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Analyticity Properties of Helicity Amplitudes and Construction of Kinematical Singularity-Free Amplitudes for Any Spin*

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The analyticity properties of helicity amplitudes for binary reactions of particles with arbitrary spins are studied using the following three properties: (i) the analyticity properties of scattering amplitudes, assuming these are correctly predicted by perturbation theory; (ii) the crossing relations of helicity amplitudes near $s=0$; (iii) the threshold behavior of partial-wave amplitudes. Making use of these properties, kinematical singularity-free amplitudes for any spin are constructed by modifying helicity amplitudes. MacDowell reciprocity is generalized to arbitrary spin. Helicity amplitudes are proved to satisfy the Froissart limit at the high-energy limit.

I. INTRODUCTION

THE S-matrix theory of strong interactions is based on the analyticity properties of scattering amplitudes.[1] Among various kinds of amplitudes, the helicity amplitude introduced by Jacob and Wick[2] is most convenient for practical applications. Until now, if one wanted to know the analyticity properties of helicity amplitudes, one had to look for the linearly-independent, Lorentz-invariant scalars built up from the four-momenta and spin parameters of the external particles, the coefficients of which are free from kinematical singularities and satisfy the Mandelstam representation. (Here, the spin parameters include Dirac matrices, polarization vectors, and fermion spinors.) Then, one had to know the relation between helicity amplitudes and these coefficients. This has been done for $\pi\pi$, πN, and NN scatterings[3,4] and their crossed reactions.[4,5] For more complicated scattering problems, a prescription for finding kinematical singularity-free amplitudes has been given.[6] However, it is not easy to follow the prescription. For example, it was difficult even for NN scattering. Therefore, this indirect method will not be used in the following. Instead, we will investigate the analyticity properties of helicity amplitudes from the beginning.

As will be shown in the following, the analyticity properties of helicity amplitudes are not so complicated for simple scattering problems.[7]

For πN scattering[2,3]

$$f_{+,+}=\cos(\theta/2)[2MA+(W^2-M^2-\mu^2)B],$$
$$f_{+,-}=\sin(\theta/2)[(W^2+M^2-\mu^2)A+(W^2-M^2+\mu^2)MB]W^{-1}. \quad (1.1)$$

For NN scattering[4]

$$\tfrac{1}{2}(f_{++,++}-f_{++,--})=E^2G_1-zp^2G_2+m^2G_3,$$
$$\tfrac{1}{2}(f_{++,++}+f_{++,--})=(E^2G_2+m^2G_4)z-p^2G_5,$$
$$\tfrac{1}{2}[(1+z)^{-1}f_{+-,+-}-(1-z)^{-1}f_{+-,-+}]=-p^2G_3,\quad (z=\cos\theta),\quad (1.2)$$
$$\tfrac{1}{2}[(1+z)^{-1}f_{+-,+-}+(1-z)^{-1}f_{+-,-+}]=m^2G_2+E^2G_4,$$
$$(m/y)f_{++,+-}=-m^2E(G_2+G_4),\quad (y=\sin\theta).$$

For the $\pi\pi\to N\bar{N}$ process[5]

$$f_{++,}=[-2pA+2mqB\cos\theta],$$
$$f_{+-,}=2EqB\sin\theta. \quad (1.3)$$

Readers will see from Eqs. (1.1) to (1.3) that the modified helicity amplitude,

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}\equiv[\cos(\theta/2)]^{-|\lambda+\mu|}[\sin(\theta/2)]^{-|\lambda-\mu|}\times s^{-\xi/2}f_{\lambda_c\lambda_d,\lambda_a\lambda_b},$$
$$[\lambda=\lambda_a-\lambda_b,\quad \mu=\lambda_c-\lambda_d,\quad \xi=0\text{ (or 1)} \quad (1.4)$$
$$\text{if}\quad \lambda_a-\lambda_b-\lambda_c+\lambda_d\text{ is even (or odd)}],$$

satisfies the Mandelstam representation if we neglect possible (kinematical) poles at $s=0$ and $p=0$. We can get rid of these by multiplying Eq. (1.4) by s^ζ and p^{2L} (ζ and L can be predicted).

The amplitude (1.4) does not satisfy the Mandelstam representation in general. However, it will be shown that we can modify helicity amplitudes for any spin in such a way that the modified amplitudes satisfy Mandelstam representations. A list of kinematical singularity-free amplitudes will be given in Sec. VII. Making use of this result, a generalized MacDowell reciprocity will be proved in Sec. V. Helicity amplitudes for any spin will be proved to satisfy the Froissart limit at high energies (Sec. VI).

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[1] See, for example, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).

[2] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[3] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[4] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[5] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

[6] A. C. Hearn, Nuovo Cimento **21**, 333 (1961).

[7] Note a difference by a factor of $(sp_i/p_f)^{1/2}$ in our definition of helicity amplitudes [see Eq. (2.4)]. In our definition, the helicity amplitude for $\pi\pi$ scattering satisfies the Mandelstam representation.

II. ANALYTICITY IN COSθ AND IN t

In the following, we assume that the analyticity properties of scattering amplitudes are correctly predicted by perturbation theory. Then it has been shown[6] that the amplitudes of binary reactions $a+b\rightarrow c+d$ can be written as[8]

$$T=\sum_j B_j(s,t,u)N_j(p_i,\beta_i)\,, \qquad (2.1)$$

where T is related to the corresponding S-matrix element through

$$\langle p_c p_d|S-1|p_a p_b\rangle=(2\pi)^4 i\delta^4(p_c+p_d-p_a-p_b)\times(p_a^0p_b^0p_c^0p_d^0)^{-1/2}T\,. \qquad (2.2)$$

$B_j(s,t,u)$ is an analytic function of s, t, and u and satisfies the Mandelstam representation. $N_j(p_i,\beta_i)$ is a polynomial in the four momenta of the external particles p_i and spin parameters β_i which include Dirac matrices, fermion spinors, and polarization vectors.

