Euclidean Approach to the Bethe-Salpeter Equation for Scattering*

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It is shown that the mass-shell scattering amplitude in the physical scattering region (two-particle branch cut) can be directly obtained from the resolvent of the Bethe-Salpeter kernel transformed according to Wick (rotation of the energy integration paths to the imaginary axis). As an illustration and under the assumption that the kernel is approximated by any *finite* set of irreducible graphs, the scattering amplitude in ϕ^3 theory is given in terms of a Fredholm formula whose convergence is explicitly demonstrated.

I. INTRODUCTION

SINCE the formulation of the field-theoretic relativistic two-body equation[1–3] known as the Bethe-Salpeter (B-S) equation, the most remarkable advance in understanding its rather unfamiliar features was made by Wick and Cutkosky[4] quite some time ago. Their investigation of the B-S equation for bound states was based on analytic properties of the wave function derived from its field-theoretic definition rather than from the equation itself. Wick[4] used these properties to transform the equation by rotating the integration path of the relative energy (in momentum space) to the imaginary axis, thereby avoiding the strong singularities of the interaction kernel due to the indefiniteness of the relativistic metric. In this way the bound-state problem in a simple case ("ladder" approximation for two spinless particles bound by a scalar field) was reduced to the familiar eigenvalue problem of a Fredholm-Schmidt operator.

Recent work on the analytic properties of scattering amplitudes in the complex angular momentum plane and their relation to high-energy limits has revived interest in the B-S equation for the off-the-mass-shell amplitude (Green's function for two-body propagation). Wick's contour rotation has been used by several authors[5–9] to avoid the aforementioned singularities of the kernel. However, as emphasized by Nakanishi,[10] this approach is not justified because the analytic properties of the Green's function in the relative energy do not seem to allow the desired contour rotation.[11]

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2 M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

3 J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

4 G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

5 B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

6 R. F. Sawyer, Phys. Rev. **131**, 1384 (1963).

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9 G. Tiktopoulos, Phys. Rev. **133**, B1231 (1964).

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11 Use of contour rotation for a scattering problem was made by N. Kemmer and A. Salam, Proc. Roy. Soc. (London) **A230**, 266 (1955). Their transformed equation is not entirely free of singularities. It should also be noted that in the Kemmer-Salam paper as well as in the work of Wick specific assumptions were made concerning the asymptotic properties of the solution in the relative energy plane.

In this paper it is shown that the *mass-shell* amplitude can be obtained directly from the resolvent of the Wick-rotated kernel and, therefore, *in this sense*, the Wick rotation in the B-S equation for the scattering amplitude is rigorously justified.

The approach used avoids any detailed consideration of the complicated analytic properties of the off-the-mass-shell amplitude. In Sec. II an alternative definition of Feynman integrals is given in an unphysical region by employing integrations over Euclidean momenta. This Euclidean representation is continued analytically to the physical scattering region (two-particle branch cut) by means of appropriate contour deformations (Sec. III). Finally, in Sec. IV, the B-S equation is formulated and its solution on the mass shell is expressed in terms of the resolvent of the kernel in its Euclidean form.

As an illustration the situation is considered in some detail within the framework of a ϕ^3 type of interaction. Under the assumption that the kernel consists of any *finite* set of irreducible graphs, the exact solution on the mass-shell is given by the Fredholm formula whose convergence is explicitly demonstrated. In particular, the connection is established between the poles of the scattering amplitude in the energy plane and the bound-state solutions obtained by Wick and Cutkosky.

II. EUCLIDEAN REPRESENTATION OF FEYNMAN INTEGRALS

We consider convergent Feynman integrals corresponding to graphs with four external lines, involving particles with no internal degrees of freedom, i.e., the propagators are of the form

$$1/(m^2-Q^2-i\epsilon).$$

Since we shall always assume that the trivial δ-function integrations have been carried out, Q will be a linear combination of the external four-momenta and the "loop" momenta whose definition depends on the particular choice of the independent loops of the graph and clearly does not affect the value of the integral. However, in what follows we shall find it convenient to make a specific choice.

Although we shall consider explicitly spinless particles only, our discussion may be easily extended to more general situations since it depends critically only on the structure of the denominators of the propagators.

