsilon(n)$, it is not possible to integrate Eq. (5.1) in terms of the specific enthalpy, but an integrated form such as

$$\phi = +\int_R^{R_s} \frac{1}{\epsilon+p} \frac{\partial p}{\partial R} dR \tag{5.3}$$

can of course be written. The boundary condition incorporated into Eq. (5.3) makes $e^{\phi}=1$ at the surface $R=R_s$ as in Eq. (1.5). The analogous generalization of Eq. (5.2) is evident.

VI. INITIAL VALUE EQUATIONS

The Einstein equations corresponding to the metric (1.2) can be found in Landau and Lifshitz.[5] Since one knows[6] that the $T_0{}^0$ and $T_i{}^0$ equations will contain no second time derivatives, one may hope to find something simple in them for a starting point. In the present case, the $T_r{}^0$ equation is the simplest. It reads

$$e^{-\phi}\dot{\lambda} = 2U'/R', \tag{6.1}$$

where we use dots and primes to indicate the partial derivatives with respect to t and r, respectively, and define U by Eqs. (1.6) and (1.7). We may use the differential operators

$$\partial/\partial R = (1/R')(\partial/\partial r)_t \tag{6.2}$$

and

$$D_t = e^{-\phi}(\partial/\partial t)_r \tag{6.3}$$

to rewrite the initial value equation (6.1) in the form

$$D_t\lambda = 2(\partial U/\partial R). \tag{6.4}$$

This equation may then be used to eliminate $\dot{\lambda}$ from all the other Einstein equations. When it is used in the $T_0{}^0$ equation, one finds

$$\begin{aligned} 8\pi\epsilon R^2 = 1 + U^2 + R(\partial U^2/\partial R) \\ -[2RR''+R'^2]e^{-\lambda} - RR'(e^{-\lambda})'. \end{aligned} \tag{6.5}$$

Since this equation is of first order, and even linear, in $e^{-\lambda}$ we try to solve it for this function. The work of Oppenheimer and Volkoff[7] in the static case, where $e^{-\lambda}=1-2mr^{-1}$, indicates something of the form the solution might take. But the boundary condition (4.2) suggests some modifications of this. The form $e^{-\lambda}=R'^{-2}-2mR^{-1}$ satisfies these boundary conditions if $(m/R)\rightarrow 0$ as $r \rightarrow 0$, but it is inappropriate for dimensional reasons: The Lagrangian coordinate r is arbitrary at time $t=0$, and therefore can be assigned a dimension independent of all other quantities in the problem, while m/R is dimensionless. This leads us to try the form $e^{-\lambda}=R'^{-2}(1+f)$ which simplifies Eq. (6.5) so that it reads

$$8\pi\epsilon R^2 = \partial(RU^2 - Rf)/\partial R \tag{6.6}$$

and thus yields the solution $f=U^2-2mR^{-1}$ as given in Eqs. (1.11) and (1.13). To interpret Eq. (1.13) it is best to rewrite the integral

$$m(r,t) = \int_0^R 4\pi R^2 \epsilon dR \tag{6.7}$$

in terms of the element of proper volume

$$d^3V = 4\pi R^2 e^{\lambda/2} dr \tag{6.8}$$

to obtain

$$m = \int_0^r \epsilon \left(1+U^2-\frac{2m}{R}\right)^{1/2} d^3V. \tag{6.9}$$

This last form reminds us that when considered as an energy, m includes contributions from the kinetic energy and the gravitational potential energy.

It is now possible to rewrite the constraint (6.4) in an interesting form by substituting for λ from Eq. (1.11). The computation involves interchanging the operators D_t and $\partial/\partial r$ to write

$$D_t \ln R' \equiv \frac{\partial U}{\partial R} + U\frac{\partial \phi}{\partial R}. \tag{6.10}$$

Using this identity and Eq. (5.1) gives

$$D_t \ln\left(1+U^2-\frac{2m}{R}\right)^{1/2} = -U\frac{1}{\epsilon+p}\frac{\partial p}{\partial R}. \tag{6.11}$$

This equation is a useful first integral in the cases considered by Oppenheimer and Snyder[1] and by Bondi[8] where $p=0$. For then, since $e^{\phi}=1$ by Eq. (1.5), it reads $\frac{1}{2}\dot{R}^2-(m/R)=E=\text{const}$ and will give Newtonian free fall for $R(t)$ when we later discover that $m(t)$ is constant with this special $p=0$ equation of state.

[5] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), Sec. 11–7, Problem 5.

[6] Y. Bruhat, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Eq. (4-1.8).

[7] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).

[8] H. Bondi, *Monthly Notices of the Royal Astronomical Society* **107**, 410 (1947).