The expectation value of (2.1) between helicity states in the center-of-mass system of the s channel (helicity amplitudes) can be written as

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}=\sum_j B_j(s,t,u)[\text{Polynomial in } p_i^0,\ p^2,\ p'^2,\ pp',\ \sin(\theta/2) \text{ and } \cos(\theta/2)][p^\eta \text{ or } p'^\eta]\times\prod_{i:\,\text{fermion}}[p_i^0+m_i]^{-1/2}. \qquad (2.3)$$

In deriving (2.3) we have made use of the fact that the helicity states are linear combinations of direct products of $\epsilon_\lambda{}^\mu(p)$ and $u_\lambda(p)$ (λ stands for helicity and μ is a Lorentz index), where

$$\begin{aligned}
\epsilon_1{}^\mu(p)&=(0;\cos\theta,\,i,\,-\sin\theta)/\sqrt{2}\,,\\
\epsilon_0{}^\mu(p)&=(p;\,p^0\sin\theta,0,p^0\cos\theta)/m\,,\\
\epsilon_{-1}{}^\mu(p)&=(0;\,-\cos\theta,\,i,\,\sin\theta)/\sqrt{2}\,,\\
u_{1/2}(p)&=[(p^0+m)\cos(\theta/2),(p^0+m)\sin(\theta/2),\\
&\quad p\cos(\theta/2),p\sin(\theta/2)][2m(p^0+m)]^{-1/2},\\
u_{-1/2}(p)&=[-(p^0+m)\sin(\theta/2),\,(p^0+m)\cos(\theta/2),\\
&\quad p\sin(\theta/2),\,-p\cos(\theta/2)]\\
&\quad\times[2m(p^0+m)]^{-1/2},
\end{aligned}$$

for $p^\mu=(p^0,p\sin\theta,0,p\cos\theta)$. For example, the helicity $\frac{3}{2}$ state of a spin $\frac{3}{2}$ particle is $\epsilon_1 u_{1/2}$.

The helicity amplitude, $T_{\lambda_c\lambda_d,\lambda_a\lambda_b}$, in (2.3) is related to the conventional helicity amplitude of Jacob and Wick[2] through

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}=2\pi(sp/p')^{1/2}f_{\lambda_c\lambda_d,\lambda_a\lambda_b}(d\sigma/d\Omega=|f|^2)\,. \qquad (2.4)$$

In (2.3), p_i^0 is the energy of the particle i in the center-of-mass system of the s channel, $p(p')$ is the momentum of the initial (final) particles in the center-of-mass system of the s channel, θ_s is the scattering angle in the center-of-mass system of the s channel, and

$$p_a^0=(s+m_a^2-m_b^2)/2(s^{1/2})\,,$$
$$p^2=[s-(m_a+m_b)^2][s-(m_a-m_b)^2]/(4s)\,, \qquad (2.5)$$

and

$$\cos\theta=[2st+s^2-s\sum_i m_i^2+(m_a^2-m_b^2)(m_c^2-m_d^2)]\times(4spp')^{-1}.$$

In (2.3), $\eta=0$ (or 1) if $\eta_a\eta_b\eta_c\eta_d=1$ (or -1) due to the P invariance of the strong interactions.[9]

Next let us consider the following amplitude:

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=[\cos(\theta/2)]^{-|\lambda+\mu|}[\sin(\theta/2)]^{-|\lambda-\mu|}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}$$
$$(\lambda=\lambda_a-\lambda_b \quad\text{and}\quad \mu=\lambda_c-\lambda_d)\,. \qquad (2.6)$$

This amplitude has been shown[10,11] to depend on $\cos\theta$, but not on $\cos(\theta/2)$ and $\sin(\theta/2)$. Then, we can write it as

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=\sum_j B_j(s,t,u)[\text{Polynomial in } p_i^0,\ p^2,\ p'^2,\ pp', \text{ and } \cos\theta][p^\eta \text{ or } p'^\eta]\times\prod_{i:\,\text{fermion}}[p_i^0+m_i]^{-1/2}. \qquad (2.7)$$

Hence, we have found that for fixed real s, T' is analytic in the $\cos\theta$ plane with cuts on the real axis. Since $\cos\theta$ is linear in t, we have also found that T' is analytic in the cut t plane (for fixed s) with cuts[1] $(t_{\min},\infty)$ and[1] $(-\infty,\sum_i m_i^2-s-u_{\min})$. Therefore, we have only to study the analyticity properties of helicity amplitudes as functions of s.

III. ANALYTICITY IN s; GENERAL CASE

Thus far, we have only considered the helicity amplitude for the s reaction ($T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^s$). As has been shown in the previous section, the helicity amplitudes for the t reaction ($T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^t$) has no kinematical singularities in s if it is divided by

$$[\cos(\theta_t/2)]^{-|\lambda+\mu|}[\sin(\theta_t/2)]^{-|\lambda-\mu|}.$$

Crossing relations between helicity amplitudes in the s channel, $T_{\delta\gamma,\beta\alpha}{}^s$, and helicity amplitudes in the t channel, $T_{\delta'\beta',\gamma'\alpha'}{}^t$, have been proposed by Trueman and Wick[12] and by Muzinich.[13] According to Trueman and Wick,[14]

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[8] $s=(p_a+p_b)^2=(p_c+p_d)^2$, $t=(p_a-p_c)^2=(p_b-p_d)^2$, and $u=(p_a-p_d)^2=(p_b-p_c)^2$.