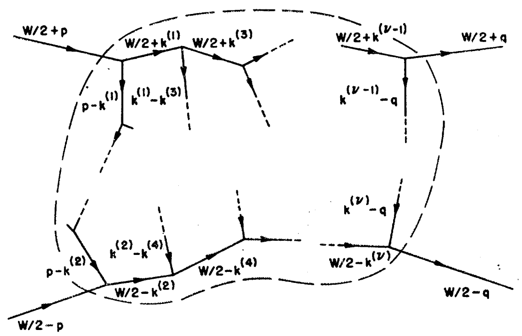

FIG. 1. Assignment of "loop" momenta in a strongly connected Feynman graph with four external lines.

Let the external masses of a graph G be equal[12] to M and let $m>0$ be the smallest internal mass. The loop momenta $k^{(1)}, k^{(2)}, \cdots, k^{(j)}, \cdots$ are chosen[13] as shown in Fig. 1. We use the notation

$$k\equiv(k_1,k_2,k_3,k_0) \quad \text{or} \quad (\mathfrak{k},k_0)\,,$$
$$k^2\equiv-\mathfrak{k}^2+k_0{}^2\,,$$

in which

$$W_1=W_2=W_3=0\,,\quad W_0=s^{1/2}\,,$$
$$p=(\mathfrak{p},0)\,,\quad q=(\mathfrak{q},0)\,,$$
$$\mathfrak{p}^2=\mathfrak{q}^2=\tfrac{1}{4}s-M^2;\quad \mathfrak{p}=|\mathfrak{p}|\,,\quad \mathfrak{q}=|\mathfrak{q}|\,.$$

Let us consider the integral $F_e(G)$ formally obtained from the Feynman integral $F(G)$ by making the following change to all *loop* momenta (not the external ones):

$$k_0\to ik_4\,,\quad dk_0\to idk_4\,,$$

where k_4 is a *real* variable integrated over from $-\infty$ to $+\infty$. This would correspond to a "Wick rotation" of energy contours. At the present stage, however, it is merely a formal definition of $F_e(G)$.

A simple inspection of the propagator denominators shows that for values of $\cos(\angle\mathfrak{pq})$ in the real interval $[-1, 1]$, the integral $F_e(G)$ represents an analytic function of s and $\mathfrak{p}$ in the region

$$\mathfrak{D}_e\colon\ \{m^2-(\mathrm{Im}\mathfrak{p})^2>0\,,\ |\mathrm{Re}s^{1/2}|<2m\}\,.$$

In particular, $\mathfrak{D}_e$ includes the "real environment"

$$R\colon\ \{\mathfrak{p}=\text{real}>0\,,\ s=\text{real}<0\}\,,$$

where all denominators are real and positive throughout the range of integration. In R the integrations over the loop momenta can be carried out as usual after a Feynman parametrization, leading to the familiar parametric integral[14]

$$I(s,\mathfrak{p})=\int_0^1\cdots\int_0^1\frac{c^{l-1}\delta(\sum x_i-1)dx_1\cdots dx_N}{[-sf_1-tf_2-uf_3-M^2f_4+f_5]^{l+1}}\,,$$

where $c, f_1, \cdots, f_5$ are polynomials ($\geqslant 0$ in the integration domain) in the real integration parameters[15] $x_1, x_2, \cdots, x_N$, and

$$t=-2\mathfrak{p}^2[1-\cos(\angle\mathfrak{pq})]\,,$$
$$u=-2\mathfrak{p}^2[1+\cos(\angle\mathfrak{pq})]\,,$$
$$M^2=\tfrac{1}{4}s-\mathfrak{p}^2\,.$$

On the other hand, it is known[16] that $I(s,\mathfrak{p})$ represents the continuation of the *Feynman* integral $F(G)$ to the region

$$\mathfrak{D}\colon\ \left\{\begin{matrix}\mathrm{Re}s<4m^2\\ \mathrm{Re}[-2\mathfrak{p}^2(1-\cos\angle\mathfrak{pq})]<4m^2\\ \mathrm{Re}[-2\mathfrak{p}^2(1+\cos\angle\mathfrak{pq})]<4m^2\\ \mathrm{Re}[\tfrac{1}{4}s-\mathfrak{p}^2]<2m^2\end{matrix}\right\}\,.$$

Since $\mathfrak{D}\supset R$ and $F_e(G)=F(G)$ in R, we conclude $F(G)\equiv F_e(G)$.