VII. EQUATION OF MOTION

It is known[9] that the Einstein equations

$$R_{ij}=8\pi(T_{ij}-\tfrac{1}{2}g_{ij}T^{\mu}{}_{\mu}) \qquad (7.1)$$

for i, $j=1$, 2, 3 contain as leading terms just $\partial K_{ij}/\partial t$, where K_{ij} is the second fundamental form of the $t=$const surface. Equivalently, the *only* second time derivative that appears in Eq. (7.1) is $\partial^2 g_{ij}/\partial t^2$. Thus, the R_{rr} equation will contain just $\ddot{\lambda}$ and will be an identity since we have eliminated λ from our scheme by solving Eq. (6.5). The $R_{\theta\theta}$ and $R_{\varphi\varphi}$ equations will be equivalent (by symmetry) and each will contain just $\ddot{R}$. They read

$$\begin{aligned}\frac{G}{c^4}4\pi R^2(\epsilon-p)&=e^{-\phi}\partial(R\dot{R}e^{-\phi})/\partial t+\tfrac{1}{2}e^{-2\phi}R\dot{R}\dot{\lambda}\\&\quad+1-e^{-\lambda/2}\partial(RR'e^{-\lambda/2})/\partial r-e^{-\lambda}RR'\phi'\\&=R_{\theta\theta}=(\sin^2\theta)^{-1}R_{\varphi\varphi}. \qquad (7.2)\end{aligned}$$

In this equation we introduce the operator D_t of Eq. (1.10) and, for some intermediate computations, the operator

$$D_r=e^{-\lambda/2}\frac{\partial}{\partial r}=\left(1+U^2-\frac{2m}{R}\right)^{1/2}\frac{\partial}{\partial R} \qquad (7.3)$$

Then $\dot{\lambda}$ is eliminated using Eq. (6.4), $\dot{R}$ is replaced by U via Eq. (1.9), and ϕ' with p' via Eq. (5.1). This result is Eq. (1.12-U) which includes the well-known Oppenheimer-Volkov[7] equation of hydrostatic equilibrium in the limiting case $U=0$.

Using the main equation (1.12-U) we can carry out some of the differentiations in Eq. (6.11) to reduce it to the form (1.12-m). It is this form which shows that $\dot{m}=0$ in the case of a $p=0$ equation of state.

VIII. EQUATION OF CONTINUITY

The continuity equation (2.2) implies quite generally that the integral

$$A=\int nu^0(-g)^{1/2}d^3x \qquad (8.1)$$

taken over a $t=$const surface is independent of t. Its value A is analogous to the mass number of a nucleus and represents the total amount of matter, or total number of baryons, in the system. In comoving coordinates satisfying Eq. (1.4) the corresponding integral over any fixed domain of the spatial coordinates $x^i(i=1, 2, 3)$ is time-independent, since Eq. (2.2) then reads

$$\partial(nu^0\sqrt{-g})/\partial t=0. \qquad (8.2)$$

For our problem we can insert expressions for u^0 and $\sqrt{-g}$ here to obtain the statement that

$$4\pi R^2nR'/[1+U^2-2mR^{-1}]^{1/2}=A'(r) \qquad (8.3)$$

is *time-independent.*

[9] Reference 6, Eq. (4-1.9).

Because we have rather thoroughly reshuffled the Einstein equations of the metric (1.2) in obtaining a system of equations for independent field variables R, m, and U, it may be of interest to prove directly that for each r, Eq. (8.3) gives a constant of motion for the system of equations (5.1), (1.12), and (1.13) supplemented by an adiabatic equation of state $\epsilon(n)$ for each r. The computation begins by forming the logarithmic derivative of Eq. (8.3)

$$\frac{D_tA'}{A'}=\frac{D_tn}{n}+\frac{2D_tR}{R}+\frac{D_tR'}{R'}-\tfrac{1}{2}D_t\ln\left(1+U^2-\frac{2m}{R}\right). \qquad (8.4)$$

In the second term of Eq. (8.4) we can write by (1.12-R) that $D_tR=U$; the third term has been rewritten in Eq. (6.10); the derivatives in the last term can all be evaluated from Eqs. (1.12) and give Eq. (6.11) which, by (5.1), reads

$$D_t\ln\left(1+U^2-\frac{2m}{R}\right)^{1/2}=U\frac{\partial\phi}{\partial R}. \qquad (8.5)$$

We thus obtain the reduced form

$$\frac{D_tA'}{A'}=\frac{D_tn}{n}+\frac{1}{R^2}\frac{\partial}{\partial R}(R^2U). \qquad (8.6)$$