[9] η_i is defined in Ref. 2.

[10] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[11] F. Calogero, J. Charap, and E. Squires (to be published); F. Calogero and J. Charap, Ann. Phys. (N. Y.) **26**, 44 (1964).

[12] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[13] I. Muzinich (to be published).

[14] A proof based on Muzinich's relations can be given in the same way.

$$T_{\delta\gamma,\beta\alpha}{}^{s}=\sum_{\delta'\gamma'\beta'\alpha'} d_{\delta'\delta}{}^{s_d}(\psi_d)d_{\gamma'\gamma}{}^{s_c}(\pi-\psi_c)d_{\beta'\beta}{}^{s_b}(\pi-\psi_b)d_{\alpha'\alpha}{}^{s_a}(\psi_a)T_{\delta'\beta',\gamma'\alpha'}{}^{t},$$

where

$$\cos\psi_a=\frac{(s+m_a^2-m_b^2)(t+m_a^2-m_c^2)+2m_a^2(m_b^2-m_a^2+m_c^2-m_d^2)}{4pp_t(st)^{1/2}},$$

$$\cos\psi_b=\frac{(s+m_b^2-m_a^2)(t+m_b^2-m_d^2)+2m_b^2(m_a^2-m_b^2+m_d^2-m_c^2)}{4pp_t'(st)^{1/2}},$$

$$\sin\psi_a=m_ap_t'\sin\theta_t/s^{1/2}p,$$

$$\sin\psi_b=m_bp_t\sin\theta_t/s^{1/2}p,$$

$$p_t=[t-(m_a+m_c)^2]^{1/2}[t-(m_a-m_c)^2]^{1/2}/2t^{1/2},$$

$$p_t'=[t-(m_b+m_d)^2]^{1/2}[t-(m_b-m_d)^2]^{1/2}/2t^{1/2},$$

$$\cos\theta_t=[2st+t^2-t\sum_i m_i^2+(m_a^2-m_c^2)(m_b^2-m_d^2)](4tp_tp_t')^{-1}.$$

At first, let us consider the general case. (We assume any two of the four external particles have unequal masses and $m_a>m_b$ and $m_c>m_d$.) In this case, $d_{\lambda\mu}(\psi_i)$, $\sin(\theta_t/2)$, $\cos(\theta_t/2)$ have no $s^{1/2}$ and s^{-1} type singularities and $\sin(\theta_s/2)\to O(s)^{1/2}$ and $\cos(\theta_s/2)\to O(1)$ as $s\to 0$. Therefore, from the crossing relations, (2.7) becomes

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)f_j(s,\cos\theta),\tag{3.1}$$

where f is a function of s and $\cos\theta$ and is analytic in s at $s=0$, and from (2.7) and the crossing relations we obtain

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}''\equiv[\prod_{i:\text{ fermion}}[2s^{1/2}(p_i^0+m_i)]^{1/2}]T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'$$
$$=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s^{1/2},\ p^2s,\ p'^2s,\text{ and }\cos\theta]\begin{bmatrix}1 & \text{or} & pp's\\ ps^{1/2} & \text{or} & p's^{1/2}\end{bmatrix}\quad\text{for}\quad \eta=\begin{bmatrix}0\\1\end{bmatrix}.\tag{3.2}$$

For the sake of simplicity, we consider only the $\eta=0$ case in the following[15]: Then[16]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos\theta]\times(1\ \text{or}\ pp's)\quad\text{for}\quad BB\to BB,$$

$$[s-(m_a-m_b)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos\theta]\times(1\ \text{or}\ pp's)\quad\text{for}\quad FF\to BB,\tag{3.3}$$

$$[s-(m_a-m_b)^2]^{1/2}[s-(m_c-m_d)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos\theta]\times(1\ \text{or}\ pp's)\quad\text{for}\quad FF\to FF,$$

since

$$\prod_{i:\text{ fermion}}[2s^{1/2}(p_i^0+m_i)]^{1/2}=(W+m_a+m_b)[s-(m_a-m_b)^2]^{1/2}\quad\text{for}\quad FF\to BB$$
$$=(W+m_a+m_b)(W+m_c+m_d)[s-(m_a-m_b)^2]^{1/2}[s-(m_c-m_d)^2]^{1/2}\quad\text{for}\quad FF\to FF,$$

and since $\bar{u}O_iu\bar{u}'O_iu'$, etc., do not have poles at $W=s^{1/2}=-(m_a+m_b)$ and $W=-(m_c+m_d)$.

For $BF\to BF$,

$$[2s^{1/2}(p_a^0+m_a)]^{1/2}[2s^{1/2}(p_c^0+m_c)]^{1/2}=\tfrac{1}{2}\{[2s^{1/2}(p_a^0+m_a)]^{1/2}[2s^{1/2}(p_c^0+m_c)]^{1/2}+[2s^{1/2}(p_a^0-m_a)]^{1/2}\times[2s^{1/2}(p_c^0-m_c)]^{1/2}\}+\tfrac{1}{2}\{[2s^{1/2}(p_a^0+m_a)]^{1/2}[2s^{1/2}(p_c^0+m_c)]^{1/2}-[2s^{1/2}(p_a^0-m_a)]^{1/2}\times[2s^{1/2}(p_c^0-m_c)]^{1/2}\}\equiv E(s)+s^{1/2}O(s).\tag{3.4}$$

[15] For the $\eta=1$ case, see Sec. VII.