By this we mean that the Feynman amplitude can be alternatively defined by $F_e(G)$ in $\mathfrak{D}_e$ and obtained in the physical region by analytic continuation. We note that for M^2 real and fixed at a value less than $2m^2$, $\mathfrak{D}_e$ includes the following region (see Fig. 2) in the s plane (for physical values of $\cos\angle\mathfrak{pq}$):

$$(\mathrm{Im}s)^2<16m^2\times\min\{4m^2-\mathrm{Re}s,\ \mathrm{Re}s-4(M^2-m^2)\}\,.$$

We note also that $F_e(G)$ exists and is continuous on the boundary of this region. This may be verified by checking the integrability of the propagator singularities in $F_e(G)$ on the boundary.

Concluding this section we remark that the union of $\mathfrak{D}_e$ with the two analogous regions corresponding to the t and u channel, respectively, cover the set of real (s,t,u) values for which a general parametric Feynman integral is defined over real Feynman parameters.[17]

III. CONTINUATION TO A PHYSICAL TWO-PARTICLE CUT

In this section we shall show that by means of contour deformations $F_e(G)$ can be analytically continued[18] in s from the region of Fig. 2 to the physical two-particle branch cut.

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[12] We have obtained similar results in the case of unequal external masses.

[13] This choice is not possible for the special class of graphs obtained from graphs with a planar skeleton [see G. Tiktopoulos, Phys. Rev. **131**, 2373 (1963)] by crossing the p and q external lines (and their iterations). The typical case is the fourth-order crossed box graph. Since, however, one can easily find other convenient loop momenta assignments for such graphs and make the appropriate straightforward modifications in the argument, we prefer not to burden the discussion by considering them explicitly.

[14] Y. Nambu, Nuovo Cimento **6**, 1064 (1957); K. Symanzik, Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958); N. Nakanishi, *ibid.* **26**, 337 (1961); Y. Shinamoto, Nuovo Cimento **25**, 1292 (1962).

[15] f_5 also depends linearly on the *internal* masses.

[16] T. T. Wu, Phys. Rev. **123**, 678 (1961).

[17] This set was obtained by Wu (Ref. 16) by means of electric-circuit methods.

[18] In what follows $\mathfrak{p}$ will no longer be used as an independent variable and the (analytic) dependence of the Feyman amplitudes on s will be meant as expressed by

$$F[s,\mathfrak{p}(s)]=F[s,(\tfrac{1}{4}s-M^2)^{1/2}]\,,$$

where M is the *physical* external mass.

We shall assume that the lowest threshold corresponds to intermediate states of two particles of mass $m>0$ and that m is also the smallest internal mass of the graph. Under these assumptions we shall carry out the continuation to the region $4m^2<s<(2m+\bar{\mu})^2$, where $\bar{\mu}$ is one of the internal masses to be defined in what follows.

By a distortion C: $(\tilde{C},C_4)$ of the integration contours associated with the loop momentum $k^{(j)}$ we shall mean the following: Let $\mathfrak{k}^{(j)}$, $k_4^{(j)}$ be the components of $k^{(j)}$. Introducing spherical coordinates for the space part we have the variables

$$\mathfrak{k}^{(j)},\, k_4^{(j)} \rightarrow \phi^{(j)},\, \theta^{(j)},\, \mathfrak{k}^{(j)},\, k_4^{(j)}; \quad \mathfrak{k}=|\mathfrak{k}|$$
$$d^{(4)}k^{(j)}=d\phi^{(j)}d\cos\theta^{(j)}(\mathfrak{k}^{(j)})^2 d\mathfrak{k}^{(j)}dk_4^{(j)}.$$

Provided no singularities of the integrand of $F_e(G)$ forbid it, we may distort the real contours for $\mathfrak{k}^{(j)}:(0,\infty)$ and $k_4^{(j)}:(-\infty,+\infty)$ along the curves $\tilde{C}$ and C_4, respectively, so that they retain the same end-points. Let us also introduce the numbers $\tilde{\eta}$ and η_4 such that

$$|\mathrm{Im}\,\mathfrak{k}^{(j)}|<\tilde{\eta} \quad \text{along} \quad \tilde{C},$$
$$|\mathrm{Im}\,k_4^{(j)}|<\eta_4 \quad \text{along} \quad C_4,$$

as bounds to be determined. For our present purpose we shall need to distort the integration path of only those momenta which (under the specific assignment made in Fig. 1) appear in the combinations $W/2\pm k^{(j)}$. We must now verify that by an appropriate such distortion we can carry out the desired analytic continuation in s.