Because of the adiabatic condition $D_ts=0$, changes in density are related according to Eq. (2.3) by

$$\frac{D_tn}{n}=\frac{D_t\epsilon}{\epsilon+p}. \qquad (8.7)$$

In the reduced system of equations for R, m, and U, the density ϵ is given by Eq. (1.13) which we differentiate to obtain $D_t\epsilon$:

$$D_t\frac{\partial m}{\partial R}=8\pi RU\epsilon+4\pi R^2D_t\epsilon. \qquad (8.8)$$

To evaluate the left-hand side of this equation we need the commutator

$$\left[D_t,\frac{\partial}{\partial R}\right]=\frac{\partial\phi}{\partial R}\left(D_t-U\frac{\partial}{\partial R}\right)-\frac{\partial U}{\partial R}\frac{\partial}{\partial R} \qquad (8.9)$$

in which $\partial\phi/\partial R$ can be eliminated using Eq. (5.1). We find then with the use of Eqs. (1.12-m) and (1.13) that

$$D_t\frac{\partial m}{\partial R}=-8\pi RpU-4\pi R^2(\epsilon+p)\frac{\partial U}{\partial R}, \qquad (8.10)$$

which allows us to rewrite Eq. (8.8) in the form

$$D_t\epsilon=-(\epsilon+p)\frac{1}{R^2}\frac{\partial}{\partial R}(R^2U). \qquad (8.11)$$

Combining this with Eqs. (8.7) and (8.6) then gives

$$D_t A' = 0 \tag{8.12}$$

as we wished to show.

IX. BOUNDARY CONDITIONS

The condition, previously discussed, that the metric or first fundamental form of the boundary surface should be the same whether obtained from the interior or exterior metric, guarantees that for some coordinate system the metric components $g_{\mu\nu}$ will be continuous across the surface. In order to guarantee that coordinates can be introduced for which the first derivatives of the metric, $g_{\mu\nu,\alpha}$, are continuous, it is sufficient that the second fundamental form be the same whether the boundary surface is considered imbedded in the interior or the exterior space-time.[10] For any hypersurface s with unit normal vector n^μ, the second fundamental form Φ is defined as[11]

$$\Phi = (-n_{\mu;\nu} dx^\mu dx^\nu)_s \,, \tag{9.1}$$

where the subscript s means that one of the coordinate differentials is to be eliminated using the equation of the surface. For example, one sets $(dR - \dot{R}_s dt)_s = 0$ in the exterior coordinates of our problem. For comparison purposes, we write

$$\Phi = K_{t't'}(e^\phi dt)^2 + K_{\theta'\theta'}[(R_s d\theta)^2 + (R_s \sin\theta d\varphi)^2] \tag{9.2}$$

and compute from the interior solution that for the hypersurface $r = r_s$ one has

$$K_{\theta'\theta'} = -R_s^{-1}[1 + U_s^2 - (2m_s/R_s)]^{1/2} \,, \tag{9.3}$$

while the exterior metric gives, for the hypersurface $R = R_s(t)$,

$$K_{\theta'\theta'} = -R_s^{-1}[1 + U_s^2 - (2M/R_s)]^{1/2} . \tag{9.4}$$

Matching these components of Φ therefore gives $M = m_s$ which is Eq. (1.16). Since M is a constant this equation can be differentiated with respect to t with $r = r_s$ to give $\dot{m}_s = 0$ which implies, through Eq. (1.12-m), that $p_s U_s = 0$. The correct boundary condition is more specifically

$$p_s \equiv p(r_s, t) = 0 \,, \tag{9.5}$$

as can be seen by comparing the interior and exterior components $K_{t't'}$ and using the field equation (1.12-U). The interior computation gives

$$K_{t't'}^{-} = -\left(1 + U^2 - \frac{2M}{R}\right)^{1/2} \frac{1}{\epsilon + p} \frac{\partial p}{\partial R} \,, \tag{9.6}$$

while the exterior gives

$$K_{t't'}^{+} = +\left(1 + U^2 - \frac{2M}{R}\right)^{-1/2} \left\{\frac{M}{R^2} + D_t U\right\} . \tag{9.7}$$

The difference of these two, using Eq. (1.12-U) and $m_s = M$, is

$$K_{t't'}^{+} - K_{t't'}^{-} = -[1 + U_s^2 - (2M/R_s)]^{-1/2} 4\pi p_s R_s . \tag{9.8}$$

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[10] D. L. Beckedorff, thesis, Princeton University, Mathematics Department, 1961 (unpublished); and C. W. Misner and D. L. Beckedorff (unpublished).

[11] E. Cartan, *Lecons sur la Geometrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1951), Sec. 207.