[16] In the following, B and F stand for bosons and fermions, respectively.

Thus[17]

$$E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos\theta](1 \quad \text{or} \quad pp's) \tag{3.5a}$$

and

$$s^{1/2}0(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=s^{1/2-|\lambda-\mu|/2}\sum_j B_j'(s,t,u)[\text{Polynomial in } s \text{ and } \cos\theta](1 \quad \text{or} \quad pp's). \tag{3.5b}$$

Therefore, we can find kinematical singularity-free amplitudes if we can separate the terms that contain the factor $pp's$ from these that do not contain it. This separation can be made by making use of the threshold behavior of partial-wave amplitudes.

For this purpose, let us consider the parity-conserving helicity amplitudes defined by

$$\begin{aligned} T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{\pm} &= T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'\pm(-1)^{\lambda+\lambda_m+s_c+s_d-v}\eta_c\eta_d T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}' \\ &= 2\pi s^{1/2}2^{|\lambda+\mu|/2+|\lambda-\mu|/2}\sum_J (2J+1)[e_{\lambda\mu}{}^{J+}(z)F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J\pm}+e_{\lambda\mu}{}^{J-}(z)F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J\mp}], \end{aligned} \tag{3.6}$$

where $\lambda_m=\max(|\lambda|,|\mu|)$ and v is $\frac{1}{2}$ (or 0) for half-integral J (or for integral J). In terms of $T^{\pm}$, we can write F matrix elements $[F_{fi}{}^J\equiv(S_{fi}{}^J-\delta_{fi})(2i)^{-1}p'^{-1/2}p^{-1/2}]$ as

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J\pm}=(16\pi^2 s)^{-1/2}2^{-|\lambda+\mu|/2-|\lambda-\mu|/2}\int_{-1}^{1}dz[c_{\lambda\mu}{}^{J+}(z)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{\pm}(z)+c_{\lambda\mu}{}^{J-}(z)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(z)^{\mp}],$$

where

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}=\pm\langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle_{\pm}$$

and

$$P|JM;\lambda_a\lambda_b\rangle_{\pm}=\pm(-1)^{J-v}|JM;\lambda_a\lambda_b\rangle_{\pm}.$$

The $c_{\lambda\mu}{}^{J+}$ can be written as linear combinations of the Legendre functions $P_{J-\lambda_m+2v}, P_{J-\lambda_m+2v+2}, \cdots, P_{J+\lambda_m-2v}$ with constant coefficients.[10] The $c_{\lambda\mu}{}^{J-}$ can be written as linear combinations of the Legendre functions $P_{J-\lambda_m+2v+1}$, $P_{J-\lambda_m+2v+3}, \cdots, P_{J+\lambda_m-2v-1}$ with constant coefficients.[10] For all reactions, $e_{\lambda\mu}{}^{J+}$ ($e_{\lambda\mu}{}^{J-}$) is an even (odd) function of $\cos\theta$ if $J+\max(|\lambda|,|\mu|)-2v$ is even and an odd (even) function of $\cos\theta$ if $J+\max(|\lambda|,|\mu|)-2v$ is odd. If we assume that $\eta_a\eta_b=\eta_c\eta_d=1$, then F^{J+} is an even (odd) function of both p and p' if $J-v$ is even (odd) and F^{J-} is an odd (even) function of both p and p' if $J-v$ is even (odd). In the following part of this section, we assume that $\max(|\lambda|,|\mu|)-v$ is even. Therefore, we find for $BB\to BB$, $FF\to BB$, and $FF\to FF$

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}''^{+}\equiv T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{+},[s-(m_a-m_b)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{+}$$

or

$$[s-(m_a-m_b)^2]^{1/2}[s-(m_c-m_d)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{+}=s^{-\xi/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta] \times(1 \quad \text{or} \quad pp's\cos\theta) \tag{3.7a}$$

and

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}''^{-}\equiv T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{-}, \quad [s-(m_a-m_b)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{-}$$

or

$$[s-(m_a-m_b)^2]^{1/2}[s-(m_c-m_d)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{-}=s^{-\xi/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta] \times(pp's \quad \text{or} \quad \cos\theta), \tag{3.7b}$$

where

$$\xi=\max(|\lambda-\mu|,|\lambda+\mu|).$$

Therefore, we have found that the amplitudes (3.7a) and $(pp's)\times$(3.7b) satisfy the Mandelstam representations if we neglect possible kinematical singularities at $pp's=0$ [$\cos\theta$ contains a factor $(pp's)^{-1}$]. The singularity at $pp's=0$ can be removed by multiplying Eqs. (3.7a) and (3.7b) by $(pp's)^{L_\pm}$. The $L_\pm$ can be determined from the threshold behavior of phase shifts.