Let us examine the location of the various singularities of the integrand of $F_e(G)$ in the complex plane of the $k_4^{(j)}$ integration variables:

(i) From the zeros of denominators of the form $\mu^2-(p-k)^2$ where $p=(\mathfrak{p},0)$ is an external relative momentum (see Fig. 1) we have singularities at

$$k_4=\pm i[\mu^2+(\mathfrak{p}_r-\mathfrak{k}_r)^2-(\mathfrak{p}_i-\mathfrak{k}_i)^2 -2i(\mathfrak{p}_r-\mathfrak{k}_r)\cdot(\mathfrak{p}_i-\mathfrak{k}_i)]^{1/2}.$$

The subscripts r and i denote the real and the imaginary part of the associated vectors, respectively. It is easy to see that these singularities are confined in the k_4-plane by the two branches of the curve (see Fig. 3)

$$(\mathrm{Im}k_4)^2=\frac{(\mathrm{Re}k_4)^2+\mu^2-(|\mathfrak{p}_i|+\tilde{\eta})^2}{1-[(\mathrm{Re}k_4)^2/(|\mathfrak{p}_i|+\tilde{\eta})^2]}.$$

(ii) Denominators of the type $\mu^2-(k^{(j)}-k^{(m)})^2$ do not vanish provided

$$\mu^2>4(\tilde{\eta}^2+\eta_4^2).$$

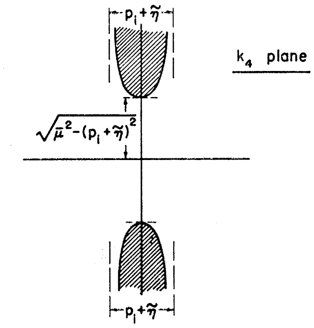

FIG. 3. Singularities of type (i) are confined in the shaded regions.

(iii) Finally from the zeros of denominators of the form $m^2-(\frac{1}{2}W\pm k)^2$ we have the singularities

$$k_4=\pm is^{1/2}/2\pm i(m^2+\mathfrak{k}^2)^{1/2}.$$

Clearly, the coincidence of $is^{1/2}/2-i(m^2+\mathfrak{k}^2)^{1/2}$ with $-is^{1/2}/2+i(m^2+\mathfrak{k}^2)^{1/2}$ at $s=4m^2$ for $\mathfrak{k}=0$ is responsible for the branch-point at $s=4m^2$. From the foregoing we see that for a continuation to the region $4m^2<s<(2m+\bar{\mu})^2$ the following conditions are necessary:

$$\text{(i)} \quad \bar{\mu}^2-(|\mathfrak{p}_i|+\tilde{\eta})^2>\eta_4^2,$$
$$\text{(ii)} \quad \bar{\mu}^2>4(\tilde{\eta}^2+\eta_4^2),$$
$$\text{(iii)} \quad \tfrac{1}{2}|\mathrm{Re}s^{1/2}|-m<\eta_4,$$

where $\bar{\mu}$ is the smallest mass occurring in propagators of type (i) and (ii).[19]

It is easily seen that the continuation to the cut from above may be carried out by means of contours of the form shown in Fig. 4. The constants $\tilde{\eta}$ and η_4 may be taken, e.g., to satisfy[20]

$$0<\tilde{\eta}<\min\left\{\frac{\frac{3}{4}\bar{\mu}^2-\mathfrak{p}_i^2}{2|\mathfrak{p}_i|},\ \tfrac{1}{4}[2m+\bar{\mu}-|\mathrm{Re}s^{1/2}|]^2\right\}$$

and

$$(\bar{\mu}^2/4-\tilde{\eta}^2)^{1/2}>\eta_4>\tfrac{1}{2}|\mathrm{Re}s^{1/2}|-m.$$

Thus, $F_e(G)$ is continued to a region of the s plane which includes (at least) the points satisfying the inequalities (Fig. 5)

$$\mathfrak{p}_i^2<3\bar{\mu}^2/4 \quad |\mathrm{Re}s^{1/2}|<2m+\bar{\mu}.$$

Actually, by means of contour deformations of the described kind one can continue $F_e(G)$ to a larger region of the complex plane extending also into the second sheet through the two-particle cut. However, going beyond $2m+\bar{\mu}$ on the real axis seems to require additional effort.