The threshold behavior of $F^{J\pm}$ is

$$F^{J\pm}\underset{p,p'\to 0}{\longrightarrow}O(p^{l_i}p'^{l_f}). \tag{3.8}$$

Let us define $\rho_\pm{}^i=\max|J-l_i|$ and $\rho_\pm{}^f=\max|J-l_f|$ for states of $T^\pm$, where l_i and l_f are the initial and final orbital angular momenta which are compatible with total angular momentum J. Then, we obtain

$$L_\pm=\max[\rho_\pm{}^i-\lambda_m+2v,\ \rho_\pm{}^f-\lambda_m+2v, \text{ and } 0]. \tag{3.9}$$

[17] In deriving Eqs. (3.5a) and (3.5b), properties of $E(s)\bar{u}O_iu$ and $O(s)\bar{u}O_iu$ have been used.

Therefore,

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\xi/2}(pp's)^{L\pm}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}''^{\pm} \tag{3.10}$$

satisfies the Mandelstam representations.

For $BF\to BF$: [we assume $(-1)^{\lambda+\lambda_m+s_c+s_d-v}\eta_c\eta_d=1$]

$$\begin{aligned}
E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+s^{1/2}O(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'&=s^{-\zeta/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta](1 \quad\text{or}\quad pp's\cos\theta)\,,\\
E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-s^{1/2}O(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'&=s^{-\zeta/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta](pp's \quad\text{or}\quad \cos\theta)\,,\\
s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+E(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'&=s^{-\zeta'/2}\sum B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta](1 \quad\text{or}\quad pp's\cos\theta)\,,\\
s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-E(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'&=s^{-\zeta'/2}\sum_j B_j(s,t,u)[\text{Polynomial in } s \text{ and } \cos^2\theta](pp's \quad\text{or}\quad \cos\theta)\,,
\end{aligned} \tag{3.11}$$

where[18]

$$\zeta=\max(|\lambda-\mu|,\ |\lambda+\mu|-1)$$

and

$$\zeta'=\max(|\lambda-\mu|-1,\ |\lambda+\mu|)\,.$$

Therefore, we find the following four kinematical singularity-free amplitudes:

$$s^{\zeta/2}(pp's)^{L+}\{E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+s^{1/2}O(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'\}\,, \tag{3.12a}$$

$$s^{\zeta/2}(pp's)^{L-}\{E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-s^{1/2}O(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'\}\,, \tag{3.12b}$$

$$s^{\zeta'/2}(pp's)^{L+}\{s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+E(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'\}\,, \tag{3.12c}$$

$$s^{\zeta'/2}(pp's)^{L-}\{s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-E(s)T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'\}\,. \tag{3.12d}$$

They are not independent, but Eqs. (3.12a) and (3.12c) are independent.

IV. ANALYTICITY IN s; SPECIAL CASES

In this section we consider the special cases: (i) $m_a=m_b=m_c=m_d$, (ii) $m_a=m_c$ and $m_b=m_d$, (iii) $m_a=m_b$ and $m_c=m_d$. In these cases, the analyticity properties of the scattering amplitudes are simpler than those in the general case.

A. Boson-Fermion Scattering ($m_a=m_c$, $m_b=m_d$)

When $m_a=m_c$ and $m_b=m_d$, $O(s)=2m$. Thus, Eq. (3.5b) becomes[19,20]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=s^{-|\lambda-\mu|/2}\sum_j B_j(s,t,u)\times[\text{Polynomial in } s \text{ and } \cos\theta]\,, \tag{4.1}$$

and the amplitude

$$s^{|\lambda-\mu|/2}(p^2s)^LT_{\lambda_c\lambda_d,\lambda_a\lambda_b}' \tag{4.2}$$

is found to satisfy the Mandelstam representation, where

$$L=\max[L_+,L_-]\,, \tag{4.3}$$

$$L_\pm=\max[\rho_\pm-\lambda_m+2v,\ 0]\,, \tag{4.4}$$

and

$$2\rho_\pm=\max|2J-l_i-l_f|\,. \tag{4.5}$$

The difference between (3.9) and (4.4) comes from the fact that (3.8) is replaced by

$$F^{J\pm}\underset{p\to0}{\longrightarrow}O(p^{l_i+l_f}) \tag{4.6}$$

in this case.

B. $BB\to BB$ and $FF\to FF$ ($m_a=m_c$, $m_b=m_d$)

In this case, (2.7) becomes[20,21]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=\sum_j B_j(s,t,u)s^{-\alpha/2}\times[\text{Polynomial in } s^{1/2} \text{ and } \cos\theta]\,, \tag{4.7}$$

and Eq. (3.1) is still valid here. Therefore, we find that

$$s^{|\lambda-\mu|/2}(p^2s)^LT_{\lambda_c\lambda_c,\lambda_a\lambda_b}' \tag{4.8}$$

satisfies the Mandelstam representation [L is given by Eq. (4.3)].

C. $BB\to BB$, $FF\to FF$, $BB\rightleftarrows FF$ ($m_a=m_b$, $m_c=m_d$)

In this case, (2.7) becomes[21,22]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=\sum_j B_j(s,t,u)\times[\text{Polynomial in } s^{1/2} \text{ and } \cos\theta]\,. \tag{4.9}$$

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[18] Unfortunately, the author has not been able to prove Eqs. (3.11) in general. The amplitudes for the reaction $\pi+p\to K+\Lambda$ can be shown to satisfy (3.11), but for other cases (3.11) is only conjectured.

[19] In this case, $p_a{}^0=p_c{}^0$, $p_b{}^0=p_d{}^0$, $p^2=p'^2=pp'$, and $\cos\theta=1+(t/2p^2)$.

[20] Here, we assume $\eta_a\eta_b=\eta_c\eta_d=1$.