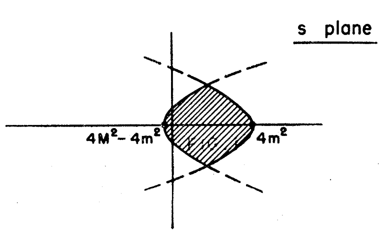

FIG. 2. Region of the complex s plane where the Euclidean integral $F_e(G)$ converges and (without any contour distortions) defines an analytic function of s. m is the (smallest) internal mass and M is the external mass.

[19] Clearly, for a given graph there exists an optimum choice of loop momenta which makes $\bar{\mu}$ largest.

[20] $\tilde{\eta}$ must be different from zero because for $\tilde{\eta}=0$ the singularities at $\pm i[s^{1/2}/2-(m^2+\mathfrak{k}^2)^{1/2}]$ would hit C_4 for $\mathfrak{k}$ sufficiently large (see Fig. 4).

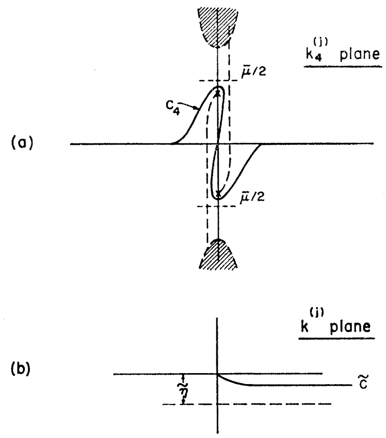

FIG. 4. Form of the contours $C_4(a)$ and $\tilde{C}(b)$ which allow an analytic continuation to the physical region $4m^2<s<(2m+\bar{\mu})^2$.

We note also that the foregoing analytic continuation was carried out for a fixed value M^2 of the external mass ($M^2<2m^2$) and for any fixed physical value of the scattering angle $\angle \mathfrak{pq}$. Consequently, our analysis is also true for the partial wave projections of the Feynman amplitude defined as

$$F_l(s)=\tfrac{1}{2}\int_{-1}^{1} d\cos\angle\mathfrak{pq}\cdot F(s,\cos\angle\mathfrak{pq})\,.$$

IV. THE BETHE-SALPETER EQUATION

The Bethe-Salpeter equation for a two-particle scattering process $a+b\rightarrow a+b$ may be formulated by noticing that the class T of all Feynman graphs with four external lines can be subdivided into subsets having the structure shown in Fig. 6, where V is the subset of all "irreducible" graphs, i.e., graphs with no intermediate state of the two particles a and b. One thus obtains the following integral equation for the *off-the-mass-shell* amplitude

$$\langle p|T(W)|q\rangle=\langle p|V(W)|q\rangle+\lambda\int\langle p|V(W)|k'\rangle \times G_1(\tfrac{1}{2}W+k')G_2(\tfrac{1}{2}W-k')\langle k'|T(W)|q\rangle d^4k'\,,\quad (1)$$

where G_1 and G_2 are the propagators of the particles a and b. Spin indices and summations over them have been omitted. Symbolically we write in operator form

$$T=V+\lambda VG_1G_2T\,,$$

where G_1, G_2 appear as "diagonal" operators.

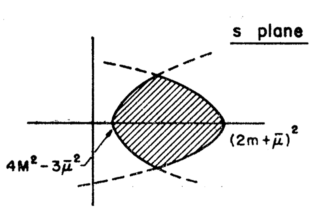

FIG. 5. Part of the region in the s plane to which analytic continuation is possible by means of the contours shown in Figs. 4(a) and 4(b).