[21] In this case, we do not need to consider the factor $\prod(p_i{}^0+m_i)^{-1/2}$ since $\bar u_{\lambda'}(p_c)O_iu_\lambda(p_a)$ and $\bar u_{\lambda'}(p_d)O_iu_\lambda(p_b)$ are polynomials in $p_i{}^0$, p, p', $\sin(\theta/2)$, and $\cos(\theta/2)$.

[22] In this case, $2p_i{}^0=s^{1/2}$, $4p^2=s-4m^2$, $4p'^2=s-4m'^2$, and $\cos\theta=(t-u)/(4pp')$.

From (3.3) and (4.9) we find

$$s^{-\xi}(pp')^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}' \quad (4.10)$$

satisfies the Mandelstam representation, where

$$L=\max[L_+,L_-],$$
$$L_\pm=\max[\rho_\pm{}^i-\lambda_m+2v,\ \rho_\pm{}^f-\lambda_m+2v,\ 0], \quad (4.11)$$

and

$\xi=0$ if $\lambda-\mu+\eta$ is even in $BB\to BB$, $FF\to FF$
or if $\lambda-\mu+\eta$ is odd in $BB \rightleftarrows FF$

and

$\xi=1$ if $\lambda-\mu+\eta$ is odd in $BB\to BB$, $FF\to FF$
or if $\lambda-\mu+\eta$ is even in $BB \rightleftarrows FF$.

D. Equal Mass Scattering ($m_a=m_b=m_c=m_d$)

In this case, (2.7) becomes[21]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=\sum_j B_j(s,t,u) \times[\text{Polynomial in } s^{1/2} \text{ and } \cos\theta]. \quad (4.12)$$

From Eqs. (4.1), (4.10), and (4.12) we find the amplitude

$$s^{-\xi}p^{2L}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}' \quad (4.13)$$

satisfies the Mandelstam representation, where ξ is given in the previous subsection and $\xi=0$ (or 1) if $\lambda-\mu+\eta$ is even (or odd) in $BF\to BF$. L is given in (4.3).

V. GENERALIZED MACDOWELL RECIPROCITY

MacDowell has found the following reciprocity relation between the partial-wave amplitudes of πN scattering[23]

$$f_l{}^+(-W)=-f_{l+1}{}^-(W), \quad (5.1)$$

where

$$f_l{}^\pm=\exp(i\delta_l{}^\pm)\sin\delta_l{}^\pm/p.$$

Making use of the results of the previous sections, we can generalize (5.1) for any $BF\to BF$ reaction to[24]

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J+}(W)=-(-1)^{\lambda-\mu}F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J-}(-W), \quad (5.2)$$

where $F^{J\pm}$ has been defined in Sec. III,

$$s^{1/2}F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{J\pm}=(1/4\pi)2^{-|\lambda+\mu|/2-|\lambda-\mu|/2} \times\int_{-1}^{1}dz[c_{\lambda\mu}{}^{J+}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{\pm}+c_{\lambda\mu}{}^{J-}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{\mp}].$$

Since

$$T^\pm\propto T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'\pm T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}',$$
$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(-W)=(-1)^{\lambda-\mu}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W),$$

and[25]

$$T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'(-W)=-(-1)^{\lambda-\mu}T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'(W), \quad (5.3)$$

we can easily obtain (5.2) for $BF\to BF$ reactions.

For other reactions,

$$|F^{J\pm}(W)|=|F^{J\pm}(-W)|.$$

VI. HIGH-ENERGY LIMIT OF HELICITY AMPLITUDES

Froissart has shown[26,27] that the scattering amplitudes of spinless particles have upper bounds at the high-energy limit,

$$|T|<(\text{const})s(\ln s)^2 \quad \text{for} \quad z=\pm1 \quad (6.1)$$

and

$$|T|<(\text{const})s^{3/4}(\ln s)^{3/2} \quad \text{for} \quad -1<z<1 \quad (6.2)$$

on the assumption that T satisfies the Mandelstam representation.

It was recognized later by Greenberg and Low[28] and by Martin[29] that it is not necessary to make use of the full analyticity assumed in the Mandelstam representation to obtain the bounds (6.1) and (6.2). It is sufficient to assume that T be analytic in an ellipse E.

Kinoshita, Loeffel, and Martin[30] have improved (6.2) and replaced it by

$$|T|<\text{const}(\ln s)^{3/2} \quad \text{for} \quad -1<z<1,$$

assuming more analyticity than was needed in the Refs. 28 and 29, but less than was used by Froissart.

For the asymptotic behavior of the scattering amplitudes for particles with any spin, Yamamoto has shown[31] that they satisfy the Froissart limit. However, his proof does not cover the most general case, and is somewhat complicated. Therefore, we have decided to give a general and simple proof using helicity amplitudes.

In Sec. II we have found that if the analyticity properties of helicity amplitudes are assumed to be correctly predicted by perturbation theory,

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=[2^{1/2}\cos(\theta/2)]^{-|\lambda+\mu|} \times[2^{1/2}\sin(\theta/2)]^{-|\lambda-\mu|}T_{\lambda_c\lambda_d,\lambda_a\lambda_b} \quad (6.3)$$

is analytic in the $\cos\theta$ plane with cuts on the real axis $(-\infty,-1-\alpha)$ and $(1+\beta,\infty)$, where α and β are real positive and approaches $2u_{\min}/s$ and $2t_{\min}/s$ at the high-energy limit ($s\to\infty$), respectively. However, for

[23] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

[24] If $m_a=m_b=m_c=m_d$, the factor $(-1)^{\lambda-\mu}$ in (5.2) and (5.3) should be replaced by $(-1)^{\lambda+\mu+\eta}$. For vector-spinor scattering, the relations $F^{J+}(W)=-F^{J-}(-W)$ have been obtained in Ref. 10. However, the factor $(-1)^{\lambda-\mu}$ is necessary if we use the phase factor of Jacob and Wick (Ref. 2).