The series represented in Fig. 6 can also be written as

$$T=V+\sum_{n=0}^{\infty}\lambda VG_1(\lambda G_2VG_1)^nG_2V\,.$$

In order to employ our previous results we take $m_a=m_b=m>0$ and we assume that m is the smallest mass occurring in our problem. Setting $\mathfrak{p}^2=\mathfrak{k}^2=s/4-m^2$ and $p_4=k_4=0$, we express the *mass-shell* amplitude as

$$\langle p|T(s)|q\rangle=\langle p|T(s)|q\rangle+\sum_{n=0}^{\infty}\lambda^{n+1}\int\langle p|VG_1|k'\rangle \times\langle k'|(G_2VG_1)^n|k''\rangle\langle k''|G_2V|q\rangle d^4k'd^4k''\,.\quad (2)$$

In this expression the integral is equal to the sum of Feynman integrals coming from the $(n+2)$th subset (or "iteration") in Fig. 6. For physical scattering angles and s in the physical region $4m^2<s<(2m+\bar{\mu})^2$, these integrals may be evaluated either according to the standard Feynman rules and the prescription $m^2-i\epsilon\rightarrow m^2$ or, as we have seen, by integrating the loop momenta over the appropriately deformed Euclidean contours $(\tilde{C},C_4)$. Attaching the subscript c to variables

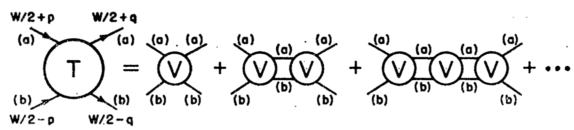

FIG. 6. Graphical scheme for the formulation of the Bethe-Salpeter equation.

which are integrated over $(\tilde{C},C_4)$, we write (2) as

$$\begin{aligned}\langle p|T(s)|q\rangle &=\langle p|V(s)|q\rangle+\sum_{n=0}^{\infty}\lambda^{n+1}\int\int\langle p|VG_1|k_c'\rangle \times\langle k_c'|(G_2VG_1)^n|k_c''\rangle\langle k_c''|G_2V|q\rangle d^4k'd^4k''\\ &=\langle p|V(s)|q\rangle+\lambda\int\langle p|VG_1|k_c'\rangle\langle k_c'|(1-\lambda G_2VG_1)^{-1}|k_c''\rangle \langle k_c''|G_2V|q\rangle d^4k_c'd^4k_c''\,,\quad (3)\end{aligned}$$

where $\langle k_c'|(1-\lambda G_2VG_1)^{-1}|k_c''\rangle$ is the familiar notation for the resolvent of the kernel

$$\lambda\langle k_c'|G_2VG_1|k_c''\rangle\,.$$

We have thus expressed the physical T-matrix

[21] The fact that the field-theoretic Green's functions are formally related to Euclidean ones by analytic continuation was discussed by J. Schwinger, Proc. Natl. Acad. Sci. U. S. 44, 956 (1958) and T. Nakano, Progr. Theoret. Phys. (Kyoto) 21, 241 (1959), independently.

element directly in terms of the resolvent of the "Euclidean" kernel.[21] Our emphasis here is on the fact that this kernel does not have the singularities which are specific to field-theoretic kernels and are connected with the necessity of following the $i\epsilon$ prescription. As sources of trouble there still remain:

(i) the behavior for large momenta (after renormalization); similar problems arise in nonrelativistic problems of scattering by singular potentials;

(ii) the fact that in an exact calculation an infinite number of irreducible graphs contribute to V.

These problems are beyond our present scope.

In connection with the relation of expression (3) to (2), it should be made clear that, although we formulated the B-S equation in terms of Feynman graphs, expression (3) has in general a meaning beyond the values of λ for which (2) converges. Therefore, having obtained the resolvent of the Euclidean kernel, one should verify that (3) is a solution of (1), e.g., by direct substitution. In the case of the ϕ^3 theory, which we shall consider below, the resolvent is given by a uniformly convergent Fredholm formula which is easily seen to satisfy Eq. (1).

Let us now illustrate our approach in more detail by considering the simple case of a trilinear scalar interaction of the ϕ^3 type. Let V consist of any *finite* set of irreducible graphs. One can then verify by simple power counting that $K=(G_2VG_1)_c$ is a Schmidt operator, namely,

$$\|K\|^2=\int\int|\langle k_c'|G_2VG_1|k_c''\rangle|^2|d^4k_c'|\times|d^4k_c''|<+\infty\,, \quad (4)$$

where[22] $|d^4k_c|=d\phi d\cos\theta|\mathfrak{k}^2d\mathfrak{k}|_c|dk_4|_{c_4}$.

Therefore, its resolvent kernel is given by the Fredholm formula[23]

$$\langle k_c'|(1-\lambda K)^{-1}-1|k_c''\rangle=\sum_{n=0}^{\infty}\lambda^{n+1}\Delta_n(k_c',k_c'')/\sum_{n=0}^{\infty}\lambda^n\delta_n\,,$$

where $\delta_0=1$, $\Delta_0(k_c',k_c'')=K(k_c',k_c'')$

$n\geqslant 1$;

$$\delta_n=\frac{(-1)^n}{n!}\begin{vmatrix} 0 & n-1 & 0 & \cdots & 0 & 0 & 0\\ \sigma_2 & 0 & n-2 & \cdots & 0 & 0 & \\ \sigma_3 & \sigma_2 & 0 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-1} & \cdots & \sigma_2 & 0 & 1\\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_3 & \sigma_2 & 0\end{vmatrix},$$

$$\Delta_n=\frac{(-1)^n}{n!}\begin{vmatrix} K & n & 0 & \cdots & 0 & 0 & 0\\ K^2 & 0 & n-1 & \cdots & 0 & 0 & 0\\ K^3 & \sigma_2 & 0 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ K^n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_2 & 0 & 1\\ K^{n+1} & \sigma_n & \sigma_{n-1} & \cdots & \sigma_3 & \sigma_2 & 0\end{vmatrix},$$

$\sigma_n=\mathrm{Trace}(K^n)$.

The convergence of the series in the numerator and denominator of the resolvent is ensured by the inequalities[23]

$$|\delta_n|\leq(e/n)^{n/2}\|K\|^n;\ \|\Delta_n\|\leq(1/n^{1/2})(e/n)^{n+1}\|K\|^{n+1}. \quad (5)$$

Our solution then reads

$$\langle p|T(s)|q\rangle=\langle p|V(S)|q\rangle+\lambda\langle p|VG_1G_2V|q\rangle+\sum_{n=0}^{\infty}\frac{1}{\lambda^n\delta_n(s)}\sum_{n=0}^{\infty}\lambda^{n+2}\int\int\langle p|VG_1|k_c'\rangle\times\Delta_n(k_c',k_c'')\langle k_c''|G_2V|q\rangle d^4k_c'd^4k_c''. \quad (6)$$

The denominator series converges absolutely and uniformly because of the first of inequalities (5). Noting the finiteness of the integrals

$$\int|\langle p|VG_1|k_c\rangle|^2|d^4k_c|=N'^2,$$

$$\int|\langle k_c|G_2V|q\rangle|^2|d^4k_c|=N''^2,$$

and using the Schwartz inequality, we find that the nth term of the numerator of (6) is not greater than

$$\lambda^{n+1}N'N''(1/n^{1/2})(e/n)^{n+1}\|K\|^{n+1}.$$

Therefore the numerator series converges[24] absolutely and uniformly in any subdomain of the region $\mathcal{E}$ in the s plane to which we were able to continue the Feynman integrals by contour deformations.[25] It should be

[22] In the sense, for example, of K. Knopp, *Theory of Functions I* (Dover Publications, Inc., New York, 1945), p. 37.

[23] We employ the form of the Fredholm series for the resolvent of a Schmidt kernel in terms of its traces due to F. Smithies, Duke Math. J. 8, 107 (1941). See also F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).

[24] This technique of analytic continuation of the Fredholm formula was developed from a suggestion by M. Rubin on a similar problem in potential scattering.