[25] For $BF\to BF$, $(-1)^{2\mu}=-1$.

[26] The proof in this section has been carried out in collaboration with Dr. Louis Balázs.

[27] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[28] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

[29] A. W. Martin, Phys. Rev. **129**, 1432 (1963).

[30] T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963).

[31] K. Yamamoto, Nuovo Cimento **27**, 1277 (1963).

our proof, we need a weaker condition, namely, that T' is analytic in an ellipse (E) with foci at -1 and 1 and with semiaxes a and $(a^2-1)^{1/2}$, where $a=\min(\alpha,\beta)$.

From (6.3), we find

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(W,z=1)=0 \quad \text{for} \quad \lambda-\mu\neq 0$$

and (6.4)

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(W,z=-1)=0 \quad \text{for} \quad \lambda+\mu\neq 0$$

since T' is finite at $|z|=1$.

Since[10]

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z)=2\pi s^{1/2}\sum_J (2J+1)F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)\times e_{\lambda\mu}{}^J(z),$$

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)=\frac{1}{4\pi s^{1/2}}\int_{-1}^{1} dz c_{\lambda\mu}{}^J(z)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(W,z),$$

and

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z)=\frac{1}{2\pi i}\oint_{\text{along ellipse } E} dz' \times\frac{T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z')}{z'-z} \quad \text{for} \quad z \text{ in } E,$$

we obtain

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)=\frac{1}{8\pi^2 i s^{1/2}}\oint_E dz' T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z')\times\int_{-1}^{1} dz\frac{c_{\lambda\mu}{}^J(z)}{z'-z}$$

$$=\frac{1}{4\pi^2 i s^{1/2}}\oint_E dz' T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z')\times C_{\lambda\mu}{}^J(z),$$

where we can write c and C in the form

$$c_{\lambda\mu}{}^J(z)=\sum_{J-\lambda m+2v}^{J+\lambda m-2v} a_{\lambda\mu}{}^{JJ'}P_{J'}(z),$$

and

$$C_{\lambda\mu}{}^J(z)=\sum_{J-\lambda m+2v}^{J+\lambda m-2v} a_{\lambda\mu}{}^{JJ'}Q_{J'}(z),$$

where the a are constants. Thus, we obtain

$$|F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)|<\frac{1}{4\pi^2 s^{1/2}}|T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'(W,z)|_{\max \text{ on } E}\times|C_{\lambda\mu}{}^J(z)|_{\max \text{ on } E}\, L,$$

where L is the length of the path along the ellipse E.

$$L<2[a+(a^2-1)^{1/2}]\equiv 2u,$$

$$u \underset{s\to\infty}{\longrightarrow} 1+(\text{const}/s^{1/2})>1.$$

We assume that $|T'(W,z)|_{\max}$ is bounded by a polynomial in W, $R_1(W)$. From the properties of Legendre functions, we find[32]

$$|C_{\lambda\mu}{}^J(z)|_{\max}<\text{const}(J)^{-1/2}(1-1/u^2)^{-1}u^{-J-1}.$$

Therefore, we find

$$|F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)|<R'(W)J^{-1/2}u^{-J},$$

where $R'(W)$ is a polynomial in W. From the unitarity of the S matrix, we know that

$$|F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)|<(pp')^{-1/2}.$$

If we use the results of Ref. 10, we find

$$|((1+z)/\sqrt{2})^{|\lambda+\mu|/2}((1-z)/\sqrt{2})^{|\lambda-\mu|/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(W,z)|$$
$$=2\pi s^{1/2}|\sum_J (2J+1)F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)c_{\lambda\mu}{}^J(z)|$$
$$<(\text{const})s^{1/2}\sum_J (2J+1)|F_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(W)|\times f_J(z), \quad (6.5)$$

where

$$f_J(1)=1$$

and

$$f_J(z) \underset{J\to\infty}{\longrightarrow} f(z)/J^{1/2} \quad \text{for} \quad -1<z<1.$$

From (6.4) and (6.5), we find

$$|T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(z)|<\text{const}\times s(\ln s)^2 \quad \text{for} \quad z=\pm 1$$

and

$$|T_{\lambda_c\lambda_d,\lambda_a\lambda_b}(z)|<\text{const}s^{3/4}(\ln s)^{3/2} \quad \text{for} \quad -1<z<1.$$

VII. SUMMARY

In this section, a list of kinematical singularity-free amplitudes $h_{\lambda_c\lambda_d,\lambda_a\lambda_b}$, is given. The properties used are as follows: (i) the analyticity properties of scattering amplitudes, assuming that these are correctly predicted by perturbation theory; (ii) the crossing relations of helicity amplitudes near $s=0$; (iii) the threshold behavior of partial-wave amplitudes.[33]

In the following

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'=[\cos(\theta/2)]^{-|\lambda+\mu|}[\sin(\theta/2)]^{-|\lambda-\mu|}T_{\lambda_c\lambda_d,\lambda_a\lambda_b},$$
$$(\lambda=\lambda_a-\lambda_b \quad \text{and} \quad \mu=\lambda_c-\lambda_d).$$

Case I. $m_a=m_b=m_c=m_d=m$

(i) $BB\to BB$, $FF\to FF$, and $BF\to BF$.