[25] Being valid for all physical angles our analysis can be readily carried over to the B-S equation for the partial-wave amplitude. For example, in the "ladder approximation," the partial-wave equation has the form of Eq. (1) with V replaced by V_l:

$$\langle k''|V_l|k'\rangle=\frac{1}{2\mathfrak{k}'\mathfrak{k}''}Q_l\left(\frac{\mu^2+\mathfrak{k}'^2+\mathfrak{k}''^2-(k_0'-k_0'')^2}{2\mathfrak{k}'\mathfrak{k}''}\right)$$

and d^4k' is replaced by $d\mathfrak{k}'dk_0'$ (see Ref. 5). For physical values of l the solution is again of the form of Eq. (6). The continuation to complex values of l is immediate since the meromorphy of the Euclidean resolvent has been established in Ref. 5 ($\mathrm{Re}l>-\frac{3}{2}$) and Ref. 9 ($\mathrm{Re}l>-\frac{5}{2}$).

emphasized at this point that although the "Schmidt norm" of K given by Eq. (4) depends on the contour C, the Fredholm expansion (6) does not. From the given expressions for δ_n and Δ_n we see that for actual calculations one has to evaluate integrals of the form

$$\int\int \langle p|VG_1|k_c'\rangle\langle k_c'|(G_2VG_1)^n|k_c''\rangle \times\langle k_c''|G_2V|q\rangle d^4k_c'd^4k_c'' ,$$

which are equal to ordinary Feynman integrals corresponding to iterations of V. They may be computed in the physical region by any convenient method, e.g., Feynman parametrization. In addition, one has to calculate what we might call "traces of Feynman graphs":

$$\mathrm{Tr}(G_2VG_1)^n=\int\langle k_c'|(G_2VG_1)^n|k_c'\rangle d^4k_c' .$$

These traces depend only on the energy variable s (and the various masses) and can also be computed by parametrization. It is then an easy matter to show by standard methods that *the traces* [*and therefore also* $\sum\lambda^n\delta_n$] *are analytic in the s plane with only a right-hand cut* starting at the lowest physical threshold.

We thus see that our Fredholm solution can be expressed in terms of *ordinary* Feynman integrals and their traces [it is only for deciding questions of convergence that we need rewrite them in the Euclidean form] Therefore, we can verify that it satisfies (1) by direct substitution and comparison of the various terms in every order in λ.

Since we are dealing with a kernel of the Schmidt type, its analytic dependence on s (in $\mathcal{E}$) implies immediately that its resolvent is meromorphic[26] with respect to s (in $\mathcal{E}$). The location of the poles is given by the zeros of the "Fredholm determinant"

$$D=\sum_{n=0}^{\infty}\lambda^n\delta_n(s) .$$

For s in the real interval $[0,4m^2]$ (where no distortion of the Euclidean contours is necessary) the zeros of D at $\lambda=\lambda_i(s)$; $i=1, 2, \cdots$ correspond to the eigenvalues of K. Apart from trivial similarity transformations, K in this region is identical to the kernel of the (homogeneous) Bethe-Salpeter equation for bound-state wave functions in which a rotation of the relative energy paths à la Wick has been carried out. Thus, by examining the poles of our solution, we arrive at the same bound-state spectrum as the one obtained by Wick and Cutkosky.

In the ϕ^3 case, instead of the standard Fredholm method that we have indicated, one may use any other method that appears more convenient in a particular problem. An alternative and perhaps more flexible technique is the Schmidt trick, which consists in approximating the given kernel by a kernel of finite rank and using perburbation theory for the remaining part. Such methods have been extensively applied to potential scattering by S. Weinberg.[27]

The point of view taken by Weinberg is that the failure of perturbation methods in field theory may be due, as in potential scattering, to the presence of bound states. Therefore, the Schmidt method (formulated as an introduction of "quasiparticles" into the theory) could always make perturbation theory work. In view of our present results this assumption is most probably correct in the simple ϕ^3 theory where the scattering problem can be formulated in terms of the Fredholm-Schmidt theory. However, in physically interesting renormalizable field theories (e.g., ϕ^4 or fermion-boson interactions of the $\bar{\psi}\Gamma\psi B$ type) the behavior of the kernels at large momenta does not allow Fredholm expansions. Accordingly, as it is indicated by simple model calculations,[6,8,28–30] singularities of a new kind may arise, e.g., square-root branch points in the coupling constant whose direct physical interpretation is not clear. (The analogy of this situation to the nonrelativistic problem with a potential having an $1/r^2$ singularity at the origin is familiar.) Thus, it seems necessary to develop a technique for treating the singular (at-large momenta) part of the kernel before any considerations of the Schmidt type are possible.

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[26] See theorem in the Appendix of Ref. 9.

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