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{-\xi/2}(s-4m^2)^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}',$$

where

$$\xi=0 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is even},$$
$$\xi=1 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is odd}.$$

(ii) $BB\rightleftarrows FF$.

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{-\xi/2}(s-4m^2)^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}',$$

[32] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1927), 4th ed., p. 322.

[33] This assumption may be replaced by the crossing relations near $p=0$ and $p'=0$.

TABLE I. The $\alpha_\pm$ and $\beta_\pm$.

$\eta_a\eta_b$	$\eta_c\eta_d$	$\max(\vert\lambda\vert,\vert\mu\vert)-v$	L_+	α_+	β_+	L_-	α_-	β_-
+	+	even	even	0	0	odd	0	0
+	+	odd	odd	0	0	even	0	0
+	−	even	even	0	1	even	1	0
			odd	1	0	odd	0	1
+	−	odd	even	1	0	even	0	1
			odd	0	1	odd	1	0
−	+	even	even	1	0	even	0	1
			odd	0	1	odd	1	0
−	+	odd	even	0	1	even	1	0
			odd	1	0	odd	0	1
−	−	even	odd	0	0	even	0	0
−	−	odd	even	0	0	odd	0	0

where

$$\xi=0 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is odd},$$
$$\xi=1 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is even}.$$

Case II. $m_a=m_c,\ m_b=m_d,\ \eta_a\eta_b=\eta_c\eta_d$

$BB\to BB$, $FF\to FF$, and $BF\to BF$.

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{|\lambda-\mu|/2}(p^2s)^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'.$$

Case III. $m_a=m_b,\ m_c=m_d$

(i) $BB\to BB$ and $FF\to FF$.

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{-\xi/2}(pp')^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}',$$

where

$$\xi=0 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is even},$$
$$\xi=1 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is odd}.$$

(ii) $BB\rightleftarrows FF$.

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{-\xi/2}(pp')^L T_{\lambda_c\lambda_d,\lambda_a\lambda_b}',$$

where

$$\xi=0 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is odd},$$
$$\xi=1 \quad \text{if} \quad \lambda-\mu+\eta \quad \text{is even}.$$

Case IV. General Case

Any two of the m_i's are not equal and $m_a>m_b$ and $m_c>m_d$.

(i) $BB\to BB$, $BB\rightleftarrows FF$, and $FF\to FF$.

We define $T_{\lambda_c\lambda_d,\lambda_a\lambda_b}''^{\pm}$ as

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}''^{\pm}=\begin{cases} T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{\pm} & \text{for} \quad BB\to BB, \\ [s-(m_a-m_b)^2]^{1/2}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{\pm} & \text{for} \quad FF\to BB, \\ [s-(m_a-m_b)^2]^{1/2}[s-(m_c-m_d)^2]^{1/2} T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{\pm} & \text{for} \quad FF\to FF, \end{cases}$$

where

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{\pm}=T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'\pm(-1)^{\lambda+\lambda_m}\eta_c\eta_d(-1)^{s_c+s_d-v}\times T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}'.$$

Then

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\xi/2}(pp's)^{L_\pm}(ps^{1/2})^{\alpha_\pm}(p's^{1/2})^{\beta_\pm}T_{\lambda_c\lambda_d,\lambda_a\lambda_b}''^{\pm},$$

where

$$\xi=\max(|\lambda-\mu|,|\lambda+\mu|)$$

and $\alpha_\pm$ and $\beta_\pm$ are given in Table I, and $L_\pm$ are given by Eq. (3.9).

(ii) $BF\to BF$.

We define $T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{(i)}$ ($i=1$, 2, 3, and 4) as

$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{(1)}=E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+s^{1/2}O(s)\nu T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}',$$
$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{(2)}=E(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-s^{1/2}O(s)\nu T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}',$$
$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{(3)}=s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'+E(s)\nu T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}',$$
$$T_{\lambda_c\lambda_d,\lambda_a\lambda_b}^{(4)}=s^{1/2}O(s)T_{\lambda_c\lambda_d,\lambda_a\lambda_b}'-E(s)\nu T_{-\lambda_c-\lambda_d,\lambda_a\lambda_b}',$$

where $E(s)$ and $s^{1/2}O(s)$ are defined in (3.4) and $\nu=(-1)^{\lambda+\lambda_m+s_c+s_d-v}\eta_c\eta_d$. Then,

$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\zeta/2}(pp's)^{L_+}(ps^{1/2})^{\alpha_+}(p's^{1/2})^{\beta_+}T^{(1)},$$
$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\zeta/2}(pp's)^{L_-}(ps^{1/2})^{\alpha_-}(p's^{1/2})^{\beta_-}T^{(2)},$$
$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\zeta'/2}(pp's)^{L_+}(ps^{1/2})^{\alpha_+}(p's^{1/2})^{\beta_+}T^{(3)},$$
$$h_{\lambda_c\lambda_d,\lambda_a\lambda_b}=s^{\zeta'/2}(pp's)^{L_-}(ps^{1/2})^{\alpha_-}(p's^{1/2})^{\beta_-}T^{(4)},$$

where

$$\zeta=\max(|\lambda-\mu|,\ |\lambda+\mu|-1),$$

and

$$\zeta'=\max(|\lambda-\mu|-1,\ |\lambda+\mu|),$$

and $\alpha_\pm$ and $\beta_\pm$ are given in Table I, and $L_\pm$ are given by Eq. (3.9).

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